2-Homogeneous bipartite distance-regular graphs and the quantum group $U_q'(\mathfrak{so}_6)$

Paul Terwilliger

University of Wisconsin-Madison

Contents

- Warmup: the Lie algebra $\mathfrak{so}_n$
- The nonstandard quantum group $U'_q(\mathfrak{so}_n)$
- The case $U'_q(\mathfrak{so}_6)$
- A 2-homogeneous bipartite distance-regular graph Γ
- The strongly balanced condition on Γ
- Using Γ to obtain an irreducible $U'_q(\mathfrak{so}_6)$-module

Throughout this talk, all vector spaces and tensor products that we encounter are understood to be over $\mathbb{C}$.

Fix an integer $n \geq 3$.

let $\mathrm{Mat}_n(\mathbb{C})$ denote the algebra of $n \times n$ matrices that have all entries in $\mathbb{C}$.

For $1 \leq i, j \leq n$ define $E_{i,j} \in \mathrm{Mat}_n(\mathbb{C})$ that has (i, j)-entry 1 and all other entries 0.

The Lie algebra $\mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C})$ consists of the vector space $\mathrm{Mat}_n(\mathbb{C})$ together with the Lie bracket $[X, Y] = XY - YX$.

The dimension of $\mathfrak{gl}_n$ is n^2.

For $R \in \mathfrak{gl}_n$ consider the transpose R^t.

We say that R is **antisymmetric** whenever $R^t = -R$.

For $R, S \in \mathfrak{gl}_n$, if each of R, S is antisymmetric then so is $[R, S]$.

The Lie algebra $\mathfrak{so}_n$

Definition

Let $\mathfrak{so}_n = \mathfrak{so}_n(\mathbb{C})$ denote the Lie subalgebra of $\mathfrak{gl}_n$ consisting of the antisymmetric matrices. We call $\mathfrak{so}_n$ a **special orthogonal** Lie algebra.

The dimension of $\mathfrak{so}_n$ is $\binom{n}{2}$.

Example

The Lie algebra $\mathfrak{so}_3$ consists of the matrices

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \qquad a, b, c \in \mathbb{C}.$$

Recall the simple Lie algebras over $\mathbb{C}$ that have finite dimension at least 2:

$$A_\ell \ (\ell \geq 1), \quad B_\ell \ (\ell \geq 2), \quad C_\ell \ (\ell \geq 3), \quad D_\ell \ (\ell \geq 4),$$
$$E_6, \quad E_7, \quad E_8, \quad F_4, \quad G_2.$$

Some Isomorphisms

Lemma (Fulton and Harris, Section 21.2)

We give some isomorphisms that involve $\mathfrak{so}_n$:

(i) $\mathfrak{so}_3$ *is isomorphic to* A_1;

(ii) $\mathfrak{so}_4$ *is isomorphic to* $A_1 \oplus A_1$;

(iii) $\mathfrak{so}_5$ *is isomorphic to* B_2;

(iv) $\mathfrak{so}_6$ *is isomorphic to* A_3;

(v) *For odd* $n = 2r + 1 \geq 7$, $\mathfrak{so}_n$ *is isomorphic to* B_r;

(vi) *For even* $n = 2r \geq 8$, $\mathfrak{so}_n$ *is isomorphic to* D_r.

A basis for $\mathfrak{so}_n$

Our next goal is to describe a basis for $\mathfrak{so}_n$.

