Finite bivariate Tratnik functions

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joint work with
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Combinatorics around q-Onsager algebra
June 24, 2025

Motivation

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P- and Q-polynomial association schemes:

$$A_i = v_i(A_1)$$

$$|X|E_i = v_i^\star(|X|E_1)$$

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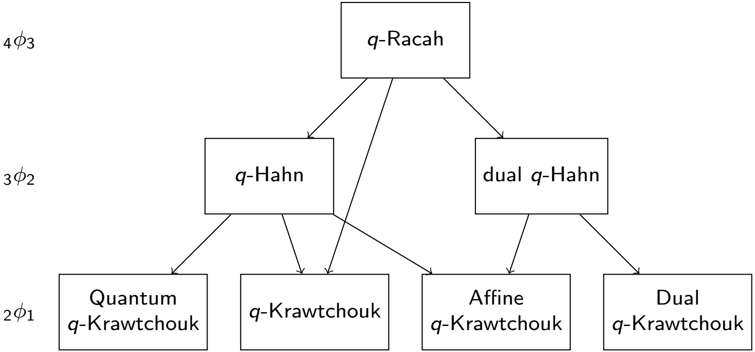
Leonard's theorem: these polynomials are finite families of the (q-)Askey scheme or Bannai-Ito polynomials.



$_4\phi_3$
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$_3\phi_2$
q-Hahn
dual q-Hahn
$_2\phi_1$
Quantum q-Krawtchouk
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Affine q-Krawtchouk
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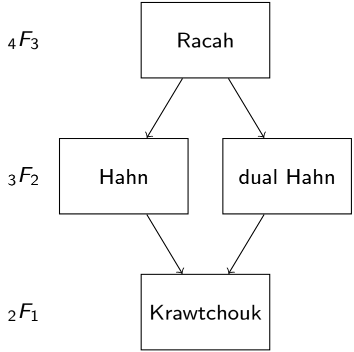
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Askey–Wilson algebra: encodes bispectrality of polynomials.

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- Factorized A_2-Leonard pairs.
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Two recurrence relations with three terms each:

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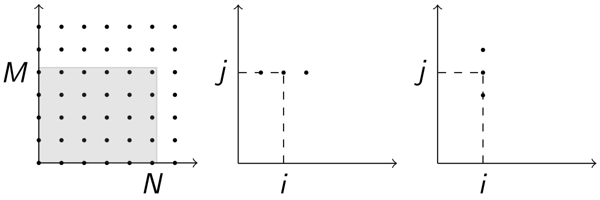

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Eigenvalues are:

$$T_{ij}(x,y) \propto \begin{bmatrix} N-x \\ j \end{bmatrix}_q \underbrace{{}_3\phi_2\left(\begin{matrix} q^{-i},\ q^{-x},\ 0 \\ q^{-\ell},\ q^{-N+j} \end{matrix} \middle|\ q,q\right)}_{\text{affine } q\text{-Krawtchouk}} \underbrace{{}_3\phi_2\left(\begin{matrix} q^{-j},\ q^{-y},\ q^{x+y-n-1} \\ q^{-n+N},\ q^{-N+x} \end{matrix} \middle|\ q,q\right)}_{\text{dual } q\text{-Hahn}}$$

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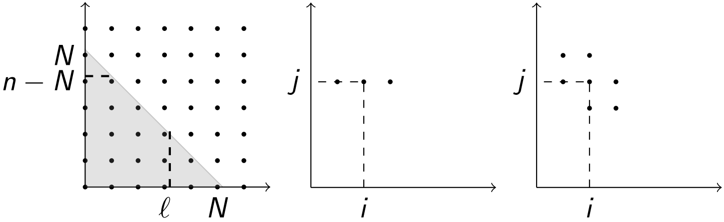

Tratnik functions

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Bispectrality of $R_i(x;\boldsymbol{\rho})$:

$$\lambda_{x,\boldsymbol{\rho}}\ R_i(x;\boldsymbol{\rho}) = A_{i,\boldsymbol{\rho}}\ R_{i+1}(x;\boldsymbol{\rho}) - (A_{i,\boldsymbol{\rho}} + C_{i,\boldsymbol{\rho}})\ R_i(x;\boldsymbol{\rho}) + C_{i,\boldsymbol{\rho}}\ R_{i-1}(x;\boldsymbol{\rho}),$$
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In order for this to be considered as a recurrence relation for $T_{ij}(x,y)$,

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We get a first recurrence relation for $T_{i,j}(x,y)$ with 3 terms.

