Four bases for the Onsager Lie algebra related by a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action

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Let $\mathbb{F}$ denote a field of characteristic zero.

Definition (Onsager 1944, Perk 1987)

Let O denote the Lie algebra over $\mathbb{F}$ with generators A, B and relations

$$[A,[A,[A,B]]] = 4[A,B], \tag{1}$$

$$[B,[B,[B,A]]] = 4[B,A]. \tag{2}$$

We call O the Onsager Lie algebra. We call A, B the standard generators for O.

The relations (1), (2) are called the Dolan-Grady relations.

The Onsager Lie algebra O is closely related to a Lie algebra $\boxtimes$, called the tetrahedron algebra. Define the set $\mathbb{I} = \{0, 1, 2, 3\}$.

Definition (Hartwig, Terwilliger 2007)

Let $\boxtimes$ denote the Lie algebra over $\mathbb{F}$ that has generators

$$\{x_{ij} \mid i, j \in \mathbb{I}, \quad i \neq j\}$$

and the following relations:

(i) For distinct $i, j \in \mathbb{I}$,
$$x_{ij} + x_{ji} = 0.$$

(ii) For mutually distinct $h, i, j \in \mathbb{I}$,
$$[x_{hi}, x_{ij}] = 2x_{hi} + 2x_{ij}.$$

(iii) For mutually distinct $h, i, j, k \in \mathbb{I}$,
$$[x_{hi}, [x_{hi}, [x_{hi}, x_{jk}]]] = 4[x_{hi}, x_{jk}].$$

We call $\boxtimes$ the tetrahedron algebra, and x_{ij} the standard generators for $\boxtimes$.

An S_4-action on $\boxtimes$

- Consider the symmetric group S_4 on the set $\mathbb{I} = \{0, 1, 2, 3\}$, with generators

$$\rho = (12)(30), \quad \tau = (12), \quad \mu = (23)(10), \quad \varphi = (123).$$

- The group S_4 acts on the generators x_{ij} of $\boxtimes$ by permuting indices:

$$\beta(x_{ij}) = x_{\beta(i)\beta(j)} \qquad (\beta \in S_4,\ i \neq j \in \mathbb{I}).$$

- This action induces an injective group homomorphism (Hartwig, Terwilliger 2007):

$$S_4 \hookrightarrow \mathrm{Aut}(\boxtimes).$$

Relationship between O and $\boxtimes$ (Hartwig, Terwilliger 2007)

- For mutually distinct $h, i, j, k \in \mathbb{I}$, there exists an injective Lie algebra homomorphism from O to $\boxtimes$ that sends

$$A \to x_{ij}, \qquad B \to x_{hk}.$$

- Call the image of O under this homomorphism an Onsager subalgebra of $\boxtimes$.

- Let $\mathcal{O}$, $\mathcal{O}'$, $\mathcal{O}''$ be the subalgebras of $\boxtimes$ such that

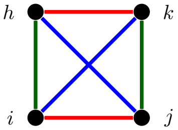
$$\mathcal{O} = \langle x_{ij}, x_{hk} \rangle, \qquad \mathcal{O}' = \langle x_{jk}, x_{hi} \rangle, \qquad \mathcal{O}'' = \langle x_{ki}, x_{hj} \rangle.$$

  Each of $\mathcal{O}$, $\mathcal{O}'$, $\mathcal{O}''$ is isomorphic to O.

- The $\mathbb{F}$-vector space $\boxtimes$ satisfies

$$\boxtimes = \mathcal{O} + \mathcal{O}' + \mathcal{O}'' \qquad \text{(direct sum)}$$

(Illustration)



Recall

- The Lie algebra $\mathfrak{sl}_2$ over $\mathbb{F}$ has a basis e, f, h and Lie brackets

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

- The Lie algebra $\mathfrak{sl}_2$ has another basis called the equitable basis x, y, z (Hartwig, Terwilliger 2007):

$$x = 2e - h, \qquad y = -2f - h, \qquad z = h,$$

  with Lie brackets

$$[x, y] = 2x + 2y, \qquad [y, z] = 2y + 2z, \qquad [z, x] = 2z + 2x.$$

- Define

$$\mathcal{A} = \mathbb{F}[t, t^{-1}, (t-1)^{-1}].$$

- There exists a unique $\mathbb{F}$-algebra automorphism $\prime$ of $\mathcal{A}$ that sends t to $1 - t^{-1}$.
- Note that

$$t' = 1 - t^{-1}, \qquad t'' = (1-t)^{-1}, \qquad t''' = t.$$

Definition (Hartwig, Terwilliger 2007)

Let $L(\mathfrak{sl}_2)^+$ denote the Lie algebra over $\mathbb{F}$ consisting of the $\mathbb{F}$-vector space

$$\mathfrak{sl}_2 \otimes \mathcal{A}, \qquad \otimes = \otimes_{\mathbb{F}}$$

and Lie bracket

$$[u \otimes a, v \otimes b] = [u, v] \otimes ab, \qquad u, v \in \mathfrak{sl}_2, \quad a, b \in \mathcal{A}.$$

We call $L(\mathfrak{sl}_2)^+$ the three-point $\mathfrak{sl}_2$ loop algebra.

