Complex Hadamard Matrices and Quantum Symmetry

TerwilligerFest

Combinatorics around the q-Onsager algebra

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Joint work with Chris Godsil and Thomás Jung Spier

Happy belated 70^{th} Birthday, Paul!!!

Thank you, Blas, Giusy, Mark, Rene, Safet and Štefko, for organizing this celebration!!!

Two graphs X and Y are isomorphic if there exists a permutation matrix P s.t.

$$A(X)P = PA(Y).$$

Two graphs X and Y are quantum isomorphic if there exists a quantum permutation matrix Q s.t.

$$A(X)Q = QA(Y).$$

Theorem (Gromada, 2024; Chan and Martin, 2024)

Any two Hadamard graphs of the same order are quantum isomorphic.

quantum quantum

Two graphs X and Y are ∨ isomorphic if there exists a ∨ permutation matrix $\not{P}$ Q s.t.

$$A(X)\not{P}Q = Q\not{P}A(Y).$$

Theorem (Gromada, 2024; Chan and Martin, 2024)

Any two Hadamard graphs of the same order are quantum isomorphic.

When $X = Y$:

Theorem (Schmidt, 2020)

The Higman-Sims graph has quantum symmetry.

Remark

The Bose-Mesner algebras of Hadamard graphs and the Higman-Sims graph contain spin models.

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Leonard pairs, spin models, and distance-regular graphs

Kazumasa Nomura [a,*], Paul Terwilliger [b]

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Spin models and distance-regular graphs of q-Racah type

Kazumasa Nomura [a], Paul Terwilliger [b]

Definition (Wang, 1998)

Let $\mathcal{A}$ be a C^*-algebra with unity $\mathbf{1}$.

A quantum permutation matrix is an $n \times n$ matrix Q with entries in $\mathcal{A}$ satisfying

i. $Q_{ij}^* = Q_{ij} = Q_{ij}^2$, and

ii. $\sum_{k=1}^{n} Q_{ik} = \sum_{k=1}^{n} Q_{kj} = \mathbf{1}$, for $1 \leq i, j \leq n$.

Remark:

If $\mathcal{A}$ is finite dimensional then $Q_{ij} \in M_d(\mathbb{C})$, for some d, and $Q_{ij}^* = \overline{Q_{ij}}^T$.

Example

Q is a permutation matrix.

Example

$$Q = \begin{bmatrix} \mathbf{p} & \mathbf{1-p} & \mathbf{0} & \mathbf{0} \\ \mathbf{1-p} & \mathbf{p} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{q} & \mathbf{1-q} \\ \mathbf{0} & \mathbf{0} & \mathbf{1-q} & \mathbf{q} \end{bmatrix}, \quad \text{for orthogonal projections } \mathbf{p}, \mathbf{q} \in M_d(\mathbb{C}).$$

Example

Q is a permutation matrix.

$$Q_{ij} \in \{0, 1\} \quad \Longrightarrow \quad Q_{ij}Q_{rs} = Q_{rs}Q_{ij}, \quad \forall i, j, r, s.$$

Example

$$Q = \begin{bmatrix} \mathbf{p} & \mathbf{1-p} & \mathbf{0} & \mathbf{0} \\ \mathbf{1-p} & \mathbf{p} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{q} & \mathbf{1-q} \\ \mathbf{0} & \mathbf{0} & \mathbf{1-q} & \mathbf{q} \end{bmatrix}, \quad \text{for orthogonal projections } \mathbf{p}, \mathbf{q} \in M_d(\mathbb{C}).$$

e.g. $\quad \mathbf{p} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{pq} \neq \mathbf{qp}.$

Theorem (Weber, 2023)

Let $\mathcal{A} \subset M_d(\mathbb{C})$, and $Q \in M_n(\mathcal{A})$ be a quantum permutation matrix.

If all Q_{ij}'s commute, then there exist permutation matrices $P_1, \ldots, P_d \in M_n(\mathbb{C})$ such that Q is unitarily equivalent to

$$\begin{bmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_d \end{bmatrix}.$$

Definition (Banica, 2005)

The quantum automorphism group of X is the compact matrix quantum group $(C(G^+_{aut}(X)), U)$, where $C\left(G^+_{aut}(X)\right)$ is the *universal* C^*-algebra with generators U_{ij}, $1 \leq i, j \leq n$, and relations

i. $U_{ij} = \left(U_{ij}\right)^* = \left(U_{ij}\right)^2, \quad \forall i, j,$

ii. $\sum_{k=1}^{n} U_{ik} = \mathbf{1} = \sum_{k=1}^{n} U_{kj}, \quad \forall i, j,$

iii. $AU = UA.$

$\Longrightarrow$ U is a quantum permutation matrix that commutes with A.