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Second recurrence relation

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We want:

$$w_{x,y}\, T_{i,j}(x,y) = \sum_{(\epsilon,\epsilon')\in\mathcal{S}} \Phi_{i,j}^{\epsilon,\epsilon'}\; T_{i+\epsilon,j+\epsilon'}(x,y)\,.$$

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- $\mathcal{S}_{A_2} = \{(0,0),(0,1),(0,-1),(1,0),(-1,0),(-1,1),(1,-1)\}$.
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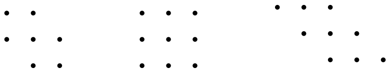
- $\mathcal{S}_{B_2'} = \{(0,0),(0,1),(0,-1),(1,0),(-1,0),(-1,1),(2,-1),(1,-1),(-2,1)\}$.



A_2 B_2 B_2'

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From the (q-)difference equation of $S_y(j;\boldsymbol{\sigma}_x)$,

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&w_{x,y}\, T_{i,j}(x,y) = k_{x,y}\nu_j(x) R_i(x;\boldsymbol{\rho}_j) \\
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$$\begin{aligned} k_{x,y}\frac{\nu_j(x)}{\nu_{j+1}(x)}\mathcal{B}_{j,\boldsymbol{\sigma}_x} R_i(x;\boldsymbol{\rho}_j) &= \sum_{\epsilon} \Phi_{i,j}^{\epsilon,1}\ R_{i+\epsilon}(x;\boldsymbol{\rho}_{j+1})\,, \\ -k_{x,y}(\mathcal{B}_{j,\boldsymbol{\sigma}_x} + \mathcal{D}_{j,\boldsymbol{\sigma}_x} - k'_{x,y}) R_i(x;\boldsymbol{\rho}_j) &= \sum_{\epsilon=0,1,-1} \Phi_{i,j}^{\epsilon,0}\ R_{i+\epsilon}(x;\boldsymbol{\rho}_j)\,, \\ k_{x,y}\frac{\nu_j(x)}{\nu_{j-1}(x)}\mathcal{D}_{j,\boldsymbol{\sigma}_x} R_i(x;\boldsymbol{\rho}_j) &= \sum_{\epsilon} \Phi_{i,j}^{\epsilon,-1} R_{i+\epsilon}(x;\boldsymbol{\rho}_{j-1})\,. \end{aligned}$$

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$$k_{x,y}\frac{\nu_j(x)}{\nu_{j+1}(x)}\mathcal{B}_{j,\boldsymbol{\sigma}_x}R_i(x;\boldsymbol{\rho}_j) = \sum_{\epsilon}\Phi^{\epsilon,1}_{i,j}\ R_{i+\epsilon}(x;\boldsymbol{\rho}_{j+1}),$$
$$-k_{x,y}(\mathcal{B}_{j,\boldsymbol{\sigma}_x} + \mathcal{D}_{j,\boldsymbol{\sigma}_x} - k'_{x,y})R_i(x;\boldsymbol{\rho}_j) = \sum_{\epsilon=0,1,-1}\Phi^{\epsilon,0}_{i,j}\ R_{i+\epsilon}(x;\boldsymbol{\rho}_j),$$
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Can see that $k_{x,y} = k_x$ and $k'_{x,y} = k'_x$.

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$$\omega_{i,j}\, T_{i,j}(x,y) = \sum_{(\epsilon,\epsilon')\in\mathcal{S}'} \Psi_{y,x}^{\epsilon,\epsilon'}\; T_{i,j}(x+\epsilon', y+\epsilon),$$

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with an eigenvalue

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Note that $\mathcal{S}'$ can be of different type than $\mathcal{S}$.

From the (q-)difference relation of $R_i(x;\boldsymbol{\rho}_j)$,

$$\omega_{i,j}\, T_{i,j}(x,y) = \kappa_{j,i}\nu_j(x) S_y(s;\boldsymbol{\sigma}_x)$$
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Contiguity relations

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Work with N. Crampé, L. Morey and L. Vinet.

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Classify solutions of equations of the form

$$\lambda^{+}_{x,\boldsymbol{\rho}}\ R_i(x;\boldsymbol{\rho}) = \sum_{\epsilon\in\mathscr{S}} \Phi_i^{\epsilon,+}\ R_{i+\epsilon}(\overline{x};\overline{\boldsymbol{\rho}}),$$

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$\overline{x} = x+\eta$, $\eta \in \{0,+1,-1\}$,
$\overline{\boldsymbol{\rho}}$ is a set of modified parameters,
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Related to Christoffel and Geronimus transformations.

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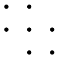
Some previous work done by R. Oste and J. Van der Jeugt (2016).