- Note that the algebra $L(\mathfrak{sl}_2)^+$ is a right $\mathcal{A}$-module with the action map:

$$L(\mathfrak{sl}_2)^+ \times \mathcal{A} \longrightarrow L(\mathfrak{sl}_2)^+, \qquad (u \otimes a, b) \longmapsto u \otimes ab.$$

- The elements

$$u \otimes 1, \quad u \otimes t^n, \quad u \otimes (t')^n, \quad u \otimes (t'')^n, \qquad u \in \{x, y, z\}, \quad n \in \mathbb{N}^+$$

form a basis for $L(\mathfrak{sl}_2)^+$.

Lemma (Hartwig, Terwilliger 2007)

There exists a unique Lie algebra isomorphism

$$\sigma : \boxtimes \longrightarrow L(\mathfrak{sl}_2)^+$$

that sends

$$\begin{aligned} x_{12} &\longmapsto x \otimes 1, & x_{03} &\longmapsto y \otimes t + z \otimes (t-1), \\ x_{23} &\longmapsto y \otimes 1, & x_{01} &\longmapsto z \otimes t' + x \otimes (t'-1), \\ x_{31} &\longmapsto z \otimes 1, & x_{02} &\longmapsto x \otimes t'' + y \otimes (t''-1), \end{aligned}$$

where x, y, z is the equitable basis for $\mathfrak{sl}_2$.

We call x_{ij}^{σ} the standard generators for $L(\mathfrak{sl}_2)^+$.

An S_4-action on $L(\mathfrak{sl}_2)^+$ (Elduque, 2007)

- Recall the S_4-action on $\boxtimes$ and the embedding $S_4 \hookrightarrow \mathrm{Aut}(\boxtimes)$.
- Using the Lie algebra isomorphism $\sigma : \boxtimes \to L(\mathfrak{sl}_2)^+$, an S_4-action is induced on $L(\mathfrak{sl}_2)^+$ as group automorphisms.
- For $\beta \in S_4$ and the standard generators x^σ_{ij} in $L(\mathfrak{sl}_2)^+$,

$$\beta(x^\sigma_{ij}) = x^\sigma_{\beta(i)\beta(j)} \qquad (i, j \in \mathbb{I}, \quad i \neq j).$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} \boxtimes & \xrightarrow{\ \sigma\ } & L(\mathfrak{sl}_2)^+ \\ \beta \big\downarrow & & \big\downarrow \beta \\ \boxtimes & \xrightarrow{\ \sigma\ } & L(\mathfrak{sl}_2)^+ \end{array} \qquad (\beta \in S_4)$$

Actions of ρ and τ on $L(\mathfrak{sl}_2)^+$

Recall $\rho = (12)(30)$ and $\tau = (12)$.

- ρ is the automorphism of $L(\mathfrak{sl}_2)^+$ given by

$$\rho(u \otimes a) = \rho(u \otimes 1)a$$

for all $u \in \mathfrak{sl}_2$ and $a \in \mathcal{A}$, where

$$\begin{aligned} \rho(x \otimes 1) &= -x \otimes 1, \\ \rho(y \otimes 1) &= (x \otimes 1 + z \otimes (1-t))t^{-1}, \\ \rho(z \otimes 1) &= (x \otimes 1 + y \otimes t)(1-t)^{-1}. \end{aligned}$$

- $\tau = \tau_{\mathfrak{sl}_2} \otimes \tau_{\mathcal{A}}$ is the automorphism of $L(\mathfrak{sl}_2)^+$ given by

$$\tau(u \otimes a) = \tau_{\mathfrak{sl}_2}(a) \otimes \tau_{\mathcal{A}}(a)$$

for all $u \in \mathfrak{sl}_2$ and $a \in \mathcal{A}$, where $\tau_{\mathfrak{sl}_2}$ is the order 2 automorphism of $\mathfrak{sl}_2$ given by

$$\tau_{\mathfrak{sl}_2}(x) = -x, \qquad \tau_{\mathfrak{sl}_2}(y) = -z, \qquad \tau_{\mathfrak{sl}_2}(z) = -y,$$

and $\tau_{\mathcal{A}}$ is the order 2 automorphism of $\mathcal{A}$ determined by $\tau_{\mathcal{A}}(t) = 1 - t$.

Actions of μ and φ on $L(\mathfrak{sl}_2)^+$

Recall $\mu = (23)(10)$ and $\varphi = (123)$.