Definition

For distinct $i, j \in \{1, 2, \ldots, n\}$ define

$$I_{i,j} = E_{i,j} - E_{j,i}.$$

Lemma

The elements

$$I_{i,j} \qquad (1 \leq i < j \leq n)$$

form a basis for $\mathfrak{so}_n$. *Moreover:*

(i) *For distinct* $i, j \in \{1, 2, \ldots n\}$,

$$I_{i,j} = -I_{j,i}.$$

(ii) *For mutually distinct* $h, i, j \in \{1, 2, \ldots n\}$,

$$[I_{h,i}, I_{i,j}] = -I_{j,h}.$$

(iii) *For mutually distinct* $h, i, j, k \in \{1, 2, \ldots n\}$,

$$[I_{h,i}, I_{j,k}] = 0.$$

A basis for $\mathfrak{so}_3$

Example

The Lie algebra $\mathfrak{so}_3$ has a basis

$$I_{1,2} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad I_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$I_{1,3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Relations for $\mathfrak{so}_3$

Example

For $\mathfrak{so}_3$ we have

$$[I_{1,2}, I_{2,3}] = -I_{3,1},$$
$$[I_{2,3}, I_{3,1}] = -I_{1,2},$$
$$[I_{3,1}, I_{1,2}] = -I_{2,3}.$$

Generators and relations for $\mathfrak{so}_n$

Next, we give a presentation of $\mathfrak{so}_n$ by generators and relations.

Generators and relations for $\mathfrak{so}_n$, cont.

Definition

Define a Lie algebra $\mathbb{L}_n$ by generators

$$B_i \qquad\qquad (1 \leq i \leq n-1)$$

and the following relations.

(i) For $1 \leq i, j \leq n-1$ with $|i-j| = 1$,

$$[B_i, [B_i, B_j]] = -B_j, \qquad\qquad [B_j, [B_j, B_i]] = -B_i.$$

(ii) For $1 \leq i, j \leq n-1$ with $|i-j| \geq 2$,

$$[B_i, B_j] = 0.$$

Lemma

There exists a Lie algebra isomorphism $\mathbb{L}_n \to \mathfrak{so}_n$ *that sends*

$$B_i \mapsto I_{i,i+1} \qquad (1 \le i \le n-1).$$

Generators and relations for $\mathfrak{so}_3$

Example

The Lie algebra $\mathfrak{so}_3$ is defined by generators B_1, B_2 and relations

$$[B_1, [B_1, B_2]] = -B_2, \qquad [B_2, [B_2, B_1]] = -B_1.$$

The nonstandard quantum group $U'_q(\mathfrak{so}_n)$

Motivated by $\mathfrak{so}_n$, we now define $U'_q(\mathfrak{so}_n)$.

From now on, we fix a nonzero $q \in \mathbb{C}$ that is not a root of unity.

Recall the q-commutator

$$[X, Y]_q = qXY - q^{-1}YX.$$

Note that

$$[X, [X, Y]_q]_{q^{-1}} = X^2Y - (q^2 + q^{-2})XYX + YX^2.$$

The definition of $U'_q(\mathfrak{so}_n)$

Definition (Gavrilik and Klimyk 1991)

Define the algebra $U'_q(\mathfrak{so}_n)$ by generators

$$B_i \qquad\qquad (1 \le i \le n-1)$$

and the following relations.

(i) For $1 \le i, j \le n-1$ with $|i-j| = 1$,

$$B_i^2 B_j - (q^2 + q^{-2}) B_i B_j B_i + B_j B_i^2 = -B_j,$$
$$B_j^2 B_i - (q^2 + q^{-2}) B_j B_i B_j + B_i B_j^2 = -B_i.$$

(ii) For $1 \le i, j \le n-1$ with $|i-j| \ge 2$,

$$[B_i, B_j] = 0.$$

Generators and relations for $U_q'(\mathfrak{so}_3)$

Example

The algebra $U_q'(\mathfrak{so}_3)$ is defined by generators B_1, B_2 and relations

$$B_1^2 B_2 - (q^2 + q^{-2}) B_1 B_2 B_1 + B_2 B_1^2 = -B_2,$$
$$B_2^2 B_1 - (q^2 + q^{-2}) B_2 B_1 B_2 + B_1 B_2^2 = -B_1.$$

The above relations are a special case of the **Askey-Wilson** relations.

A PBW basis for $U'_q(\mathfrak{so}_n)$

Our next goal is to describe a PBW basis for $U'_q(\mathfrak{so}_n)$.