Definition (Banica and Bichon, 2007)

The graph X has no quantum symmetry if

$$C\left(G^+_{aut}(X)\right) = C\left(Aut(X)\right),$$

equivalently, $U_{ij}U_{rs} = U_{rs}U_{ij}, \quad \forall i, j, r, s.$

Example ($X = K_2$)

The rows and columns of U sum to $\mathbf{1}$

$$\implies \quad U = \begin{bmatrix} \mathbf{p} & \mathbf{1}-\mathbf{p} \\ \mathbf{1}-\mathbf{p} & \mathbf{p} \end{bmatrix}.$$

$\implies$ The entries of U commute.

$\implies$ K_2 has no quantum symmetry.

Example ($X = C_4$)

$$Q = \begin{bmatrix} \mathbf{p} & \mathbf{1-p} & \mathbf{0} & \mathbf{0} \\ \mathbf{1-p} & \mathbf{p} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{q} & \mathbf{1-q} \\ \mathbf{0} & \mathbf{0} & \mathbf{1-q} & \mathbf{q} \end{bmatrix}, \text{ with } \mathbf{p} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } \mathbf{q} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\Longrightarrow \quad \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix} Q = Q \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$\mathbf{pq} \neq \mathbf{qp} \quad \Longrightarrow \quad C_4$ has quantum symmetry.

Remark

If $Q_{ij} \in M_d(\mathbb{C})$ then $AQ = QA$ becomes

$$(A \otimes I_d)\, Q = Q\, (A \otimes I_d)\,.$$

Theorem (Schmidt, 2020)

The Hamming graph $H(n,q)$ has quantum symmetry if and only if

i. $n \geq 2$ and $q = 2$, or

ii. $n \geq 1$ and $q \geq 4$.

We prove Schmidt's result for $q = 2$, $q = 4$ and $q \geq 6$:

- construct a quantum permutation matrix Q from a complex Hadamard matrix,
- determine A's satisfying $QA = AQ$,
- detemine when $\exists\, i, j, k, l$ where $Q_{ij}Q_{kl} \neq Q_{kl}Q_{ij}$.

Definition

An $n \times n$ matrix W is type-II if

$$WW^{(-)T} = nI,$$

where $W^{(-)}$ denotes the Schur-inverse of W.

Equivalently,

$$\sum_{k=1}^{n} \frac{W_{ki}}{W_{kj}} = n\delta_{ij} = \sum_{k=1}^{n} \frac{W_{ik}}{W_{jk}}.$$

Definition

A complex Hadamard matrix is a type-II matrix W with $|W_{ij}| = 1, \ \forall i, j$.

Example

Spin models and four-weight spin models are type-II matrices.

Example

For $\lambda \neq 0$, $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \lambda & -\lambda \\ 1 & -1 & -\lambda & \lambda \end{bmatrix}$ is type II.

It is a complex Hadamard matrix if $|\lambda| = 1$.

Example

The character table of a finite Abelian group is a complex Hadamard matrix.

For type-II matrix W, define $W_{i/j} = \begin{bmatrix} \frac{W_{1i}}{W_{1j}} \\ \vdots \\ \frac{W_{ni}}{W_{nj}} \end{bmatrix}$ $(i, j = 1, \ldots, n)$

Example: $W = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \lambda & -\lambda \\ 1 & -1 & -\lambda & \lambda \end{bmatrix}$ $W_{3/1} = \begin{bmatrix} 1 \\ -1 \\ \lambda \\ -\lambda \end{bmatrix}$, $W_{1/3} = (W_{3/1})^{(-)} = \begin{bmatrix} 1 \\ -1 \\ \frac{1}{\lambda} \\ -\frac{1}{\lambda} \end{bmatrix}$

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Example: $W = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \lambda & -\lambda \\ 1 & -1 & -\lambda & \lambda \end{bmatrix} \qquad W_{3/1} = \begin{bmatrix} 1 \\ -1 \\ \lambda \\ -\lambda \end{bmatrix}, \quad W_{1/3} = (W_{3/1})^{(-)} = \begin{bmatrix} 1 \\ -1 \\ \frac{1}{\lambda} \\ -\frac{1}{\lambda} \end{bmatrix}$