- μ is the automorphism of $L(\mathfrak{sl}_2)^+$ given by

$$\mu(u \otimes a) = \mu(u \otimes 1)a$$

for all $u \in \mathfrak{sl}_2$ and $a \in \mathcal{A}$, where

$$\mu(x \otimes 1) = y \otimes t + z \otimes (t-1),$$
$$\mu(y \otimes 1) = -y \otimes 1,$$
$$\mu(z \otimes 1) = (x \otimes 1 + y \otimes t)(t-1)^{-1}.$$

- $\varphi = \varphi_{\mathfrak{sl}_2} \otimes \varphi_{\mathcal{A}}$ is the automorphism of $L(\mathfrak{sl}_2)^+$ given by

$$\varphi(u \otimes a) = \varphi_{\mathfrak{sl}_2}(a) \otimes \varphi_{\mathcal{A}}(a)$$

for all $u \in \mathfrak{sl}_2$ and $a \in \mathcal{A}$, where $\varphi_{\mathfrak{sl}_2}$ is the order 3 automorphism of $\mathfrak{sl}_2$ given by

$$\varphi_{\mathfrak{sl}_2}(x) = y, \qquad \varphi_{\mathfrak{sl}_2}(y) = z, \qquad \varphi_{\mathfrak{sl}_2}(z) = x,$$

and $\varphi_{\mathcal{A}}$ is the order 3 automorphism of $\mathcal{A}$ determined by $\varphi_{\mathcal{A}}(t) = 1 - t^{-1}$.

- Recall the Onsager Lie algebra O with the standard generators A and B.

- By an Onsager subalgebra of $L(\mathfrak{sl}_2)^+$, we mean the σ-image of an Onsager subalgebra of ⊠.

- For the rest of this talk, we identify O with one of the three Onsager subalgebras of $L(\mathfrak{sl}_2)^+$, namely,

$$O = \langle A, B \rangle \subset L(\mathfrak{sl}_2)^+,$$

where

$$A = x_{12}^{\sigma} = x \otimes 1, \qquad B = x_{03}^{\sigma} = y \otimes t + z \otimes (t - 1).$$

The subgroup G of S_4

Define the subgroup G of S_4:

$$G = \langle \rho, \tau \rangle \approx \mathbb{Z}_2 \times \mathbb{Z}_2,$$

where $\rho = (12)(30)$ and $\tau = (12)$.

Lemma

Recall the standard generators A, B of O. Then ρ, τ act on A, B as follows:

$$\begin{aligned} \rho(A) &= -A, & \rho(B) &= -B, \\ \tau(A) &= A, & \tau(B) &= -B. \end{aligned}$$

Moreover, O is invariant under the G-action.

Remark:

- The action of $\varphi = (123)$ on O was studied (L., 2024).
- The action of $\mu = (23)(10)$ fixes O and interchanges A and B.

The x_{ij}-like elements in ⊠

Defintion

By an x_{ij}-like element in ⊠, we mean an element ξ in ⊠ that satisfies the following conditions:

$$[x_{ij}, \xi] = 0,$$
$$[x_{hk}, [x_{hk}, [x_{hk}, \xi]]] = 4[x_{hk}, \xi],$$

where $h, i, j, k \in \mathbb{I}$ are mutually distinct .

- X_{ij}: the subset of ⊠ consisting of all x_{ij}-like elements in ⊠.
- Note that X_{ij} is a subspace of ⊠.

Lemma

For mutually distinct $h, i, j, k \in \mathbb{I}$, the vector space ⊠ satisfies

$$⊠ = X_{kh} + X_{hi} + X_{ij} \qquad \text{(direct sum)}.$$

Proof. Sufficient to show that $L(\mathfrak{sl}_2)^+ = X^\sigma_{03} + X^\sigma_{31} + X^\sigma_{12}$ is a direct sum. ■

Consider the decomposition:

$$\boxtimes = X_{03} + X_{31} + X_{12}.$$

We will use this decomposition to analyze the structure of the Onsager Lie algebra O and present four decompositions of O.

To this end, we carry out our computations in $L(\mathfrak{sl}_2)^+$.

Recall the Onsager subalgebra $O = \langle A, B \rangle$ of $L(\mathfrak{sl}_2)^+$, where

$$A = x_{12}^{\sigma} = x \otimes 1, \qquad B = x_{03}^{\sigma} = y \otimes t + z \otimes (t-1).$$

Lemma

We have

$$O = (X_{03}^{\sigma} \cap O) + (X_{31}^{\sigma} \cap O) + (X_{12}^{\sigma} \cap O) \qquad \text{(direct sum)}.$$

The following (i)–(iii) hold.

(i) $X_{03}^{\sigma} \cap O$ has a basis $\{(y \otimes t + z \otimes (t-1))t^n\}_{n \in \mathbb{N}}$.