The definition of a PBW basis

Definition

Let $\mathcal{A}$ denote an algebra. A **Poincaré-Birkoff-Witt basis** (or **PBW basis**) for $\mathcal{A}$ is a subset Ω of $\mathcal{A}$ together with a linear order $\leq$ on Ω such that the following is a linear basis for the vector space $\mathcal{A}$:

$$a_1 a_2 \cdots a_r \qquad r \in \mathbb{N}, \qquad a_1, a_2, \ldots, a_r \in \Omega,$$
$$a_1 \leq a_2 \leq \cdots \leq a_r.$$

A PBW basis for $U_q'(\mathfrak{so}_n)$

Definition

For distinct $i, j \in \{1, 2, \ldots, n\}$ we define $I_{i,j} \in U_q'(\mathfrak{so}_n)$ as follows.

(i) For $j = i + 1$,

$$I_{i,i+1} = B_i.$$

(ii) For $j \geq i + 2$,

$$I_{i,j} = [B_i, I_{i+1,j}]_q.$$

(iii) For $j < i$,

$$I_{i,j} = -I_{j,i}.$$

A PBW basis for $U_q'(\mathfrak{so}_n)$

Lemma (Gavrilik and Klimyk 1991)

A PBW basis for $U_q'(\mathfrak{so}_n)$ is obtained by the elements

$$I_{i,j} \qquad\qquad (1 \le i < j \le n)$$

in the following order:

$$I_{1,2} < I_{1,3} < \cdots < I_{1,n} < I_{2,3} < I_{2,4} < \cdots < I_{2,n} < \cdots < I_{n-1,n}.$$

Our next goal is to describe some relations for the elements $I_{i,j}$ of $U'_q(\mathfrak{so}_n)$.

To facilitate this description, we give a definition.

Clockwise and counter-clockwise paths

Definition

Consider a regular n-gon P_n with vertices labeled clockwise $1, 2, \ldots, n$. We orient the edges $1 \to 2 \to 3 \to \cdots \to n \to 1$. Consider a sequence of at least 3 mutually distinct vertices of P_n, written $v_1, v_2, \ldots, v_t$ $(3 \leq t \leq n)$. Let p denote the directed path of length $n-1$ that starts at v_1 and runs clockwise around P_n. The sequence $v_1, v_2, \ldots, v_t$ is said to **run clockwise** whenever the path p encounters $v_1, v_2, \ldots, v_t$ in that order. The sequence $v_1, v_2, \ldots, v_t$ is said to run **counter-clockwise** whenever the inverted sequence $v_t, \ldots, v_2, v_1$ is runs clockwise. For distinct vertices i, j in P_n, by the **diagonal** $\overline{ij}$ we mean the line segment with endpoints i, j.

Lemma (Gavrilik and Klimyk 1991)

The following relations are satisfied by the elements $I_{i,j} \in U'_q(\mathfrak{so}_n)$.

(i) *For distinct $i, j \in \{1, 2, \dots n\}$, $I_{i,j} = -I_{j,i}$.*

(ii) *For mutually distinct $h, i, j \in \{1, 2, \dots n\}$,*

$$[I_{h,i}, I_{i,j}]_q = -I_{j,h} \qquad \text{if the sequence } h, i, j \text{ runs clockwise;}$$
$$[I_{h,i}, I_{i,j}]_{q^{-1}} = -I_{j,h} \qquad \text{if the sequence } h, i, j \text{ runs c-clockwise.}$$

(iii) *For mutually distinct $h, i, j, k \in \{1, 2, \dots n\}$,*

$$[I_{h,i}, I_{j,k}] = 0 \qquad \text{if the diagonals } \overline{hi} \text{ and } \overline{jk} \text{ do not overlap;}$$
$$[I_{h,i}, I_{j,k}] = (q^{-2} - q^2)(I_{h,j}I_{i,k} + I_{j,i}I_{k,h}) \quad \text{if } h, j, i, k \text{ runs cl;}$$
$$[I_{h,i}, I_{j,k}] = (q^2 - q^{-2})(I_{h,j}I_{i,k} + I_{j,i}I_{k,h}) \quad \text{if } h, j, i, k \text{ runs c-cl.}$$

In the previous lemma we see $\mathbb{Z}_n$-cyclic symmetry in the relations.

