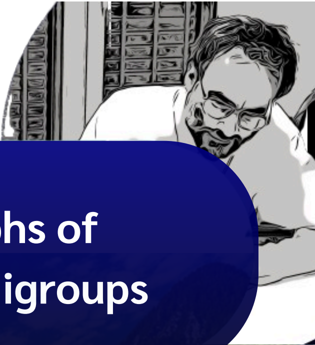
Khon Kaen University

Rajamangala University of Technology Lanna

Thammasat University

TerwilligerFest - Combinatorics around the q-Onsager algebra

Kranjska Gora, Slovenia | 23 - 27 June 2025



Cayley Regularity Graphs of Full Transformation Semigroups

Presenter: Nuttawoot Nupo

Co-authors:

Chollawat Pookpienlert and Yanisa Chaiya

Table of contents

1. Introduction and Preliminaries
   - Regular elements and regular semigroups
   - Cayley regularity graphs of semigroups
   - Full transformation semigroups
2. Cayley Regularity Graphs of Full Transformation Semigroups
   - Characterization of arcs
   - Connectedness
   - Completeness
   - Planarity
   - Out-degree and in-degree
   - Traversability
   - Domination parameters

Table of contents

1. Introduction and Preliminaries
   - Regular elements and regular semigroups
   - Cayley regularity graphs of semigroups
   - Full transformation semigroups
2. Cayley Regularity Graphs of Full Transformation Semigroups
   - Characterization of arcs
   - Connectedness
   - Completeness
   - Planarity
   - Out-degree and in-degree
   - Traversability
   - Domination parameters

Table of contents

1. Introduction and Preliminaries
   - Regular elements and regular semigroups
   - Cayley regularity graphs of semigroups
   - Full transformation semigroups
2. Cayley Regularity Graphs of Full Transformation Semigroups
   - Characterization of arcs
   - Connectedness
   - Completeness
   - Planarity
   - Out-degree and in-degree
   - Traversability
   - Domination parameters

Table of contents

1. Introduction and Preliminaries
    - Regular elements and regular semigroups
    - Cayley regularity graphs of semigroups
    - Full transformation semigroups
2. Cayley Regularity Graphs of Full Transformation Semigroups
    - Characterization of arcs
    - Connectedness
    - Completeness
    - Planarity
    - Out-degree and in-degree
    - Traversability
    - Domination parameters

Table of contents

1. Introduction and Preliminaries
   - Regular elements and regular semigroups
   - Cayley regularity graphs of semigroups
   - Full transformation semigroups
2. Cayley Regularity Graphs of Full Transformation Semigroups
   - Characterization of arcs
   - Connectedness
   - Completeness
   - Planarity
   - Out-degree and in-degree
   - Traversability
   - Domination parameters

Table of contents

1. Introduction and Preliminaries
   - Regular elements and regular semigroups
   - Cayley regularity graphs of semigroups
   - Full transformation semigroups
2. Cayley Regularity Graphs of Full Transformation Semigroups
   - Characterization of arcs
   - Connectedness
   - Completeness
   - Planarity
   - Out-degree and in-degree
   - Traversability
   - Domination parameters

Table of contents

1. Introduction and Preliminaries
   - Regular elements and regular semigroups
   - Cayley regularity graphs of semigroups
   - Full transformation semigroups
2. Cayley Regularity Graphs of Full Transformation Semigroups
   - Characterization of arcs
   - Connectedness
   - Completeness
   - Planarity
   - Out-degree and in-degree
   - Traversability
   - Domination parameters

Table of contents

1. Introduction and Preliminaries
    - Regular elements and regular semigroups
    - Cayley regularity graphs of semigroups
    - Full transformation semigroups
2. Cayley Regularity Graphs of Full Transformation Semigroups
    - Characterization of arcs
    - Connectedness
    - Completeness
    - Planarity
    - Out-degree and in-degree
    - Traversability
    - Domination parameters

Table of contents

1. Introduction and Preliminaries
    - Regular elements and regular semigroups
    - Cayley regularity graphs of semigroups
    - Full transformation semigroups
2. Cayley Regularity Graphs of Full Transformation Semigroups
    - Characterization of arcs
    - Connectedness
    - Completeness
    - Planarity
    - Out-degree and in-degree
    - Traversability
    - Domination parameters