(ii) $X_{31}^{\sigma} \cap O$ has a basis $\{z \otimes (t-1)t^n\}_{n \in \mathbb{N}}$.

(iii) $X_{12}^{\sigma} \cap O$ has a basis $\{x \otimes t^n\}_{n \in \mathbb{N}}$.

By the G-action on O, we have

$$O = O^{\rho} = (X_{30}^{\sigma} \cap O) + (X_{02}^{\sigma} \cap O) + (X_{21}^{\sigma} \cap O),$$
$$O = O^{\tau} = (X_{03}^{\sigma} \cap O) + (X_{32}^{\sigma} \cap O) + (X_{21}^{\sigma} \cap O),$$
$$O = O^{\rho\tau} = (X_{30}^{\sigma} \cap O) + (X_{01}^{\sigma} \cap O) + (X_{12}^{\sigma} \cap O).$$

Each of the three sums above is direct.

Four direct sum decompositions of O

Corollary

In ⊠, we have

$$\begin{aligned} O &= (X_{03} \cap O) + (X_{31} \cap O) + (X_{12} \cap O) \\ &= (X_{30} \cap O) + (X_{02} \cap O) + (X_{21} \cap O) \\ &= (X_{03} \cap O) + (X_{32} \cap O) + (X_{21} \cap O) \\ &= (X_{30} \cap O) + (X_{01} \cap O) + (X_{12} \cap O). \end{aligned}$$

Each of the four sums above is direct.

Next goal: For each decomposition, we provide a basis for each summand.

A basis $[0312]$ for O

Consider the direct sum decomposition of O:

$$O = (X^\sigma_{03} \cap O) + (X^\sigma_{31} \cap O) + (X^\sigma_{12} \cap O)$$
$$= x^\sigma_{03}\mathbb{F}[t] + x^\sigma_{31}(t-1)\mathbb{F}[t] + x^\sigma_{12}\mathbb{F}[t].$$

Define the elements $A_i^{\uparrow\uparrow}, B_i^{\uparrow\uparrow}, \psi_i^{\uparrow\uparrow} \in L(\mathfrak{sl}_2)^+$ as follows.

$$A_i^{\uparrow\uparrow} := x^\sigma_{12}(t-1)^i, \quad B_i^{\uparrow\uparrow} := x^\sigma_{03}(t-1)^i, \quad \psi_i^{\uparrow\uparrow} := x^\sigma_{31}(t-1)^i \quad (i \geq 0).$$

Observe that

$$A_0^{\uparrow\uparrow} = x^\sigma_{12} = A, \qquad B_0^{\uparrow\uparrow} = x^\sigma_{03} = B, \qquad \psi_0^{\uparrow\uparrow} = x^\sigma_{31} \notin O.$$

Lemma

Subspace U	$X^\sigma_{03} \cap O$	$X^\sigma_{31} \cap O$	$X^\sigma_{12} \cap O$
Basis for U	$\{B_i^{\uparrow\uparrow}\}_{i\in\mathbb{N}}$	$\{\psi_{i+1}^{\uparrow\uparrow}\}_{i\in\mathbb{N}}$	$\{A_i^{\uparrow\uparrow}\}_{i\in\mathbb{N}}$

A basis $[0312]$ for O (Cont'd)

Theorem (L. 2025+)

The Onsager Lie algebra O has a basis

$$A_i^{\uparrow\uparrow}, \qquad B_i^{\uparrow\uparrow}, \qquad \psi_{i+1}^{\uparrow\uparrow}, \qquad\qquad i \in \mathbb{N},$$

where $A_0^{\uparrow\uparrow} = A$, $B_0^{\uparrow\uparrow} = B$.

The Lie bracket acts on this basis as follows. For $i, j \in \mathbb{N}$,

$$\begin{aligned}
[A_i^{\uparrow\uparrow}, A_j^{\uparrow\uparrow}] &= 0, \\
[B_i^{\uparrow\uparrow}, B_j^{\uparrow\uparrow}] &= 0, \\
[\psi_i^{\uparrow\uparrow}, \psi_j^{\uparrow\uparrow}] &= 0, \\
[\psi_i^{\uparrow\uparrow}, A_j^{\uparrow\uparrow}] &= 2\psi_{i+j}^{\uparrow\uparrow} + 2A_{i+j}^{\uparrow\uparrow}, \\
[B_i^{\uparrow\uparrow}, \psi_j^{\uparrow\uparrow}] &= 2B_{i+j}^{\uparrow\uparrow} + 2\psi_{i+j}^{\uparrow\uparrow}, \\
[A_i^{\uparrow\uparrow}, B_j^{\uparrow\uparrow}] &= 2A_{i+j}^{\uparrow\uparrow} + 2B_{i+j}^{\uparrow\uparrow} - 4\psi_{i+j+1}^{\uparrow\uparrow}.
\end{aligned}$$

We denote this basis by $[0312]$.