Observe: $\begin{bmatrix} W_{1/3} & W_{2/3} & W_{3/3} & W_{4/3} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & \frac{1}{\lambda} & \\ & & & -\frac{1}{\lambda} \end{bmatrix} W$

$$\begin{bmatrix} W_{3/1} & W_{3/2} & W_{3/3} & W_{3/4} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & \lambda & \\ & & & \lambda \end{bmatrix} W^{(-)}$$

$\Longrightarrow \quad \{W_{1/j}, \dots, W_{n/j}\}$ and $\{W_{j/1}, \dots, W_{j/n}\}$ are bases of $\mathbb{C}^n$

From Soffía's talk:

Definition (Musto and Vicary, 2015)

A quantum Latin square is an $n \times n$ array of elements in $\mathbb{C}^n$ such that every row and every column is an orthonormal basis of $\mathbb{C}^n$.

Theorem (Godsil, < 2020)

Let W be a complex Hadamard matrix of order n. Then

$$\mathcal{L} = \left[\frac{1}{\sqrt{n}} W_{i/j} \right]_{i,j}$$

is a quantum Latin square.

Type-II matrices to Association schemes:

Definition (Nomura, 1997)

Let W be a type-II matrix. The Nomura algebra of W is

$$\mathcal{N}_W = \left\{ M \in M_n(\mathbb{C}) \ : \ W_{i/j} \text{ is an eigenvector of } M, \text{ for } i, j = 1, \ldots, n \right\}.$$

Theorem (Jaeger, Matsumoto, Nomura, 1998)

Let W be a type-II matrix. Then $\mathcal{N}_W$ and $\mathcal{N}_{W^T}$ are a formally dual pair of Bose-Mesner algebras.

Moreover, $W \in \mathcal{N}_W$ if and only if cW is a spin model, for some $c \in \mathbb{C}$.

Example (Jaeger, Matsumoto, Nomura, 1998)

Let $W = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \lambda & -\lambda \\ 1 & -1 & -\lambda & \lambda \end{bmatrix}$. Then $\dim \mathcal{N}_W = \begin{cases} 4 & \text{if } \lambda^4 = 1, \\ 3 & \text{otherwise.} \end{cases}$

Example (Jaeger, Matsumoto, Nomura, 1998)

Let W be a type-II matrix of order n.

Then $\dim \mathcal{N}_W = n$ if and only if W is type-II equivalent to the character table of an abelian group of order n.

Construction of quantum permutation matrix:

Definition

Let W be a type-II matrix of order n. For $i, j = 1, \ldots, n$, let $Y_{ij} = \frac{1}{n} W_{i/j} W_{j/i}^T$.

The matrix of idempotent of W is the $n \times n$ block matrix

$$\mathcal{Y} = [Y_{ij}]_{i,j}.$$

Example (Complex Hadamard matrix H)

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & e^{i\theta} & -e^{i\theta} \\ 1 & -1 & -e^{i\theta} & e^{i\theta} \end{bmatrix}$$

$$H_{1/3} = \overline{H_{3/1}} \implies Y_{31} = \frac{1}{n} H_{3/1} H_{3/1}^* = \begin{bmatrix} 1 & -1 & e^{-i\theta} & -e^{-i\theta} \\ -1 & 1 & -e^{-i\theta} & e^{-i\theta} \\ e^{i\theta} & -e^{i\theta} & 1 & -1 \\ -e^{i\theta} & e^{i\theta} & -1 & 1 \end{bmatrix}.$$

Let H be a complex Hadamard matrix of order $n \implies H_{j/i} = \overline{H_{i/j}}$.

- $Y_{ij}^* = \frac{1}{n}\left(H_{i/j}H_{i/j}^*\right)^* = Y_{ij}$
- $Y_{ij}^2 = \left(\frac{1}{n}H_{i/j}H_{i/j}^*\right)\left(\frac{1}{n}H_{i/j}H_{i/j}^*\right) = Y_{ij}$

Let H be a complex Hadamard matrix of order n $\implies$ $H_{j/i} = \overline{H_{i/j}}$.