Table of contents

1. Introduction and Preliminaries
   - Regular elements and regular semigroups
   - Cayley regularity graphs of semigroups
   - Full transformation semigroups
2. Cayley Regularity Graphs of Full Transformation Semigroups
   - Characterization of arcs
   - Connectedness
   - Completeness
   - Planarity
   - Out-degree and in-degree
   - Traversability
   - Domination parameters

Table of contents

1. Introduction and Preliminaries
    - Regular elements and regular semigroups
    - Cayley regularity graphs of semigroups
    - Full transformation semigroups
2. Cayley Regularity Graphs of Full Transformation Semigroups
    - Characterization of arcs
    - Connectedness
    - Completeness
    - Planarity
    - Out-degree and in-degree
    - Traversability
    - Domination parameters

Table of contents

1. Introduction and Preliminaries
   - Regular elements and regular semigroups
   - Cayley regularity graphs of semigroups
   - Full transformation semigroups
2. Cayley Regularity Graphs of Full Transformation Semigroups
   - Characterization of arcs
   - Connectedness
   - Completeness
   - Planarity
   - Out-degree and in-degree
   - Traversability
   - Domination parameters

Introduction and Preliminaries

Regular elements and regular semigroups

Definition

Let S be a semigroup and $x \in S$.

- x is said to be *regular* if there exists $y \in S$ such that $x = xyx$.
- y is called a *regular part* of x.
- S is called a *regular semigroup* if all of its elements are regular.

Cayley regularity graphs of semigroups

In 2023, Nupo and Panma introduced the Cayley regularity graph of a semigroup based on the connection between its elements and their regular parts as follows:

Definition

The *Cayley regularity graph* $\mathrm{CR}(S)$ of a semigroup S is defined to be a digraph with vertex set S and arc set containing all ordered pairs (x, y) in which y is a regular part of x.

Full transformation semigroup on X_n

Let $X_n = \{1, 2, \ldots, n\}$, and let $T_n = \{\alpha \mid \alpha : X_n \to X_n\}$. Then T_n is a semigroup under function composition and it is called the *full transformation semigroup on X_n*.

Generally, every semigroup can be embedded into a transformation semigroup on an appropriate set. As a result, the transformation semigroup is the well-known system widely studied in semigroup theory.

Full transformation semigroup on X_n

Let $X_n = \{1, 2, \ldots, n\}$, and let $T_n = \{\alpha \mid \alpha : X_n \to X_n\}$. Then T_n is a semigroup under function composition and it is called the *full transformation semigroup on X_n*.

Generally, every semigroup can be embedded into a transformation semigroup on an appropriate set. As a result, the transformation semigroup is the well-known system widely studied in semigroup theory.

Full transformation semigroup on X_n

Let $\alpha \in T_n$.

- $x\alpha$ is the image of $x \in X_n$ under α.
- $\mathrm{im}\,\alpha = \{x\alpha : x \in X_n\}$ is the range of α.
- $x\alpha^{-1} = \{y \in X_n : y\alpha = x\}$ is the inverse image of x under α.
- The notation $\alpha = a_1 a_2 \ldots a_n$, where $a_1, a_2, \ldots, a_n \in X_n$, means that $1\alpha = a_1, 2\alpha = a_2, \ldots,$ and $n\alpha = a_n$.

Full transformation semigroup on X_n

Let $\alpha \in T_n$.

- $x\alpha$ is the image of $x \in X_n$ under α.
- $\operatorname{im}\alpha = \{x\alpha : x \in X_n\}$ is the range of α.
- $x\alpha^{-1} = \{y \in X_n : y\alpha = x\}$ is the inverse image of x under α.
- The notation $\alpha = a_1a_2 \ldots a_n$, where $a_1, a_2, \ldots, a_n \in X_n$, means that $1\alpha = a_1, 2\alpha = a_2, \ldots,$ and $n\alpha = a_n$.

Full transformation semigroup on X_n

Let $\alpha \in T_n$.

- $x\alpha$ is the image of $x \in X_n$ under α.
- $\operatorname{im}\alpha = \{x\alpha : x \in X_n\}$ is the range of α.
- $x\alpha^{-1} = \{y \in X_n : y\alpha = x\}$ is the inverse image of x under α.
- The notation $\alpha = a_1 a_2 \ldots a_n$, where $a_1, a_2, \ldots, a_n \in X_n$, means that $1\alpha = a_1, 2\alpha = a_2, \ldots,$ and $n\alpha = a_n$.