We now make this symmetry more explicit.

Definition

For notational convenience, define $B_n \in U'_q(\mathfrak{so}_n)$ by

$$B_n = I_{n,1}.$$

Cyclic symmetry for $U'_q(\mathfrak{so}_n)$

Lemma

There exists an automorphism ρ of $U'_q(\mathfrak{so}_n)$ that sends $B_i \mapsto B_{i+1}$ for $1 \le i \le n-1$ and $B_n \mapsto B_1$. For distinct $i, j \in \{1, 2, \ldots, n\}$ this automorphism sends $I_{i,j} \mapsto I_{i+1,j+1}$ where we understand $I_{i,n+1} = I_{i,1}$ and $I_{n+1,j} = I_{1,j}$.

The example $U_q'(\mathfrak{so}_3)$

To illustrate, we highlight the cyclic symmetry for $U_q'(\mathfrak{so}_3)$.

Example (Odesskii 1986, Fairlie 1990)

The algebra $U_q'(\mathfrak{so}_3)$ is defined by generators B_1, B_2, B_3 and relations

$$[B_1, B_2]_q = -B_3,$$
$$[B_2, B_3]_q = -B_1,$$
$$[B_3, B_1]_q = -B_2.$$

The representation theory of $U_q'(\mathfrak{so}_n)$ was carefully studied by Gavrilik, Iorgov, Klimyk during 1990–2005.

We now summarize their results.

On each finite-dimensional $U_q'(\mathfrak{so}_n)$-module the generators $\{B_i | 1 \leq i \leq n-1, \ i \text{ odd}\}$ are simultaneously diagonalizable.

Each finite-dimensional $U_q'(\mathfrak{so}_n)$-module is a direct sum of irreducible $U_q'(\mathfrak{so}_n)$-submodules.

The representation theory of $U_q'(\mathfrak{so}_n)$, cont.

The finite-dimensional irreducible $U_q'(\mathfrak{so}_n)$-modules are classified up to isomorphism.

According to the classification, there are two types of finite-dimensional irreducible $U_q'(\mathfrak{so}_n)$-modules, called **classical type** and **nonclassical type**.

The type is determined by the form of the eigenvalues for the generators $\{B_i | 1 \leq i \leq n-1, \ i \text{ odd}\}$ acting on the module.

The finite-dimensional irreducible $U_q'(\mathfrak{so}_6)$-modules

We now consider $n = 6$.

The isomorphism classes of finite-dimensional irreducible $U_q'(\mathfrak{so}_6)$-modules of classical type are in bijection with the 3-tuples (n_1, n_2, n_3) such that:

(i) $2n_i \in \mathbb{Z}$ for $i \in \{1, 2, 3\}$;

(ii) $n_i - n_j \in \mathbb{Z}$ for $i, j \in \{1, 2, 3\}$;

(iii) $n_1 \geq n_2 \geq |n_3|$.

Given a 3-tuple (n_1, n_2, n_3) that satisfies (i)–(iii) above, the corresponding finite-dimensional irreducible $U_q'(\mathfrak{so}_6)$-module of classical type was constructed by Klimyk by giving a Gelfand-Tsetlin basis for the module and the action of the generators $\{B_i\}_{i=1}^{5}$ on the basis.

For a finite-dimensional irreducible $U_q'(\mathfrak{so}_6)$-module of classical type, the corresponding 3-tuple (n_1, n_2, n_3) is called the **highest weight** of the module.

Our next goal is to use a 2-homogeneous bipartite distance-regular graph of diameter D to obtain an irreducible $U_q'(\mathfrak{so}_6)$-module.

