Eigenspace embeddings of imprimitive association schemes

Janoš Vidali

University of Ljubljana
Faculty of Mathematics and Physics

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Relation schemes

- Let X be a set of n vertices and $\mathcal{R} = \{R_i \mid i \in \mathcal{I}\}$ a set of symmetric relations partitioning X^2 such that $R_0 = \mathrm{id}_X \in \mathcal{R}$.
- $\mathcal{A} = (X, \mathcal{R})$ is said to be a *d-class relation scheme* if $|\mathcal{I}| = d + 1$.
- We define the *relation graphs* $\Gamma_i = (X, R_i)$ $(i \in \mathcal{I})$ and denote their adjacency matrices by $A_i \in \{0, 1\}^{X \times X}$.
- Note that for a subset $Y \subseteq X$, the *induced subscheme* $\mathcal{A}|_Y = (Y, \mathcal{R}|_Y)$ is again a relation scheme.

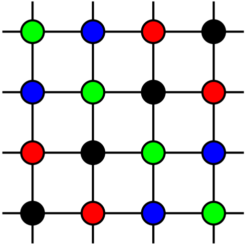
Association schemes

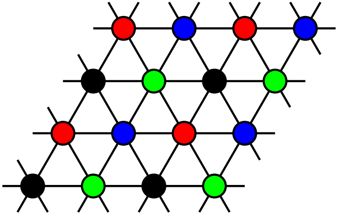
- Let $\mathcal{A} = (X, \mathcal{R})$ be a d-class relation scheme.
- $\mathcal{A}$ is said to be a *d-class association scheme* if there exist numbers p^h_{ij} $(h, i, j \in \mathcal{I})$ such that, for all $x, y \in X$,

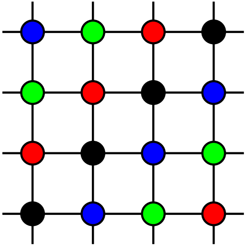
$$x \ R_h \ y \Rightarrow |\{z \in X \mid x \ R_i \ z \ R_j \ y\}| = p^h_{ij}$$

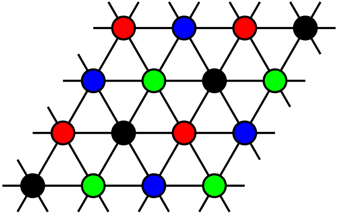
- We call the numbers p^h_{ij} $(h, i, j \in \mathcal{I})$ the *intersection numbers*.
- The values $k_i = p^0_{ii}$ $(i \in \mathcal{I})$ are called the *valencies* of the relations of $\mathcal{A}$.

Examples









Bose-Mesner algebra

- The adjacency matrices of $\mathcal{A}$ can be diagonalized simultaneously and they share $d+1$ eigenspaces $\{S_j \mid j \in \mathcal{J}\}$ with multiplicities m_j $(j \in \mathcal{J})$.
- Let $P \in \mathbb{R}^{\mathcal{J} \times \mathcal{I}}$ be a matrix with P_{ji} being the eigenvalue of A_i corresponding to the eigenspace S_j.
- Let $Q \in \mathbb{R}^{\mathcal{I} \times \mathcal{J}}$ be such that $PQ = nI$.
- We call P the *eigenmatrix*, and Q the *dual eigenmatrix*.
- The matrices $\{A_i \mid i \in \mathcal{I}\}$ are the basis of the *Bose-Mesner algebra* $\mathcal{M}$, which has a second basis $\{E_j \mid j \in \mathcal{J}\}$ of minimal idempotents for each eigenspace.

Krein parameters

- In the Bose-Mesner algebra $\mathcal{M}$,
  the following relations are satisfied:

$$A_i = \sum_{j \in \mathcal{J}} P_{ji} E_j \quad \text{and} \quad E_j = \frac{1}{n} \sum_{i \in \mathcal{I}} Q_{ij} A_i.$$

- We also have

$$A_i A_j = \sum_{h \in \mathcal{I}} p_{ij}^h A_h \quad \text{and} \quad E_i \circ E_j = \frac{1}{n} \sum_{h \in \mathcal{J}} q_{ij}^h E_h,$$

  where $\circ$ is the entrywise matrix product.
- The numbers q_{ij}^h are called the *Krein parameters* and are nonnegative algebraic real numbers.
- Each of the parameter sets p_{ij}^h, P, Q and q_{ij}^h determines the others.

