On the classification of thin distance-regular graphs with classical parameters

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(Based on joint work with Jack H. Koolen)

Terwilliger Fest

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Outline

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1. Introduction
   - Definitions
   - Thin DRGs
2. Our result
   - Twisted Latin Square graphs
   - Co-edge-regular graphs with $\lambda_{\min} = -3$
   - Thin DRGs with $b = 2$
3. Open problem

Definition 1

$\Gamma = (V, E)$ is a graph with a vertex set V and an edge set $E \subseteq \binom{V}{2}$.

- The adjacency matrix $A(\Gamma)$ of a graph Γ is the matrix whose rows and columns are indexed by its vertices, such that $A_{xy}(\Gamma) = 1$ if xy is an edge and 0 otherwise.
- The eigenvalues of Γ are the eigenvalues of its adjacency matrix.
- An eigenvalue of a graph is called non-principal if it has an eigenvector orthogonal to the all-ones vector.
- Two graphs are called cospectral if they have the same spectrum.

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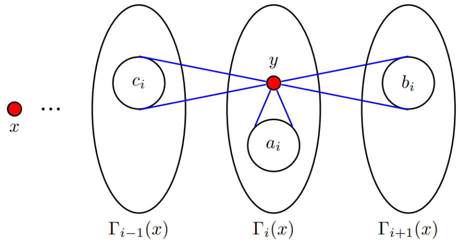
- Two graphs are called cospectral if they have the same spectrum.
- A graph Γ is called k-regular if every vertex in Γ has k neighbours.
- The neighborhood $N(x)$ of x is the set of vertices adjacent to x.
- For a vertex x of Γ, the subgraph induced on $N(x)$ is called the local graph of Γ at x denoted by $\Gamma_1(x)$.

Definition 2

A connected graph Γ with diameter D is called distance-regular if for $0 \le i \le D$ there are integers b_i and c_i such that for every pair of vertices $x, y \in V(\Gamma)$ with $d(x,y) = i$, among the neighbors of y, there are precisely c_i (resp. b_i) at distance $i-1$ (resp. $i+1$) from x.



Definition 3

A DRG Γ of diameter D has classical parameters (D, b, α, β) if the intersection numbers of Γ satisfy $c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_b (1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b)$ and $b_i = (\begin{bmatrix} D \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b)(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b)$ for $0 \leq i \leq D$, where $\begin{bmatrix} i \\ 1 \end{bmatrix}_b = \frac{b^i - 1}{b - 1}$.

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- There are a lot of DRGs with classical parameters such as the Johnson graphs, the Hamming graphs, the Grassmann graphs, the bilinear forms graphs and so on.
- All the known DRGs with diameter large enough are either DRGs with classical parameters or derived from DRGs with classical parameters

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For a pair x, y of adjacent vertices in a graph G, let $\lambda(x, y)$ be the number of common neighbours of x and y. We say that a co-edge-regular graph is of level t if $\#\{\lambda(x, y) \mid x, y \text{ are adjacent vertices } \} = t$.

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- A co-edge-regular graph of level 1 is a strongly-regular graph

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Proposition 2: The non-principal eigenvalues of $\Gamma_1(x)$ are in the set $\{\beta - \alpha - 1, \frac{b^D-b}{b-1}\alpha - 1, -1, -b-1\}$

P. Terwilliger, Lecture notes on Terwilliger algebra, 1993.

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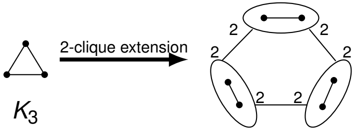
- We only need focus on co-edge-regular graph with at least four distinct eigenvalues.
- For example, clique-extensions of strongly-regular graphs (Sterner graphs or Latin Square graphs) have exactly four distinct eigenvalues and co-edge-regular.

Definition 6

For a positive integer s, the *s-clique extension* of a graph G is the graph $\tilde{G}$ obtained from G by replacing each vertex $x \in V(G)$ by a clique $\tilde{X}$ with s vertices, such that $\tilde{x} \sim \tilde{y}$ (for $\tilde{x} \in \tilde{X}$, $\tilde{y} \in \tilde{Y}$) in $\tilde{G}$ if and only if $x \sim y$ in G.



