Spherical m-stiff configurations and related combinatorial structures

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Summary

E. Bannai, H. Kurihara, and H. Nozaki, On the existence and non-existence of spherical m-stiff configurations, arXiv:2504.17184.

- Spherical m-stiff configurations are spherical $(2m-1)$-designs that lie on m parallel hyperplanes.
- Non-existence results:
  1. For each fixed integer $m > 5$, there exists no m-stiff configuration in S^{d-1} for sufficiently large d.
  2. For each fixed integer $d > 10$, there exists no m-stiff configuration in S^{d-1} for sufficiently large m.
- Existence results:
  1. We provide a complete classification of the dimensions where m-stiff configurations exist for $m = 2, 3, 4, 5$

Spherical t-design

Definition 1 (Delsarte–Goethals–Seidel (1977))

A finite subset $X \subset S^{d-1}$ is a spherical t-design if

$$\frac{1}{|S^{d-1}|}\int_{S^{d-1}} f(x)d\mu(x) = \frac{1}{|X|}\sum_{x\in X} f(x)$$

for each polynomial $f(x_1,\ldots,x_d)$ of degree at most t.

Absolute bound [Delsarte–Goethals–Seidel (1977)]

- If $X \subset S^{d-1}$ is a spherical $2s$-design then $|X| \geq \binom{d+s-1}{s} + \binom{d+s-2}{s-1}$.
- If $X \subset S^{d-1}$ is a spherical $(2s+1)$-design then $|X| \geq 2\binom{d+s-1}{s}$. X is tight if equality holds.

Spherical designs with only few distances

For a finite subset X of S^{d-1},

$$D(X) = \{\langle x, y \rangle \mid x, y \in X, x \neq y\}.$$

X is an s-distance set, if $|D(X)| = s$.

Suppose X is a spherical s-distance t-design.

- $t \leq 2s$ (equality $\Leftrightarrow$ tight $2s$-design)
- If $t \geq s - 1$ holds, then X is distance invariant [DGS (1977)] ($|\{y \in X \mid \langle x, y \rangle = a\}|$ is independent of $x \in X$).
- If $t \geq 2s - 2$ holds, then X has the sturucture of a Q-polynomial association scheme [DGS (1977)].
- If $t \geq 2s - 1$ holds, then X is a universally optimal code [Cohn–Kumar (2007)].

Classification of tight spherical designs

Theorem 2 (Bannai–Damerell (1979, 1980))

If a tight t-design on $\mathbb{S}^{d-1}$ for $d \geq 3$ exists, then $t \leq 5$ or $t = 7, 11$.

- $t = 2, 3, 11$: classified
- $t = 4, 5, 7$: open

dim.	size	t	$D(X)$	name
n	$n+1$	2	$-1/n$	simplex
n	$2n$	3	$-1, 0$	cross polytope
8	240	7	$-1, \pm\frac{1}{2}, 0$	E_8 root
7	56	5	$-1, \pm\frac{1}{3}$	kissing
6	27	4	$-1/2, 1/4$	kissing
24	196560	11	$-1, \pm\frac{1}{2}, \pm\frac{1}{4}, 0$	Leech lattice
23	4600	7	$-1, \pm\frac{1}{3}, 0$	kissing
23	552	5	$-1 \pm \frac{1}{5}$	equiangular lines
22	275	4	$-1/4, 1/6$	kissing

Spherical m-stiff configurations

For a given point $z \in S^{d-1}$,

$$D(z, X) = \{\langle x, z\rangle \colon\ x \in X\}.$$

$$D_m(X) = \{z \in S^{d-1} \colon\ |D(z, X)| \leq m\}$$

Definition 3 (m-stiff configurations [Borodachov (2024)])

$X \subset S^{d-1}$ is an m-stiff if

1. X is a spherical $(2m-1)$-design.
2. $D_m(X)$ is non-empty.

An m-stiff minimizes f-potential energy

Let $f : [0, 4] \to (-\infty, \infty]$ be a function that is continuous on $(0, 4]$ with $f(0) = \lim_{t\to 0^+} f(t)$, differentiable on $(0, 4)$, and whose $(2m - 2)$-th derivative $f^{(2m-2)}$ is convex on $(0, 4)$.