A basis $[0312]$ for O (Cont'd)

By Theorem, the basis $[0312]$ for O is recursively obtained in the following order

$$A_0^{\uparrow\uparrow}, \quad B_0^{\uparrow\uparrow}, \quad \psi_1^{\uparrow\uparrow}, \quad A_1^{\uparrow\uparrow}, \quad B_1^{\uparrow\uparrow}, \quad \psi_2^{\uparrow\uparrow}, \quad A_2^{\uparrow\uparrow}, \quad B_2^{\uparrow\uparrow}, \quad \psi_3^{\uparrow\uparrow}, \quad \ldots$$

using $A_0^{\uparrow\uparrow} = A$, $B_0^{\uparrow\uparrow} = B$ and the following equations for $i \geq 1$:

$$\psi_i^{\uparrow\uparrow} = \frac{A_{i-1}^{\uparrow\uparrow}}{2} + \frac{B_{i-1}^{\uparrow\uparrow}}{2} - \frac{[A_{i-1}^{\uparrow\uparrow}, B]}{4},$$

$$A_i^{\uparrow\uparrow} = \frac{[\psi_i^{\uparrow\uparrow}, A]}{2} - \psi_i^{\uparrow\uparrow},$$

$$B_i^{\uparrow\uparrow} = \frac{[B, \psi_i^{\uparrow\uparrow}]}{2} - \psi_i^{\uparrow\uparrow}.$$

Example: $A_i^{\uparrow\uparrow}$, $B_i^{\uparrow\uparrow}$, $\psi_{i+1}^{\uparrow\uparrow}$ for $0 \le i \le 2$

Set

$$A_0^{\uparrow\uparrow} = A, \qquad B_0^{\uparrow\uparrow} = B.$$

Then,

$$\psi_1^{\uparrow\uparrow} = \frac{A}{2} + \frac{B}{2} - \frac{1}{4}[A,B],$$

$$A_1^{\uparrow\uparrow} = -\frac{A}{2} - \frac{B}{2} - \frac{1}{8}[A,[B,A]],$$

$$B_1^{\uparrow\uparrow} = -\frac{A}{2} - \frac{B}{2} - \frac{1}{8}[B,[A,B]],$$

$$\psi_2^{\uparrow\uparrow} = -\frac{A}{2} - \frac{B}{2} - \frac{1}{8}[B,A] - \frac{1}{16}[A,[B,A]] - \frac{1}{16}[B,[A,B]] - \frac{1}{32}[B,[A,[B,A]]],$$

$$A_2^{\uparrow\uparrow} = \frac{A}{2} + \frac{B}{2} + \frac{1}{8}[A,[B,A]] + \frac{1}{16}[B,[A,B]] + \frac{1}{64}[A,[B,[A,[B,A]]]],$$

$$B_2^{\uparrow\uparrow} = \frac{A}{2} + \frac{B}{2} + \frac{1}{8}[B,[A,B]] + \frac{1}{16}[A,[B,A]] + \frac{1}{64}[B,[A,[B,[A,B]]]],$$

$$\psi_3^{\uparrow\uparrow} = \frac{A}{2} + \frac{B}{2} + \frac{1}{16}[B,A] + \frac{3}{32}[A,[B,A]] + \frac{3}{32}[B,[A,B]] + \frac{1}{32}[B,[A,[B,A]]]$$
$$+ \frac{1}{128}[A,[B,[A,[B,A]]]] + \frac{1}{128}[B,[A,[B,[A,B]]]] + \frac{1}{256}[B,[A,[B,[A,[B,A]]]]].$$

A basis $[3021]$ for O

Apply $\rho = (12)(03)$ to $A_i^{\uparrow\uparrow}$, $B_i^{\uparrow\uparrow}$, $\psi_i^{\uparrow\uparrow}$ $(i \geq 0)$:

$$\begin{aligned} A_i^{\uparrow\uparrow} &= x_{12}^{\sigma}(t-1)^i \\ B_i^{\uparrow\uparrow} &= x_{03}^{\sigma}(t-1)^i \\ \psi_i^{\uparrow\uparrow} &= x_{31}^{\sigma}(t-1)^i \end{aligned} \quad \xrightarrow{\rho} \quad \begin{aligned} A_i^{\downarrow\downarrow} &:= x_{21}^{\sigma}(t-1)^i \\ B_i^{\downarrow\downarrow} &:= x_{30}^{\sigma}(t-1)^i \\ \psi_i^{\downarrow\downarrow} &:= x_{02}^{\sigma}(t-1)^i \end{aligned}$$

Note that

Subspace U	$X_{30}^{\sigma} \cap O$	$X_{02}^{\sigma} \cap O$	$X_{21}^{\sigma} \cap O$
Basis for U	$\{B_i^{\downarrow\downarrow}\}_{i\in\mathbb{N}}$	$\{\psi_{i+1}^{\downarrow\downarrow}\}_{i\in\mathbb{N}}$	$\{A_i^{\downarrow\downarrow}\}_{i\in\mathbb{N}}$

The Onsager Lie algebra O has a basis

$$A_i^{\downarrow\downarrow}, \qquad B_i^{\downarrow\downarrow}, \qquad \psi_{i+1}^{\downarrow\downarrow}, \qquad\qquad i \in \mathbb{N},$$

where $A_0^{\downarrow\downarrow} = -A$, $B_0^{\downarrow\downarrow} = -B$.