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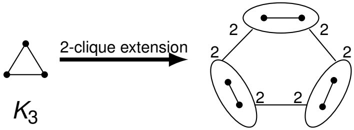

$$A(\tilde{G}) = (A(G) + I_v) \otimes J_s - I_{sv}$$

Remark 1

If $Spec(G) = \{[\theta_0]^{m_0}, [\theta_1]^{m_1}, \ldots, [\theta_r]^{m_r}\}$, *then*

$$Spec(\tilde{G}) = \{[s(\theta_0+1)-1]^{m_0}, [s(\theta_1+1)-1]^{m_1}, \ldots, [s(\theta_r+1)-1]^{m_r}, [-1]^{(s-1)v}\}.$$

Conjecture 1 (Tan-Koolen-Xia 2020)

Let G be a connected co-edge-regular graph with parameters (v,k,μ) and exactly four distinct eigenvalues. For fixed integer $t \geq 2$, there exists a constant n_t such that, if $\lambda_{\min}(G) \geq -t$, $v \geq n_t$ and $k < v - 2 - \frac{(t-1)^2}{4}$, then

- *G is the s-clique extension of a strongly regular graph for $2 \leq s \leq t-1$, or*
- *G is a $p \times q$-grid with $p > q \geq 2$.*

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- However, we will show that this conjecture is false, by providing an infinite family of co-edge-regular graphs, called twisted Latin square graphs.

Y. Tan and J.H. Koolen and Z. Xia, A spectral characterization of the s-clique extension of the triangular graphs. Discuss. Math. Graph Theory, 2020.

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- In the following, q is a prime power.
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- The non-principal eigenvalues of $TLS(q,n)$ are $q^2(n-1)-1$, -1, and $-q^2-1$, corresponding to the classical parameters $\alpha = b = q^2$.

H.-J. Ge and J.H. Koolen, On co-edge-regular graphs with 4 distinct eigenvalues. arXiv:2503.12025, 2025+.

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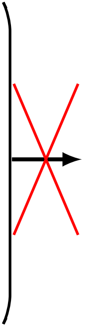

Uniquely determined

Problem 2

What is known about co-edge-regular graphs with at most five distinct eigenvalues and smallest eigenvalue equal to -3 or -4?

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Theorem 3 (Ge-Koolen 2025+)

Let G be a co-edge-regular graph with parameters (v, k, μ) and non-principal eigenvalues in the set $\{\beta - \alpha - 1, (2^D - 2)\alpha - 1, -1, -3\}$, where $v = (2^D - 1)\beta$ and $k = \beta - 1 + (2^D - 2)\alpha$. There exists a positive integer κ such that if $k \geq \kappa$, then one of the following holds:

1. $\mu = 6$, $\alpha = 2$, *and G is the Latin square graph;*
2. $\mu = 9$, $\alpha = 3$, *and G is the Steiner graph;*
3. $\mu = 4$, $\alpha = 2$, *and G is the 2-clique extension of the $(\frac{k+3}{4} \times \frac{k+3}{4})$-grid;*
4. $\mu = 8$, $\alpha = 4$, *and G is the 2-clique extension of the triangular graph $T(\frac{k+7}{4})$;*
5. $\mu = 4$, $\alpha = 2$, *and G is the 2-clique extension of the $(\frac{\beta}{2} \times (2^D - 1))$-grid.*

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Case 1: $\Gamma_1(x)$ is a strongly-regular graph

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Lemma 7 (Jurišić-Koolen-Terwilliger 2000)

If Γ is a distance-regular graph with classical parameters (D, b, α, β). Then Γ is tight if and only if there exists a vertex x such that $\Gamma_1(x)$ is connected strongly-regular with eigenvalues $\{a_1, \beta - \alpha - 1, -b - 1\}$.