Type A_2



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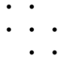

$$X_{i,\rho} := A_{i-1,\rho} C_{i,\rho}\,, \qquad Y_{i,\rho} := -(A_{i,\rho} + C_{i,\rho}).$$

Definition

The constraints $(\mathfrak{C}_{A_2})$ are defined by the requirement that the following expressions

$$\zeta^{2i} \frac{\zeta Y_{i,\overline{\rho}} - Y_{i,\rho} - \xi}{X_{i+1,\rho} - \zeta^2 X_{i,\overline{\rho}}} \prod_{k=1}^{i} \frac{X_{k,\overline{\rho}}}{X_{k,\rho}}\,,$$

$$\zeta^{2i} \frac{X_{i+1,\rho} - \zeta^2 X_{i+1,\overline{\rho}}}{(\zeta Y_{i,\overline{\rho}} - Y_{i+1,\rho} - \xi) X_{i+1,\rho}} \prod_{k=1}^{i} \frac{X_{k,\overline{\rho}}}{X_{k,\rho}}\,,$$

are equal and independent of i for any $i \geq 0$.

Proposition

The polynomials $R_i(x;\boldsymbol{\rho})$ chosen such that $\lambda_{x,\boldsymbol{\rho}} = \zeta\lambda_{\overline{x},\overline{\boldsymbol{\rho}}} - \xi$ satisfy the contiguity relation of type A_2

$$\lambda^{+}_{x,\boldsymbol{\rho}} R_i(x;\boldsymbol{\rho}) = \Phi_i^{0,+} R_i(\overline{x};\overline{\boldsymbol{\rho}}) + \Phi_i^{-1,+} R_{i-1}(\overline{x};\overline{\boldsymbol{\rho}}), \qquad i \geq 0,$$

(with the convention $\Phi_0^{-1,+} = 0$ and with $\Phi_0^{0,+} \neq 0$), if and only if the coefficients are given by (up to a global normalization)

$$\lambda^{+}_{x,\boldsymbol{\rho}} = 1,$$

$$\Phi_i^{0,+} = \zeta^i \prod_{k=0}^{i-1} \frac{A_{k,\overline{\boldsymbol{\rho}}}}{A_{k,\boldsymbol{\rho}}}, \qquad i \geq 0,$$

$$\Phi_i^{-1,+} = \zeta^{-i+1} \left(1 - \frac{\zeta A_{0,\overline{\boldsymbol{\rho}}} + \xi}{A_{0,\boldsymbol{\rho}}}\right) \prod_{k=1}^{i-1} \frac{C_{k+1,\boldsymbol{\rho}}}{C_{k,\overline{\boldsymbol{\rho}}}}, \qquad i \geq 1,$$

and the constraints $(\mathfrak{C}_{A_2})$ are satisfied.

Proposition

The polynomials $R_i(x;\boldsymbol{\rho})$ chosen such that $\lambda_{x,\boldsymbol{\rho}} = \zeta\lambda_{\overline{x},\overline{\boldsymbol{\rho}}} - \xi$ satisfy the contiguity relation of type A_2

$$\lambda^-_{x,\boldsymbol{\rho}} R_i(\overline{x};\overline{\boldsymbol{\rho}}) = \Phi^{1,-}_i R_{i+1}(x;\boldsymbol{\rho}) + \Phi^{0,-}_i R_i(x;\boldsymbol{\rho}), \qquad i \geq 0,$$

(with $\Phi^{0,-}_0 \neq 0$) if and only if the coefficients are given by (up to a global normalization)

$$\lambda^-_{x,\boldsymbol{\rho}} = (A_{0,\boldsymbol{\rho}} - \zeta A_{0,\overline{\boldsymbol{\rho}}} - \xi)\left(\frac{\lambda_{x,\boldsymbol{\rho}}}{A_{0,\boldsymbol{\rho}}} + 1\right) + C_{1,\boldsymbol{\rho}}\,,$$

$$\Phi^{1,-}_i = (A_{0,\boldsymbol{\rho}} - \zeta A_{0,\overline{\boldsymbol{\rho}}} - \xi)\zeta^{-i}\prod_{k=1}^{i}\frac{A_{k,\boldsymbol{\rho}}}{A_{k-1,\overline{\boldsymbol{\rho}}}}\,, \qquad i \geq 0,$$

$$\Phi^{0,-}_i = \zeta^i C_{1,\boldsymbol{\rho}}\prod_{k=1}^{i}\frac{C_{k,\overline{\boldsymbol{\rho}}}}{C_{k,\boldsymbol{\rho}}}\,, \qquad i \geq 0,$$

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q-Racah case

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$$R_i^{(qR)}(x;\boldsymbol{\rho}=\alpha,\beta,\gamma,N,q) = {}_4\phi_3\left(\begin{matrix} q^{-i},\ \alpha\beta q^{i+1},\ q^{-x},\ \gamma q^{x-N} \\ \alpha q,\ \beta\gamma q,\ q^{-N} \end{matrix}\ \middle|\ q;q\right).$$