For a given m-stiff configuration X, the f-potential

$$\sum_{x \in X} f(|z - x|^2)$$

attains its global minimum over S^{d-1} at every point z of $D_m(X)$ [Borodachov (2024)].

Known examples of m-stiff configurations

- $d = 2$: the regular $2m$-gon is the m-stiff.
- 1-stiff ($t = 1$): spherical 1-designs in S^{d-1} on a hyper plane
- 2-stiff ($t = 3$): regular cross polytopes and cubes in $S^{d-1}(d \geq 2)$, demicubes in S^{d-1} ($d \geq 5$), other examples (embedding of orthogonal array of strength 3 with 2 levels)

Name	$\|X\|$	d	m	t
24-cell	24	4	3	5
Symmetrized Schläfli	54	6	3	5
E_6 lattice	72	6	3	5
*Kissing (from E_8)	56	7	3	5
E_7 lattice	126	7	3	5
*Kissing (from Leech)	4600	23	4	7

*: tight spherical design

Existence theorem of m-stiff configurations

$P_m^{(\alpha,\alpha)}(x)$: Jacobi polynomial (Gegenbauer poly.), $\alpha = (d-3)/2$.

Theorem 4 (Borodachov (2024))

Suppose $m \geq 1$, $d \geq 2$. Let $x_1, \ldots, x_m$ be the zeros of $P_m^{(\alpha,\alpha)}(x)$. Let $\varphi_k(x) = \prod_{i \neq k}(x - x_i)/(x_k - x_i)$, which satisfies $\varphi_k(x_\ell) = \delta_{k\ell}$. Let

$$a_0(\varphi_k) = \frac{1}{\int_{-1}^{1}(1-x^2)^\alpha dx} \int_{-1}^{1} \varphi_k(x)(1-x^2)^\alpha dx.$$

There exists an m-stiff configuration of S^{d-1} if and only if $a_0(\varphi_k)$ is a positive rational number for each $k \in \{1, \ldots, m\}$.

- This theorem depends on the existence theorem of spherical designs, like Seymour and Zaslavsky (1984).
- $a_0(\varphi_k)$ coinsides with the Christoffel number λ_k of $P_m^{(\alpha,\alpha)}(x)$.

Layer structure in m-stiff Configurations

x_i is a zero of $P_m(x)$ and λ_i is its Christoffel number.

Suppose $X = X_1 \sqcup \cdots \sqcup X_m$ is an m-stiff (X_i is on a hyperplane).

- $|X_i| = \lambda_i |X|$. In particular, $|X_i|$ is independent of $z \in D_m(X)$.
- For any $x \in X_i$ and $z \in D_m(X)$, one has $\langle z, x \rangle = x_i$.

Example: Let $X = (\pm 1/\sqrt{2}, \pm 1/\sqrt{2}, 0, 0)^P$ be the 24-cell in S^3.

$$x_1 = -\frac{1}{\sqrt{2}}, \qquad x_2 = 0, \qquad x_3 = \frac{1}{\sqrt{2}}$$

$$\lambda_1 = \frac{6}{24}, \qquad \lambda_2 = \frac{12}{24}, \qquad \lambda_3 = \frac{6}{24}$$

$$|X_1| = 6, \qquad |X_2| = 12, \qquad |X_3| = 6$$

$z = (1, 0, 0, 0) \in D_3(X)$

Non-exsitence theorem (Main results)

Lemma 5

If all Christoffel numbers λ_i are rational numbers, then the squared zeros x_i^2 are rational.

Theorem 6 (Bannai–Kurihara–Nozaki (2025+))

1. *For each fixed integer $m > 5$, there exists no m-stiff configuration in S^{d-1} for $d > C_m$ with some C_m.*
2. *For each fixed integer $d > 10$, there exists no m-stiff configuration in S^{d-1} for $m > C_d$ with some C_d.*

Both results are established by analyzing the zeros of Gegenbauer polynomials $P_n^{(\alpha,\alpha)}(x)$.