We denote this basis by $[3021]$.

A basis $[0321]$ for O

Apply $\tau = (12)$ to $A_i^{\uparrow\uparrow}$, $B_i^{\uparrow\uparrow}$, $\psi_i^{\uparrow\uparrow}$ $(i \geq 0)$:

$$\begin{aligned} A_i^{\uparrow\uparrow} &= x_{12}^{\sigma}(t-1)^i \\ B_i^{\uparrow\uparrow} &= x_{03}^{\sigma}(t-1)^i \\ \psi_i^{\uparrow\uparrow} &= x_{31}^{\sigma}(t-1)^i \end{aligned} \quad \xrightarrow{\tau} \quad \begin{aligned} A_i^{\downarrow\uparrow} &:= x_{21}^{\sigma}(-t)^i \\ B_i^{\downarrow\uparrow} &:= x_{03}^{\sigma}(-t)^i \\ \psi_i^{\downarrow\uparrow} &:= x_{32}^{\sigma}(-t)^i \end{aligned}$$

Note that

Subspace U	$X_{03}^{\sigma} \cap O$	$X_{32}^{\sigma} \cap O$	$X_{21}^{\sigma} \cap O$
Basis for U	$\{B_i^{\downarrow\uparrow}\}_{i\in\mathbb{N}}$	$\{\psi_{i+1}^{\downarrow\uparrow}\}_{i\in\mathbb{N}}$	$\{A_i^{\downarrow\uparrow}\}_{i\in\mathbb{N}}$

The Onsager Lie algebra O has a basis

$$A_i^{\downarrow\uparrow}, \qquad B_i^{\downarrow\uparrow}, \qquad \psi_{i+1}^{\downarrow\uparrow}, \qquad\qquad i \in \mathbb{N},$$

where $A_0^{\downarrow\uparrow} = -A$, $B_0^{\downarrow\uparrow} = B$.

We denote this basis by $[0321]$.

Apply $\rho\tau = (03)$ to $A_i^{\uparrow\uparrow}$, $B_i^{\uparrow\uparrow}$, $\psi_i^{\uparrow\uparrow}$ $(i \geq 0)$:

$$\begin{aligned} A_i^{\uparrow\uparrow} &= x_{12}^{\sigma}(t-1)^i \\ B_i^{\uparrow\uparrow} &= x_{03}^{\sigma}(t-1)^i \\ \psi_i^{\uparrow\uparrow} &= x_{31}^{\sigma}(t-1)^i \end{aligned} \quad \xrightarrow{\rho\tau} \quad \begin{aligned} A_i^{\uparrow\downarrow} &:= x_{12}^{\sigma}(-t)^i \\ B_i^{\uparrow\downarrow} &:= x_{30}^{\sigma}(-t)^i \\ \psi_i^{\uparrow\downarrow} &:= x_{01}^{\sigma}(-t)^i \end{aligned}$$

Note that

Subspace U	$X_{30}^{\sigma} \cap O$	$X_{01}^{\sigma} \cap O$	$X_{12}^{\sigma} \cap O$
Basis for U	$\{B_i^{\uparrow\downarrow}\}_{i\in\mathbb{N}}$	$\{\psi_{i+1}^{\uparrow\downarrow}\}_{i\in\mathbb{N}}$	$\{A_i^{\uparrow\downarrow}\}_{i\in\mathbb{N}}$

The Onsager Lie algebra O has a basis

$$A_i^{\uparrow\downarrow}, \qquad B_i^{\uparrow\downarrow}, \qquad \psi_{i+1}^{\uparrow\downarrow}, \qquad\qquad i \in \mathbb{N},$$

where $A_0^{\uparrow\downarrow} = A$, $B_0^{\uparrow\downarrow} = -B$.

We denote this basis by $[0321]$.