Full transformation semigroup on X_n

Let $\alpha \in T_n$.

- $x\alpha$ is the image of $x \in X_n$ under α.
- $\operatorname{im} \alpha = \{x\alpha : x \in X_n\}$ is the range of α.
- $x\alpha^{-1} = \{y \in X_n : y\alpha = x\}$ is the inverse image of x under α.
- The notation $\alpha = a_1 a_2 \ldots a_n$, where $a_1, a_2, \ldots, a_n \in X_n$, means that $1\alpha = a_1, 2\alpha = a_2, \ldots,$ and $n\alpha = a_n$.

Cayley Regularity Graphs of T_n

Characterization of arcs

We first start by proving the characterization of arcs of $\mathrm{CR}(T_n)$.

Theorem

Let $\alpha, \beta \in T_n$. Then

$$(\alpha, \beta) \in A(\mathrm{CR}(T_n)) \text{ if and only if } x\beta \in x\alpha^{-1}$$

for all $x \in \operatorname{im} \alpha$.

Connectedness

Definition

A digraph D is said to be

1. *strongly connected* if a $[u, v]$-dipath exists in D,
2. *weakly connected* if D contains a $[u, v]$-semidipath,
3. *locally connected* if D satisfies the condition that whenever a $[u, v]$-dipath exists in D, a $[v, u]$-dipath must exist in D,
4. *unilaterally connected* if D contains either a $[u, v]$-dipath or a $[v, u]$-dipath,

for all $u, v \in V(D)$.

Connectedness

Theorem

The following statements hold.

1. $\mathrm{CR}(T_n)$ *is always a weakly connected digraph.*
2. $\mathrm{CR}(T_n)$ *is not a unilaterally, locally and strongly connected digraph.*

Completeness

Definition

A digraph D is said to be

1. *complete* if arcs (u, v) and (v, u) occur in D,
2. *semi-complete* if $A(D)$ contains (u, v) or (v, u),

for all distinct vertices $u, v \in V(D)$.

Now, we present the completeness and semi-completeness of $\mathrm{CR}(T_n)$ as follows.

Theorem

$\mathrm{CR}(T_n)$ *is not semi-complete. Consequently, it is not complete.*

Completeness

Definition

A digraph D is said to be

1. *complete* if arcs (u, v) and (v, u) occur in D,
2. *semi-complete* if $A(D)$ contains (u, v) or (v, u),

for all distinct vertices $u, v \in V(D)$.

Now, we present the completeness and semi-completeness of $\mathrm{CR}(T_n)$ as follows.

Theorem

$\mathrm{CR}(T_n)$ *is not semi-complete. Consequently, it is not complete.*

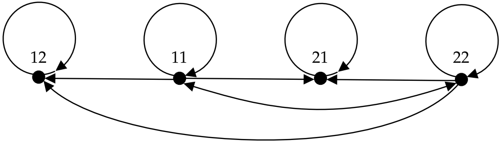
Planarity

Definition

A digraph D is called a *planar graph* if D can be drawn on the plane in such a way that no arcs cross each other.

Theorem

$\mathrm{CR}(T_n)$ *is planar if and only if* $n = 2$.



Out-degree and in-degree

We now determine the out-degree and in-degree of vertices in $\mathrm{CR}(T_n)$.

Definition

Let v be a vertex of a digraph D.

1. $d^+(v) = |N^+(v)| = \left|\{u \in V(D) \setminus \{v\} : (v,u) \in A(D)\}\right|$ is called an *out-degree* of v in D.
2. $d^-(v) = |N^-(v)| = \left|\{u \in V(D) \setminus \{v\} : (u,v) \in A(D)\}\right|$ is called an *in-degree* of v in D.

Definition

For each $\alpha \in T_n$, we say that α is *coregular* if $\alpha = \alpha\alpha\alpha$.

Out-degree and in-degree

We now determine the out-degree and in-degree of vertices in $\mathrm{CR}(T_n)$.

Definition

Let v be a vertex of a digraph D.