Key polynomials [Bannai–Damerell (1980)]

Define $S_m(x)$ as the monic polynomial whose zeros are the reciprocals of the non-zero zeros of $P_m^{(\alpha+1,\alpha+1)}(x)$.
(We give a sketch of proof only for $d = 2k$ and $m = 2n$.)
For $X = x^2$,

$$S_m(X) = X^n + \sum_{r=1}^{n} (-1)^r u_r X^{n-r},$$

where

$$u_r = \binom{n}{r} \frac{h(h+2)(h+4)\cdots(h+2r-2)}{1\cdot 3\cdot 5\cdots(2r-1)}, \qquad h = d + 2n$$

Lemma 7

Suppose $m = 2n$ with $n \geq 1$. If a zero of $S_m(X)$ is rational, then the zero is an integer. $\to$ u_r should be integer.

Key idea

Lemma 8

For $d = 2k$ with $k \geq 2$, it holds that

$$u_n = 2^{2n+\lfloor (k-1)/2 \rfloor} \prod_{i=1}^{\lfloor k/2 \rfloor} \frac{2n+2i-1}{n+\lfloor (k-1)/2 \rfloor + i}$$

for each $n \in \mathbb{N}$.

Key idea for the non-integrality of u_n (given $d = 2k$, $m = 2n \to \infty$):

For $k = 5$ and n odd, we have $n + \lfloor (k-1)/2 \rfloor + 1 = n + 3$ is odd.

$$\prod_{i=1}^{\lfloor k/2 \rfloor} (2n+2i-1) = (2n+1)(2n+2)$$

$$\equiv (-5)(-4)(\text{depends only } k) \pmod{n+3}$$

For large odd n, this is not 0.

The complete list of the dimensions for $m \leq 5$

Let d be an integer greater than 2.

1. There exist m-stiff configurations in S^{d-1} for any d when $m = 1, 2, 3$.
2. There exist 4-stiff configurations in S^{d-1} if and only if
$$d = \frac{(5+2\sqrt{6})^{\ell} + (5-2\sqrt{6})^{\ell} - 6}{4}$$
for $\ell \in \mathbb{N}$ with $\ell > 1$. $\quad d = 23, 241, 2399, 23761 \ldots$
3. There exist 5-stiff configurations in S^{d-1} if and only if
$$\begin{aligned} d_5 = &\frac{3((19+6\sqrt{10})^{\ell_1} + (19-6\sqrt{10})^{\ell_1}) - 10}{4}, \\ &\frac{(7+2\sqrt{10})(19+6\sqrt{10})^{\ell_2} + (7-2\sqrt{10})(19-6\sqrt{10})^{\ell_2} - 10}{4}, \\ &\frac{(7-2\sqrt{10})(19+6\sqrt{10})^{\ell_3} + (7+2\sqrt{10})(19-6\sqrt{10})^{\ell_3} - 10}{4} \end{aligned}$$
for $\ell_1 \in \mathbb{N}$ and $\ell_2, \ell_3 \in \mathbb{N} \cup \{0\}$. $d = 4, 26, 124, 241, 1079 \ldots$

Concluding

Results:

1. For each fixed integer $m > 5$, there exists no m-stiff configuration in S^{d-1} for sufficiently large d.
2. For each fixed integer $d > 10$, there exists no m-stiff configuration in S^{d-1} for sufficiently large m.
3. We obtained the complete lists of the dimensions where m-stiff configurations exist for $m \leq 5$.

Conjecture: there is no m-stiff configuration in S^{d-1} for (d, m) with $d \geq 3$ and $m \geq 6$. (Newton polygon methods?)

Problem:

1. Find explicit examples of m-stiff configurations for $m = 4, 5$.
2. Investigate under what conditions tight spherical $(2m-1)$-designs become m-stiff.

Thank you for your attention.

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