The resulting module has classical type and highest weight $(D/2, D/2, D/2)$.

2-homogeneous bipartite distance-regular graphs

From now on, Γ denotes a distance-regular graph with vertex set X and diameter $D \geq 3$.

Definition (Nomura, Curtin)

Assume that Γ is bipartite. Then Γ is said to be 2-**homogeneous** whenever both:

(i) Γ is an antipodal 2-cover;

(ii) Γ is Q-polynomial.

A characterization of 2-homogeneous bipartite

A Q-polynomial structure $\{E_i\}_{i=0}^{D}$ is called **dual bipartite** whenever $a_i^* = 0$ for $0 \leq i \leq D$.

Lemma (Curtin 1998)

Assume that Γ is bipartite. Then the following are equivalent:

(i) Γ *is 2-homogeneous;*

(ii) Γ *has at least one Q-polynomial ordering of the primitive idempotents that is dual bipartite.*

Assume that (i), (ii) *hold. Then every Q-polynomial ordering of the primitive idempotents is dual bipartite.*

Our assumption

From now on, we assume that Γ is 2-homogeneous bipartite, but not a hypercube nor a cycle.

We fix a Q-polynomial ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents of Γ.

The corresponding eigenvalue (resp. dual eigenvalue) sequence is denoted $\{\theta_i\}_{i=0}^D$ (resp. $\{\theta^*_i\}_{i=0}^D$).

Some formulas

Lemma (Curtin 1998)

There exists a nonzero $q \in \mathbb{C}$ that is not a root of unity such that:

$$\theta_i = \theta_i^* = H\sqrt{-1}\left(q^{D-2i} - q^{2i-D}\right) \qquad (0 \le i \le D),$$

$$c_i = c_i^* = H\sqrt{-1}\,\frac{q^{2i} - q^{-2i}}{q^{D-2i} + q^{2i-D}}, \qquad (0 \le i \le D),$$

$$b_i = b_i^* = H\sqrt{-1}\,\frac{q^{2D-2i} - q^{2i-2D}}{q^{D-2i} + q^{2i-D}} \qquad (0 \le i \le D)$$

where

$$H = \sqrt{-1}\,\frac{q^{D-2} + q^{2-D}}{q^{-2} - q^{2}}.$$

Some notation

Let ∂ denote the path-length distance function for Γ.

For $x \in X$ and $0 \leq i \leq D$ define

$$\Gamma_i(x) = \{y \in X | \partial(x, y) = i\}.$$

We abbreviate $\Gamma(x) = \Gamma_1(x)$.

The strongly balanced condition

Abbreviate $E = E_1$. The following dependency is called the **strongly balanced** condition.

Lemma (Ter 1987)

For $x, y \in X$ and $0 \le i, j \le D$,

$$\sum_{\xi \in \Gamma_i(x) \cap \Gamma_j(y)} E\xi \in \mathrm{Span}\{Ex, Ey\}.$$

We will use the strongly balanced condition to obtain our main results.

The standard module V

Let V denote the vector space with basis X.

We call V the **standard module**.

Definition

Define the vector space $V^{\otimes 3} = V \otimes V \otimes V$ and the set

$$X^{\otimes 3} = \{x \otimes y \otimes z | x, y, z \in X\}.$$

Note that $X^{\otimes 3}$ is a basis for $V^{\otimes 3}$.

Definition

We define $A^{(1)}, A^{(2)}, A^{(3)} \in \mathrm{End}(V^{\otimes 3})$ as follows. For $x \otimes y \otimes z \in X^{\otimes 3}$,

$$A^{(1)}(x \otimes y \otimes z) = \sum_{\xi \in \Gamma(x)} \xi \otimes y \otimes z,$$

$$A^{(2)}(x \otimes y \otimes z) = \sum_{\xi \in \Gamma(y)} x \otimes \xi \otimes z,$$

$$A^{(3)}(x \otimes y \otimes z) = \sum_{\xi \in \Gamma(z)} x \otimes y \otimes \xi.$$