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Type A_2 solutions:

(I) $\eta = 0\,,\quad \overline{\alpha} = \alpha\,,\quad \overline{\beta} = q\beta\,,\quad \overline{\gamma} = \gamma/q\,,\quad \overline{N} = N-1\,,\quad q^{\overline{x}} = q^x;$

(II) $\eta = 0\,,\quad \overline{\alpha} = \alpha\,,\quad \overline{\beta} = q\beta\,,\quad \overline{\gamma} = \gamma\,,\quad \overline{N} = N\,,\quad q^{\overline{x}} = q^x;$

(III) $\eta = -1\,,\quad \overline{\alpha} = q\alpha\,,\quad \overline{\beta} = \beta\,,\quad \overline{\gamma} = q\gamma\,,\quad \overline{N} = N-1\,,\quad q^{\overline{x}} = q^{x-1};$

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All solutions of type B_2 or B_2' are obtained from the solutions of type A_2.

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Similar results for other families of (q-)Askey scheme.

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Similar results for other families of (q-)Askey scheme.
Bannai-Ito: no type A_2 relations, unless we consider "complementary" BI.

Back to Tratnik functions

$$T_{i,j}(x,y) = \nu_j(x)\ R_i^{(P)}(x;\boldsymbol{\rho}_j)\ R_y^{(Q)}(j;\boldsymbol{\sigma}_x)\,,$$

(P,Q) is a pair of families of polynomials in the (q-)Askey scheme.

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Search for solutions to:

$$k_x \frac{\nu_j(x)}{\nu_{j+1}(x)} \mathcal{B}_{j,\boldsymbol{\sigma}_x} = \phi_j^+ \lambda_{x,\boldsymbol{\rho}_j}^+$$

$$- k_x(\mathcal{B}_{j,\boldsymbol{\sigma}_x} + \mathcal{D}_{j,\boldsymbol{\sigma}_x} - k_x') = \phi_j \lambda_{x,\boldsymbol{\rho}} + \phi_j'$$

$$k_x \frac{\nu_{j+1}(x)}{\nu_j(x)} \mathcal{D}_{j+1,\boldsymbol{\sigma}_x} = \phi_j^- \lambda_{xj,\boldsymbol{\rho}_j}^-$$

$$\kappa_j \frac{\nu_j(x)}{\nu_j(x+1)} B_{x,\boldsymbol{\rho}_j} = f_x^+ \ell_{j,\boldsymbol{\sigma}_x}^+$$

$$- \kappa_j(B_{x,\boldsymbol{\rho}_j} + D_{x,\boldsymbol{\rho}_j} - \kappa_j') = f_x \ell_{j,\boldsymbol{\sigma}} + f_x'$$

$$\kappa_j \frac{\nu_j(x+1)}{\nu_j(x)} D_{x+1,\boldsymbol{\rho}_j} = f_x^- \ell_{j,\boldsymbol{\sigma}_x}^-$$

Preliminary results: A_2/A_2 Tratnik functions (polynomials)

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Triangular domains ($q = 1$)
$i + j \leq N$ and $x + y \leq N$:

$$T_{i,j}(x, y) = \frac{(x - N)_j}{(-N)_j} \; R_i^{(H)}(x; \alpha, \beta + j, N - j) R_j^{(dH)}(y; a, b + x, N - x)$$

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Rectangular domains ($q = 1$)
$i, x \leq N$ and $j, y \leq M$:

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$$T_{i,j}(x, y) = R_i^{(H)}(x; \alpha, \beta + j, N) R_j^{(dH)}(y; a, x - M - 1 - \beta, M)$$

It appears there are also solutions of the form:

$$T_{i,j}(x, y) \propto R_i^{(R)}(x; \boldsymbol{\rho}_j) R_y^{(R)}(j; \boldsymbol{\sigma}_x).$$

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For example:

$$\boldsymbol{\rho}_j = (\alpha, \beta + j, -M-1, M),$$
$$\boldsymbol{\sigma}_x = (\beta - \frac{1}{2}, -\beta - M - N - 1 + x, N + 2\beta, N).$$

$$\boldsymbol{\rho}_j = (\alpha, \beta + j, -N - j - 2\beta - 1, N - j),$$
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Also of type B_2'/B_2':

$$\begin{aligned}\boldsymbol{\rho}_j &= (\alpha, 2j + c - N, \gamma - j, N - j),\\ \boldsymbol{\sigma}_x &= (a, -N + \gamma + 2x, c - x, N - x).\end{aligned}$$

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