Imprimitivity

- Let $\tilde{0} \subseteq \mathcal{I}$ be such that $R_{\tilde{0}} := \bigcup_{i \in \tilde{0}} R_i$ is an equivalence relation, and let $\tilde{X} = \{X_\ell \mid 1 \leq \ell \leq \tilde{n}\}$ be its equivalence classes.
- Then $|X_\ell| = \overline{n}$ $(1 \leq \ell \leq \tilde{n})$ such that $n = \overline{n} \cdot \tilde{n}$, and the *subschemes* $\mathcal{A}|_{X_\ell}$ $(1 \leq \ell \leq \tilde{n})$ are association schemes with parameters determined by those of $\mathcal{A}$.
- This relation induces an equivalence relation on $\mathcal{I}$ whose equivalence classes index the relations of the *quotient scheme* $\mathcal{A}/\tilde{0} = (\tilde{X}, \{\tilde{R}_{\tilde{\imath}} \mid i \in \mathcal{I}\})$, whose parameters are also determined by those of $\mathcal{A}$.
- $\mathcal{A}$ is called *imprimitive* if there is such a *nontrivial imprimitivity set* $\tilde{0}$ with $2 \leq |\tilde{0}| \leq d$.

Quotient-polynomial graphs (QPGs)

Definition ([Fio16])

An undirected graph $\Gamma = (X, R)$ with adjacency matrix A is *quotient-polynomial* if $\langle A \rangle$ is the Bose-Mesner algebra of an association scheme $\mathcal{A} = (X, \mathcal{R})$.

- Γ is a *relational QPG* if $R \in \mathcal{R}$.
- The parameters of a relational QPG are determined by its *parameter array* [HM24]

  $$[[k_1, k_2, \ldots, k_d], \\ [p^2_{11}, p^3_{11}, \ldots p^d_{11}; p^3_{12}, p^4_{12}, \ldots p^d_{12}; \ldots; p^{d-1}_{1,d-2}, p^d_{1,d-2}; p^d_{1,d-1}]].$$

- Special case of QPGs: distance-regular graphs.

Tables of feasible parameter arrays

- Recently, tables of parameter arrays for relational QPGs have been published [HM23a].
- Each parameter array is marked as
  - existent, if a corresponding association scheme has been constructed,
  - nonexistent, if some existence condition is not met, or
  - feasible, if no construction has been found.
- Using `sage-drg` [Vid19], we check for many more existence conditions and show nonexistence for many cases marked as feasible.
- In some cases, we find constructions as products of smaller association schemes.
- We will consider cases that remain feasible after performing the above checks.

Spherical representations

- Let S_j be an eigenspace of an association scheme $\mathcal{A} = (X, \mathcal{R} = \{R_i \mid i \in \mathcal{I}\})$.
- The vectors $u_x = \sqrt{\frac{n}{m_j}} E_j \mathbb{1}_x$ $(x \in X)$ are unit vectors such that

$$(x, y) \in R_i \;\Rightarrow\; \langle u_x, u_y \rangle = \frac{Q_{ij}}{m_j}.$$

- The map $x \mapsto u_x$ is called a *spherical representation* of $\mathcal{A}$ in S_j.

Embeddings of subschemes

- Let $\mathcal{A}' = (Y, \mathcal{R}' = \{R'_i \mid i \in \mathcal{I}'\})$ be a relation scheme such that $Y \subseteq X$ and $\mathcal{I}' \subseteq \mathcal{I}$.
- $\mathcal{A}'$ *admits an embedding into* S_j if there exist unit vectors $u'_x \in S_j$ $(x \in Y)$ such that

$$(x, y) \in R'_i \;\Rightarrow\; \langle u'_x, u'_y \rangle = \frac{Q_{ij}}{m_j}.$$

- Clearly, a relation scheme isomorphic to $\mathcal{A}|_Y$ admits an embedding into S_j.
- Conversely, if $\mathcal{A}'$ does not admit an embedding into S_j, then $\mathcal{A}'$ is not isomorphic to a subscheme of $\mathcal{A}$.

Finding embeddings

- Let $U \in \mathbb{R}^{Y \times m_j}$ be a matrix whose rows contain coefficients of the vectors u'_x w.r.t. an orthonormal basis of S_j.
- We may attempt to build U incrementally and thus obtain a matrix in reduced column echelon form.
- If $Q \in \mathbb{F}^{\mathcal{I} \times \mathcal{J}}$ for some field $\mathbb{F} \subseteq \mathbb{R}$, then the h-th column of U has entries from $\mathbb{F}\sqrt{\beta_h}$ for some $\beta_h \in \mathbb{F}$ with $\beta_h > 0$.