Some maps on $V^{\otimes 3}$

Definition

We define $A^{*(1)}, A^{*(2)}, A^{*(3)} \in \mathrm{End}(V^{\otimes 3})$ as follows. For $x \otimes y \otimes z \in X^{\otimes 3}$,

$$A^{*(1)}(x \otimes y \otimes z) = \theta^*_{\partial(y,z)} x \otimes y \otimes z,$$

$$A^{*(2)}(x \otimes y \otimes z) = \theta^*_{\partial(z,x)} x \otimes y \otimes z,$$

$$A^{*(3)}(x \otimes y \otimes z) = \theta^*_{\partial(x,y)} x \otimes y \otimes z.$$

The vectors $P_{h,i,j}$

Definition

For $0 \leq h, i, j \leq D$ define a vector

$$P_{h,i,j} = \sum x \otimes y \otimes z,$$

where the sum is over the 3-tuples of vertices x, y, z such that

$$h = \partial(y, z), \qquad i = \partial(z, x), \qquad j = \partial(x, y).$$

The vector space Λ

Definition

Let Λ denote the subspace of $V^{\otimes 3}$ spanned by

$$P_{h,i,j} \qquad (0 \leq h, i, j \leq D).$$

The dimension of Λ is $\binom{D+3}{3}$.

The vector space Λ, cont.

Lemma

The subspace Λ is invariant under $A^{(i)}$ and $A^{*(i)}$ for $i \in \{1, 2, 3\}$.

Some relations

Theorem (Ter 2025)

The following relations hold on Λ:

(i) *For distinct $i, j \in \{1,2,3\}$,*

$$[A^{(i)}, A^{(j)}] = 0, \qquad [A^{*(i)}, A^{*(j)}] = 0.$$

(ii) *For $i \in \{1,2,3\}$,*

$$[A^{(i)}, A^{*(i)}] = 0.$$

(iii) *For distinct $i, j \in \{1,2,3\}$,*

$$A^{(i)2}A^{*(j)} - (q^2+q^{-2})A^{(i)}A^{*(j)}A^{(i)} + A^{*(j)}A^{(i)2}$$
$$= -H^2(q^2-q^{-2})^2A^{*(j)},$$
$$A^{*(j)2}A^{(i)} - (q^2+q^{-2})A^{*(j)}A^{(i)}A^{*(j)} + A^{(i)}A^{*(j)2}$$
$$= -H^2(q^2-q^{-2})^2A^{(i)}.$$

Theorem

(Continued...)

(iv) *For mutually distinct* $h, i, j \in \{1, 2, 3\}$,

$$[A^{(h)}, [A^{*(i)}, A^{(j)}]_q]_q = [A^{*(h)}, [A^{(i)}, A^{*(j)}]_q]_q.$$

The main result

Theorem (Ter 2025)

The vector space Λ becomes a $U_q'(\mathfrak{so}_6)$-module on which

$$B_1 = \frac{A^{(1)}}{H(q^2 - q^{-2})}, \quad B_3 = \frac{A^{(2)}}{H(q^2 - q^{-2})}, \quad B_5 = \frac{A^{(3)}}{H(q^2 - q^{-2})},$$

$$B_2 = \frac{A^{*(3)}}{H(q^{-2} - q^2)}, \quad B_4 = \frac{A^{*(1)}}{H(q^{-2} - q^2)}, \quad B_6 = \frac{A^{*(2)}}{H(q^{-2} - q^2)}.$$

This module is irreducible, with classical type and highest weight $(D/2, D/2, D/2)$.

Conclusion

In this talk, we discussed the nonstandard quantum group $U'_q(\mathfrak{so}_n)$.

We then considered a 2-homogeneous bipartite distance-regular graph Γ.

In our main result, we used Γ to construct a finite-dimensional irreducible module for $U'_q(\mathfrak{so}_6)$.

THANK YOU FOR YOUR ATTENTION!