1. $d^+(v) = |N^+(v)| = \big|\{u \in V(D)\backslash\{v\} : (v,u) \in A(D)\}\big|$ is called an *out-degree* of v in D.
2. $d^-(v) = |N^-(v)| = \big|\{u \in V(D)\backslash\{v\} : (u,v) \in A(D)\}\big|$ is called an *in-degree* of v in D.

Definition

For each $\alpha \in T_n$, we say that α is *coregular* if $\alpha = \alpha\alpha\alpha$.

Out-degree and in-degree

Theorem

Let $\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_k \\ a_1 & a_2 & \dots & a_k \end{pmatrix} \in T_n$, *where* $|A_i| = n_i$ *for all* $i = 1, 2, \dots, k$. *Then*

$$d^+(\alpha) = \begin{cases} n_1 n_2 \cdots n_k n^{n-k} - 1 & \text{if } \alpha \text{ is coregular}; \\ n_1 n_2 \cdots n_k n^{n-k} & \text{otherwise.} \end{cases}$$

Out-degree and in-degree

Theorem

Let $\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_k \\ a_1 & a_2 & \dots & a_k \end{pmatrix} \in T_n$, *where* $|A_i| = n_i$ *for all* $i = 1, 2, \dots, k$. *Then*

$$d^{-}(\alpha) = \begin{cases} n - 1 + \sum\limits_{t=2}^{k} \sum\limits_{i_1 < i_2 < \cdots < i_t} n_{i_1} n_{i_2} \cdots n_{i_t} t^{n-t} & \text{if } \alpha \text{ is coregular;} \\ n + \sum\limits_{t=2}^{k} \sum\limits_{i_1 < i_2 < \cdots < i_t} n_{i_1} n_{i_2} \cdots n_{i_t} t^{n-t} & \text{otherwise,} \end{cases}$$

where $i_1, i_2, \dots, i_t \in \{1, 2, \dots, k\}$.

Traversability

Definition

Let D be a weakly connected digraph.

1. D is said to be *Eulerian* if there exists a closed trail containing all arcs of D.
2. D is said to be *Hamiltonian* if there exists a directed cycle that includes all vertices of D.

Theorem

$\mathrm{CR}(T_n)$ *is neither an Eulerian digraph nor a Hamiltonian digraph.*

Traversability

Definition

Let D be a weakly connected digraph.

1. D is said to be *Eulerian* if there exists a closed trail containing all arcs of D.
2. D is said to be *Hamiltonian* if there exists a directed cycle that includes all vertices of D.

Theorem

$\mathrm{CR}(T_n)$ *is neither an Eulerian digraph nor a Hamiltonian digraph.*

Domination parameters

Definition

A nonempty subset W of $V(D)$ is called

1. a *dominating set* of D if for each $v \in V(D) \setminus W$, there exists $u \in W$ such that $(u, v) \in A(D)$
2. a *total dominating set* if for each $v \in V(D)$, there exists $u \in W$ such that $(u, v) \in A(D)$

Domination parameters

Definition

A dominating set W of D is called

1. an *independent dominating set* if every pair of distinct $u, v \in W$ satisfies that $(u, v), (v, u) \notin A(D)$
2. a *connected dominating set* if the induced subdigraph $D[W]$ of D is weakly connected in D
3. a *split dominating set* of D if the induced subdigraph $D[V(D) \setminus W]$ is not weakly connected

Domination parameters

Definition

The *domination number* of D is the minimum cardinality of a dominating set of D which is denoted by $\gamma(D)$.

The *total, independent, connected, split domination numbers*

$$\gamma_t(D),\ \gamma_i(D),\ \gamma_c(D),\ \gamma_s(D)$$

of D are defined in sense of the minimum cardinality similar to the definition of $\gamma(D)$, respectively.

Domination parameters

Theorem

Let $n \geq 2$. Then

$$\gamma(\mathrm{CR}(T_n)) = \gamma_t(\mathrm{CR}(T_n)) = \gamma_i(\mathrm{CR}(T_n)) = \gamma_c(\mathrm{CR}(T_n)) = 1.$$

Theorem

$$n \leq \gamma_s(\mathrm{CR}(T_n)) \leq n - 1 + \sum_{t=2}^{n} t^{n-t}.$$

Thank you
for your kind attention!