Table summarizing the four bases

$[khij]$	$X_{kh} \cap O$	$X_{hi} \cap O$	$X_{ij} \cap O$
$[0312]$	$\{B_i^{\uparrow\uparrow}\}_{i\in\mathbb{N}}$	$\{\psi_{i+1}^{\uparrow\uparrow}\}_{i\in\mathbb{N}}$	$\{A_i^{\uparrow\uparrow}\}_{i\in\mathbb{N}}$
$[3021]$	$\{B_i^{\downarrow\downarrow}\}_{i\in\mathbb{N}}$	$\{\psi_{i+1}^{\downarrow\downarrow}\}_{i\in\mathbb{N}}$	$\{A_i^{\downarrow\downarrow}\}_{i\in\mathbb{N}}$
$[0321]$	$\{B_i^{\downarrow\uparrow}\}_{i\in\mathbb{N}}$	$\{\psi_{i+1}^{\downarrow\uparrow}\}_{i\in\mathbb{N}}$	$\{A_i^{\downarrow\uparrow}\}_{i\in\mathbb{N}}$
$[3012]$	$\{B_i^{\uparrow\downarrow}\}_{i\in\mathbb{N}}$	$\{\psi_{i+1}^{\uparrow\downarrow}\}_{i\in\mathbb{N}}$	$\{A_i^{\uparrow\downarrow}\}_{i\in\mathbb{N}}$

- $[khij]$: a label that describes the direct sum decomposition of O:

$$O = (X_{kh} \cap O) + (X_{hi} \cap O) + (X_{ij} \cap O).$$

- The four main rows correspond to the four decompositions of O.
- For each entry of the table, the vectors displayed form a basis for the column-index summand in the row-index decomposition.
- In each row, the union of the vectors displayed forms a basis for O.

The action of G on the basis is summarized in the following diagram:

$$\begin{array}{ccc} [0312] & \overset{\rho}{\longleftrightarrow} & [3021] \\ \tau \updownarrow & & \updownarrow \tau \\ [0321] & \overset{\rho}{\longleftrightarrow} & [3012] \end{array}$$

Moreover, A, B of O are expressed in the following basis elements:

basis	$[0312]$	$[3021]$	$[0321]$	$[3012]$
A	$A_0^{\uparrow\uparrow}$	$-A_0^{\downarrow\downarrow}$	$-A_0^{\downarrow\uparrow}$	$A_0^{\uparrow\downarrow}$
B	$B_0^{\uparrow\uparrow}$	$-B_0^{\downarrow\downarrow}$	$B_0^{\downarrow\uparrow}$	$-B_0^{\uparrow\downarrow}$

Next, we display the transition matrices between each pair of the four bases that are adjacent in the above diagram.

Transition matrices between $[0312]$ and $[3021]$

Recall the bases for O:

$$[0312]: \quad \{A_i^{\uparrow\uparrow}, \quad B_i^{\uparrow\uparrow}, \quad \psi_{i+1}^{\uparrow\uparrow}\}_{i\in\mathbb{N}},$$
$$[3021]: \quad \{A_i^{\downarrow\downarrow}, \quad B_i^{\downarrow\downarrow}, \quad \psi_{i+1}^{\downarrow\downarrow}\}_{i\in\mathbb{N}}$$

The transition matrix from $[0312]$ to $[3021]$ is given as follows.

$$A_i^{\downarrow\downarrow} = -A_i^{\uparrow\uparrow},$$
$$B_i^{\downarrow\downarrow} = -B_i^{\uparrow\uparrow},$$
$$\psi_{i+1}^{\downarrow\downarrow} = -A_i^{\uparrow\uparrow} - B_i^{\uparrow\uparrow} + \psi_{i+1}^{\uparrow\uparrow}.$$

Moreover, the transition matrix from $[3021]$ to $[0312]$ is given as follows.

$$A_i^{\uparrow\uparrow} = -A_i^{\downarrow\downarrow},$$
$$B_i^{\uparrow\uparrow} = -B_i^{\downarrow\downarrow},$$
$$\psi_{i+1}^{\uparrow\uparrow} = -A_i^{\downarrow\downarrow} - B_i^{\downarrow\downarrow} + \psi_{i+1}^{\downarrow\downarrow}.$$

Recall the bases for O:

$$
\begin{aligned}
[0321]: &\quad \{A_i^{\downarrow\uparrow}, \quad B_i^{\downarrow\uparrow}, \quad \psi_{i+1}^{\downarrow\uparrow}\}_{i\in\mathbb{N}}, \\
[3012]: &\quad \{A_i^{\uparrow\downarrow}, \quad B_i^{\uparrow\downarrow}, \quad \psi_{i+1}^{\uparrow\downarrow}\}_{i\in\mathbb{N}}
\end{aligned}
$$

The transition matrix from $[0321]$ to $[3012]$ is given as follows.

$$
\begin{aligned}
A_i^{\uparrow\downarrow} &= -A_i^{\downarrow\uparrow}, \\
B_i^{\uparrow\downarrow} &= -B_i^{\downarrow\uparrow}, \\
\psi_{i+1}^{\uparrow\downarrow} &= -A_i^{\downarrow\uparrow} - B_i^{\downarrow\uparrow} + \psi_{i+1}^{\downarrow\uparrow}.
\end{aligned}
$$