- $Y_{ij}^* = \frac{1}{n}\left(H_{i/j}H_{i/j}^*\right)^* = Y_{ij}$
- $Y_{ij}^2 = \left(\frac{1}{n}H_{i/j}H_{i/j}^*\right)\left(\frac{1}{n}H_{i/j}H_{i/j}^*\right) = Y_{ij}$
- $\left(\sum_{k=1}^{n} Y_{ik}\right)_{rs} = \frac{1}{n}\sum_{k=1}^{n}\frac{H_{ri}}{H_{rk}}\frac{H_{sk}}{H_{si}} = \delta_{rs} \implies \sum_k Y_{ik} = I_n$.
- Similarly, $\sum_k Y_{kj} = I_n$.

Let H be a complex Hadamard matrix of order n $\implies$ $H_{j/i} = \overline{H_{i/j}}$.

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- Similarly, $\sum_k Y_{kj} = I_n.$

Lemma

The matrix of idempotent $\mathcal{Y}$ of H is a quantum permutation matrix.

Lemma

Let W be a type-II matrix. Then

$$\begin{aligned}\mathcal{N}_W &= \left\{M \,:\, (I_n \otimes M)\,\mathcal{Y} = \mathcal{Y}\,(I_n \otimes M)\right\}, \quad \text{and}\\ \mathcal{N}_{W^T} &= \left\{N \,:\, (N \otimes I_n)\,\mathcal{Y} = \mathcal{Y}\,(N \otimes I_n)\right\}\end{aligned}$$

Any 01-matrix $A \in \mathcal{N}_{W^T}$ satisfies $(A \otimes I_n)\,\mathcal{Y} = \mathcal{Y}\,(A \otimes I_n)$.

Lemma

Let W be a type-II matrix. Then

$$\begin{aligned}\mathcal{N}_W &= \left\{M \;:\; (I_n \otimes M)\,\mathcal{Y} = \mathcal{Y}\,(I_n \otimes M)\right\}, \quad \text{and}\\ \mathcal{N}_{W^T} &= \left\{N \;:\; (N \otimes I_n)\,\mathcal{Y} = \mathcal{Y}\,(N \otimes I_n)\right\}\end{aligned}$$

Any 01-matrix $A \in \mathcal{N}_{W^T}$ satisfies $(A \otimes I_n)\,\mathcal{Y} = \mathcal{Y}\,(A \otimes I_n)$.

Lemma

Let H be a complex Hadamard matrix of order n.
Then $\exists\, i, j, k, l$ such that $Y_{ij}Y_{kl} \neq Y_{kl}Y_{ij}$ if and only if $\dim \mathcal{N}_H < n$.

Theorem

If H be a complex Hadamard matrix that is inequivalent to the character table of a finite Abelian group, then every graph in $\mathcal{N}_{H^T}$ has quantum symmetry.

Theorem (de la Harpe and Jones, 1990; Munemasa and Watatani, 1992)

For every prime $p \geq 7$, there exists a circulant matrix of order p, inequivalent to the Fourier matrix of order p.

Theorem (Craigen, 1991)

If n is a composite, then there an uncountable number of equivalence classes of complex Hadamard matrices of order n.

When $n \in \{2, 3, 5\}$, the Fourier matrix is the unique complex Hadamard matrix, up to equivalence.

Theorem

For $n = 4$ or $n \geq 6$, the Hamming graph $H(n, q)$ has quantum symmetry.

Proof. Let H be a complex Hadamard matrix of order q that is inequivalent to the character table of a finite abelian group. Then

$$A\left(H(n, q)\right) \in \left(\mathcal{N}_{H^T}\right)^{\otimes n} = \mathcal{N}_{(H^T)^{\otimes n}},$$

and $\dim \mathcal{N}_{(H^T)^{\otimes n}} < q^n$. □

Theorem

For $n \geq 2$ and $q = 2$, the Hamming graph $H(n, q)$ has quantum symmetry.

Proof. Use $H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & e^{\frac{2\pi i}{3}} & -e^{\frac{2\pi i}{3}} \\ 1 & -1 & -e^{\frac{2\pi i}{3}} & e^{\frac{2\pi i}{3}} \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{\otimes(n-2)}$. □

Theorem (Schmidt, 2020)

The Higman-Sims graph has quantum symmetry.

Question: Quantum permutation matrix??

(The Higman-Sims spin model is not a complex Hadamard matrix.)

Thank you for your attention!