Embeddings of imprimitive association schemes

- Let $\mathcal{A}$ be imprimitive with imprimitivity set $\tilde{0}$, and let $\{X_\ell \mid 1 \leq \ell \leq \tilde{n}\}$ be the equivalence classes of $R_{\tilde{0}}$.
- If the subschemes $\mathcal{A}|_{X_\ell}$ $(1 \leq \ell \leq \tilde{n})$ are characterized by their parameters, this gives us their embeddings into eigenspaces of $\mathcal{A}$.
- We may attempt to find *candidate subschemes* on a *small number* of X_ℓ $(1 \leq \ell \leq \tilde{n})$ such that they admit embeddings into eigenspaces of $\mathcal{A}$.
- If there are no candidates for an induced subscheme on a subset of X, then $\mathcal{A}$ does not exist.
- On the other hand, if we find an embedding such that U has full column rank, then we may consider all candidates vectors for the remaining vertices of $\mathcal{A}$.

Results

Using such a technique, we derive the following results.

- Nonexistence for two parameter sets of 4-class association schemes and one parameter set of 5-class association schemes.
- Uniqueness for two parameter sets of 5-class association schemes.

QPG with parameter array $[[12, 4, 4, 24], [6, 0, 3; 0, 1; 2]]$

- $\mathcal{A}$ has 45 vertices
  and is imprimitive with imprimitivity set $\{0, 2, 3\}$.
- The five subschemes
  are isomorphic to the Hamming scheme $H(2, 3)$.
- The graphs $\Gamma_1|_{X_\ell \cup X_{\ell'}}$ are isomorphic to $3K_{3,3}$.
- There are six nonisomorphic candidates for $\mathcal{A}|_{X_1 \cup X_2 \cup X_3}$,
  but only one admits an embedding into S_1.
- The resulting matrix of coefficients has full column rank,
  but no further vectors can be found among 216 candidates.
- We therefore conclude that $\mathcal{A}$ does not exist.

QPG with parameter array $[[8, 8, 4, 24], [1, 0, 2; 2, 1; 1]]$

- $\mathcal{A}$ has 45 vertices, is imprimitive with imprimitivity set $\{0, 3\}$ and has noncyclotomic eigenvalues [HM23b].
- The nine subschemes are 5-cliques.
- The graph Γ_1 is triangle-free, while Γ_2 does have triangles; their restriction to two subschemes are matchings.
- We find 6665 nonisomorphic candidates for $\mathcal{A}|_{X_1 \cup X_2 \cup X_3}$, of which 100 admit an embedding into S_1.
- The resulting matrices of coefficients have full column rank, but no further vectors can be found among $100 \cdot 8000$ candidates.
- We therefore conclude that $\mathcal{A}$ does not exist.

QPG with array $[[6, 18, 2, 6, 12], [1, 0, 2, 0; 0, 0, 3; 0, 1; 2]]$

- $\mathcal{A}$ has 45 vertices and is imprimitive with $\tilde{0} = \{0, 3, 4\}$, $\tilde{1} = \{1, 5\}$, $\tilde{2} = \{2\}$.
- The five subschemes are isomorphic to the scheme of $K_{3\times 3}$, and the quotient scheme is the cyclic scheme C_5.
- The graph $\Gamma_1|_{X_1 \cup X_2}$ such that $(X_1, X_2) \in \tilde{R}_{\tilde{1}}$ is a bipartite cubic graph whose distance-2 graph is 3-colorable.
- We find 18 such graphs, of which 7 corresponding schemes admit an embedding into S_1.
- However, none of the extensions to $\mathcal{A}|_{X_1 \cup X_2 \cup X_3}$ with $(X_1, X_3), (X_2, X_3) \in \tilde{R}_{\tilde{2}}$ admits an embedding into S_1.
- We therefore conclude that $\mathcal{A}$ does not exist.

QPG with array $[[12, 2, 1, 12, 12], [6, 0, 4, 1; 0, 0, 1; 0, 1; 4]]$

- $\mathcal{A}$ has 40 vertices and is imprimitive with $\tilde{0} = \{0, 2, 3\}$, $\tilde{1} = \{1, 5\}$, $\tilde{4} = \{4\}$.
- The five subschemes are isomorphic to the cyclic scheme C_4, and the quotient scheme is the Johnson scheme $J(5, 2)$.
- The graph $\Gamma_1|_{X_1 \cup X_2}$ such that $(X_1, X_2) \in \tilde{R}_{\tilde{1}}$ must be isomorphic to C_8 or $2C_4$.
- The extension to $\mathcal{A}|_{X_1 \cup X_2 \cup X_3}$ with $(X_1, X_3), (X_2, X_3) \in \tilde{R}_{\tilde{4}}$ admits an embedding into S_1 only if $\Gamma_1|_{X_1 \cup X_2} \cong 2C_4$.
- The resulting matrix of coefficients has full column rank, and we find 28 more vectors among 112 candidates.