Moreover, the transition matrix from $[3012]$ to $[0321]$ is given as follows.

$$
\begin{aligned}
A_i^{\downarrow\uparrow} &= -A_i^{\uparrow\downarrow}, \\
B_i^{\downarrow\uparrow} &= -B_i^{\uparrow\downarrow}, \\
\psi_{i+1}^{\downarrow\uparrow} &= -A_i^{\uparrow\downarrow} - B_i^{\uparrow\downarrow} + \psi_{i+1}^{\uparrow\downarrow}.
\end{aligned}
$$

Transition matrices between $[0312]$ and $[0321]$

The transition matrix from $[0312]$ to $[0321]$ is given as follows.

$$A_i^{\downarrow\uparrow} = (-1)^{i+1} \sum_{j=0}^{i} \binom{i}{j} A_j^{\uparrow\uparrow},$$

$$B_i^{\downarrow\uparrow} = (-1)^{i} \sum_{j=0}^{i} \binom{i}{j} B_j^{\uparrow\uparrow},$$

$$\psi_{i+1}^{\downarrow\uparrow} = (-1)^{i} \sum_{j=0}^{i} \binom{i}{j} B_j^{\uparrow\uparrow} + (-1)^{i+1} \sum_{j=0}^{i} \binom{i}{j} \psi_{j+1}^{\uparrow\uparrow} \qquad (i \in \mathbb{N}).$$

Moreover, the transition matrix from $[0321]$ to $[0312]$ is given as follows.

$$A_i^{\uparrow\uparrow} = (-1)^{i+1} \sum_{j=0}^{i} \binom{i}{j} A_j^{\downarrow\uparrow},$$

$$B_i^{\uparrow\uparrow} = (-1)^{i} \sum_{j=0}^{i} \binom{i}{j} B_j^{\downarrow\uparrow},$$

$$\psi_{i+1}^{\uparrow\uparrow} = (-1)^{i} \sum_{j=0}^{i} \binom{i}{j} B_j^{\downarrow\uparrow} + (-1)^{i+1} \sum_{j=0}^{i} \binom{i}{j} \psi_{j+1}^{\downarrow\uparrow} \qquad (i \in \mathbb{N}).$$

Transition matrices between $[3021]$ and $[3012]$

The transition matrix from $[3021]$ to $[3012]$ is given as follows.

$$A_i^{\uparrow\downarrow} = (-1)^{i+1} \sum_{j=0}^{i} \binom{i}{j} A_j^{\downarrow\downarrow},$$

$$B_i^{\uparrow\downarrow} = (-1)^{i} \sum_{j=0}^{i} \binom{i}{j} B_j^{\downarrow\downarrow},$$

$$\psi_{i+1}^{\uparrow\downarrow} = (-1)^{i} \sum_{j=0}^{i} \binom{i}{j} B_j^{\downarrow\downarrow} + (-1)^{i+1} \sum_{j=0}^{i} \binom{i}{j} \psi_{j+1}^{\downarrow\downarrow} \qquad (i \in \mathbb{N}).$$

Moreover, the transition matrix from $[3012]$ to $[3021]$ is given as follows.

$$A_i^{\downarrow\downarrow} = (-1)^{i+1} \sum_{j=0}^{i} \binom{i}{j} A_j^{\uparrow\downarrow},$$

$$B_i^{\downarrow\downarrow} = (-1)^{i} \sum_{j=0}^{i} \binom{i}{j} B_j^{\uparrow\downarrow},$$

$$\psi_{i+1}^{\downarrow\downarrow} = (-1)^{i} \sum_{j=0}^{i} \binom{i}{j} B_j^{\uparrow\downarrow} + (-1)^{i+1} \sum_{j=0}^{i} \binom{i}{j} \psi_{j+1}^{\uparrow\downarrow} \qquad (i \in \mathbb{N}).$$

Summary

- We reviewed the definitions of the Onsager Lie algebra O, the tetrahedron algebra $\boxtimes$, and the three-point $\mathfrak{sl}_2$ loop algebra $L(\mathfrak{sl}_2)^+$.
- We showed that the vector space $\boxtimes$ decomposes into a direct sum of three subspaces, each consisting of x_{ij}-like, x_{jk}-like, and x_{kh}-like elements.
- We obtained four direct sum decompositions of O, each with three summands.
- We examined the action of $G(\approx \mathbb{Z}_2 \times \mathbb{Z}_2)$ on these decompositions and provided a basis for each summand.
- Moreover, we described the Lie bracket action on these bases and showed how they are recursively constructed from the standard generators A, B of O.
- Finally we showed the action of G on these four bases and displayed some transition matrices among the bases.

Thank you for your attention.

Happy 70th Birthday, Professor Paul Terwilliger!

Celebrating a lifetime of inspiration,
mathematical excellence, and lasting impact.