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Theorem 8 (Koolen-Abdullah-Gebremichel-Lee 2024)

Let Γ be a tight distance-regular graph with classical parameters (D,b,α,β). If $D\geq 10$, then one of the following holds:

- *Γ is a Johnson graph $J(2D,D)$,*
- *Γ is a halved ℓ-cube where $\ell\in\{2D,2D+1\}$,*

A. Jurišić and J. H. Koolen and P. Terwilliger, Tight distance-regular graphs. J. Algebraic Combin., 2000.

J.H. Koolen and M. Abdullah and B. Gebremichel and J.-H. Lee, Towards a classification of 1-homogeneous distance-regular graphs with positive intersection number a_1. 2024.

Case 2: $\Gamma_1(x)$ has 4 distinct eigenvalues

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Theorem 9 (Ge-Koolen 2025+)

Let Γ be a thin Q-polynomial DRG with classical parameters (D, b, α, β), where $D \geq 12$, $b \geq 2$ and $\alpha > 0$. Assume that $\Gamma_1(x)$ is (b_0, a_1, μ)-co-edge-regular with exactly four eigenvalues $a_1 = \theta_0 > \theta_1 > -1 > -b-1$. If $\mu = 2\alpha$. Then Γ is the Grassmann graph $J_q(2D, D)$.

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Theorem 10 (Ge-Koolen 2025+)

Let Γ be a thin Q-polynomial DRG with classical parameters (D, b, α, β), where $D \geq 12$, $b \geq 2$ and $\alpha > 0$. If $\Gamma_1(x)$ has exactly four eigenvalues, then it must contains -1 as an eigenvalue.

Brouwer, A.E. and Cohen, A.M. and Neumaier, A., Distance-Regular Graphs. Springer-Verlag, 1989.

Case 3: $\Gamma_1(x)$ is a clique-extensions of $p \times q$-grid

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Theorem 11 (Numata-Cohen-Cooperstein)

Let Γ be a finite connected graph such that

1. $\mu_\Gamma(x,y) = \Gamma[N(x) \bigcap N(y)]$ *is a non-degenerate grid for every pair of vertices x and y with distance 2.*
2. *If $\{x,y,z\} \subset V(\Gamma)$ is an independent set, then $N(x) \cap N(y) \cap N(z)$ is an independent set.*

Then Γ is isomorphic to one of the following graphs:

- *A complete graph K_t,*
- *A Johnson graph $J(n,k)$,*
- *A folded Johnson graph,*
- *A Grassmann graph $J_q(n,D)$.*

Theorem 4 (Ge-Koolen 2025+)

Let Γ be a thin Q-polynomial DRG with classical parameters (D, b, α, β), where $D \geq 12$, $b = 2$ and $\alpha \geq 0$.

1. *$\alpha = 0$ and Γ is a dual polar graph;*
2. *$\alpha = 2$ and Γ is a Grassmann graph $J_2(n, D)$.*

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Let G be a co-edge-regular graph with level t and exactly $d+1$ distinct eigenvalues. Is $t \leq f(d)$ bounded by a function of d? Even for $d = 3$ we do not know it.

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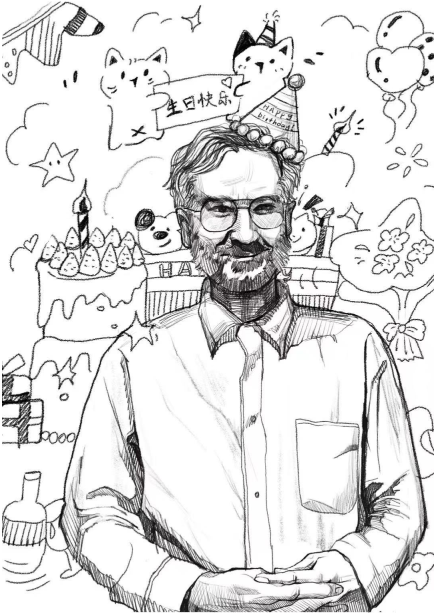
生日快乐
HAPPY Birthday!
HA