Construction

- The resulting vectors form an association scheme with the parameters of $\mathcal{A}$.
- We therefore conclude that there is, up to isomorphism, precisely one such association scheme.
- A nicer construction is given by the vectors $\frac{\sqrt{2}}{2}(\pm e_i \pm e_j)$ $(1 \leq i < j \leq 5)$ with
  - $(u, v) \in R_1 \;\Leftrightarrow\; \langle u, v \rangle = \frac{1}{2}$,
  - $(u, v) \in R_2 \;\Leftrightarrow\; \langle u, v \rangle = 0 \wedge \mathrm{supp}(u) = \mathrm{supp}(v)$,
  - $(u, v) \in R_3 \;\Leftrightarrow\; \langle u, v \rangle = -1$,
  - $(u, v) \in R_4 \;\Leftrightarrow\; \langle u, v \rangle = 0 \wedge \mathrm{supp}(u) \cap \mathrm{supp}(v) = \emptyset$,
  - $(u, v) \in R_5 \;\Leftrightarrow\; \langle u, v \rangle = -\frac{1}{2}$.

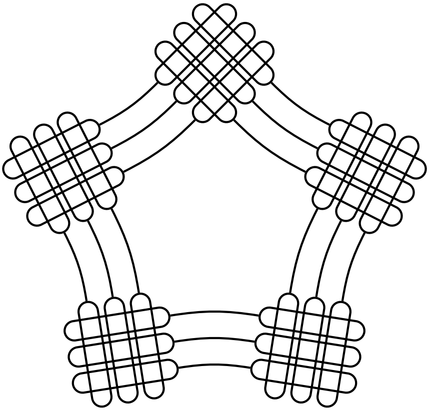
QPG with array $[[6, 4, 4, 12, 18], [3, 0, 0, 1; 0, 1, 0; 2, 0; 2]]$

- $\mathcal{A}$ has 45 vertices and is imprimitive with $\tilde{0} = \{0, 2, 3\}$, $\tilde{1} = \{1, 4\}$, $\tilde{5} = \{5\}$.
- The five subschemes are isomorphic to the Hamming scheme $H(2, 3)$, and the quotient scheme is the cyclic scheme C_5.
- The graphs $\Gamma_1|_{X_\ell \cup X_{\ell'}}$ are isomorphic to $3K_{3,3}$.
- There is a unique scheme $\mathcal{A}|_{X_1 \cup X_2 \cup X_3}$ with $(X_1, X_2) \in \tilde{R}_{\tilde{1}}$ and $(X_1, X_3), (X_2, X_3) \in \tilde{R}_{\tilde{5}}$, and it admits an embedding into S_1.
- The resulting matrix of coefficients has full column rank, and we find 36 more vectors among 72 candidates.

Construction

- We find two subsets of 18 vectors extending $\mathcal{A}|_{X_1 \cup X_2 \cup X_3}$ into an association scheme with the parameters of $\mathcal{A}$.
- Since the two obtained schemes are isomorphic, we conclude that there is, up to isomorphism, precisely one such association scheme.

The obtained association scheme



The graph Γ_1 is the Praeger-Xu graph $C(3,5,2)$ [PX89].

References I

[Fio16] Miquel Àngel Fiol.
"Quotient-polynomial graphs".
In: *Linear Algebra Appl.* 488 (2016).
doi:10.1016/j.laa.2015.09.053, pp. 363–376.

[HM23a] Allen Herman and Roghayeh Maleki.
QPG database.
https://github.com/RoghayehMaleki/QPGdatabase-.
2023.

[HM23b] Allen Herman and Roghayeh Maleki.
"The search for small association schemes with noncyclotomic eigenvalues".
In: *Ars Math. Contemp.* 23.3 (2023).
doi:10.26493/1855-3974.2724.83d, P3.02.

[HM24] Allen Herman and Roghayeh Maleki.
"Parameters of Quotient-Polynomial Graphs".
In: *Graphs Combin.* 40 (3 2024).
doi:10.1007/s00373-024-02789-2, p. 60.

References II

[PX89] Cheryl E. Praeger and Ming-Yao Xu.
"A Characterization of a Class of Symmetric Graphs of Twice Prime Valency".
In: *European J. Combin.* 10 (1 1989).
doi:10.1016/S0195-6698(89)80037-X, pp. 91–102.

[Vid19] Janoš Vidali.
jaanos/sage-drg*:* sage-drg *Sage package v0.9*.
https://github.com/jaanos/sage-drg/,
doi:0.5281/zenodo.3350856.
2019.