An improved bound for strongly regular graphs with smallest eigenvalue $-m$

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TerwilligerFest
June 23–28

Outline

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1. Definitions
   - Definitions
   - Examples
2. Bounds on SRG
   - Neumaier
   - Our bound
3. Proof
   - Definitions
   - Sketch of the proof

Notations

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- The eigenvalues of Γ are the eigenvalues of its adjacency matrix.
- $d(x, y)$:the distance between x and y.
- A graph is regular if there exists an integer k such each vertex has exactly k neighbours.

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- A strongly regular graph with given parameters will be referred to as *srg*(v, k, λ, μ), where $v \geq 2$ is the number of vertices and the graph is regular with valency k.
- To dispense with trivialities, we will be concerned only with *primitive* strongly regular graphs, which are connected strongly regular graphs whose complement is also connected.
- Examples of strongly regular graphs: The Petersen graph, $(t \times t)$-grid, etc.

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- A strongly regular graph is either a conference graph or has all eigenvalues being integers.
- In this talk, we always assume that Γ is not a conference graph.

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Steiner graphs

- A *Steiner system* $S(2, m, v)$ is a 2-$(v, m, 1)$ *design*, that is, a collection of m-sets taken from a set of size v, satisfying the property that every pair of elements from the v-set is contained in exactly one m-set.

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- The *block graph of a Steiner system* $S(2, m, nm + m - n)$ is defined as the graph whose vertices correspond to the blocks of the system and two blocks are adjacent in this graph if and only if they intersect at exactly one point.

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- We refer to this block graph as the *Steiner graph* $S_m(n)$, and this graph is an SRG with smallest eigenvalue $-m$ and $\mu = m^2$.

Now we introduce the Latin Square graphs.

Latin Square graphs

- An *orthogonal array*, denoted as $OA(m, n)$, is a $m \times n^2$ array with entries chosen from the set $\{1, \ldots, n\}$ with the property that the columns of any $2 \times n^2$ subarray contain all possible n^2 pairs exactly once.

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- The *block graph of an orthogonal array*, denoted as $\Gamma_{OA(m,n)}$, is a graph whose vertices are the columns of $OA(m, n)$ and two columns are adjacent if and only if there exists a row where they share the same entry.

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- The block graph of an $OA(m, n)$ is called a *Latin square graph* $LS_m(n)$ and this graph is an SRG with smallest eigenvalue $-m$ and $\mu = m(m-1)$.

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- Sims (unpublished) showed that for fixed integer $m \geq 2$, there are only finitely many primitive SRG besides the Steiner graphs and Latin Square graphs.
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- We only know of two examples where equality holds, namely for $m = 2$ and $m = 3$.

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Theorem

Let Γ be a primitive strongly regular graph with smallest eigenvalue $-m$, where m is a positive integer. Let $f(m,\mu) = \frac{1}{2}m(m-1)(\mu+1)+\mu-m-1$. Then the following hold:

(i) *(Bruck (1963)) If $\mu = m(m-1)$ and $\lambda > f(m,\mu)$, then Γ is a Latin square graph $\mathrm{LS}_m(\lambda - m(m-3))$.*

(ii) *(Bose (1963)) If $\mu = m^2$ and $\lambda > f(m,\mu)$, then Γ is the block graph of a 2-$(\lambda(m-1)-m(m-1)(m-2)+m, m, 1)$ design, i.e. a Steiner graph.*

(iii) *(Neumaier (1979)) If $\mu \neq m(m-1)$ and $\mu \neq m^2$, then $\lambda \leq f(m,\mu)$.*

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Theorem

Let Γ be a primitive strongly regular graph with smallest eigenvalue $-m$, where m is a positive integer. Let $f(m,\mu) = \frac{8}{3}m(\mu-1) - \frac{2}{3}\mu + 3m - \frac{10}{3}$. Then the following hold:

(i) *If $\mu = m(m-1)$ and $\lambda > f(m,\mu)$, then Γ is a Latin square graph $\mathrm{LS}_m(\lambda - m(m-3))$.*

(ii) *If $\mu = m^2$ and $\lambda > f(m,\mu)$, then Γ is the the block graph of a 2-$(\lambda(m-1) - m(m-1)(m-2) + m, m, 1)$ design.*

(iii) *If $\mu \neq m(m-1)$ and $\mu \neq m^2$, then $\lambda \leq f(m,\mu)$.*

- The bound of Neumaier is $\lambda = O(m^2\mu)$ and our bound is $\lambda = O(m\mu)$, but for small m, the bound by Neumaier is better.

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- The Cameron graph, a $srg(231, 30, 9, 3)$ with smallest eigenvalue -3, gives an example where $\lambda > m^2$ and $\mu = m$ both holds. Note that the GQ(t^2, t) has $m = t + 1$ and $\lambda = m(m-2)$ and there exists infinitely many m for which such a GQ exists.

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- Metsch (1991) showed this result for $\mu = m(m-1)$.

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- First we need some definitions.
- A *partial linear space* is an incidence structure such that each pair of distinct points are both incident with at most one line, and a *linear space* is a partial linear space such that every pair of points is contained in a unique line.
- The *point graph* Γ of an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is the graph with vertex set $\mathcal{P}$, and two distinct points are adjacent if and only if they are on a common line.

- A *partial geometry* $pg(K, R, T)$ is a partial linear space with the property that every line contains exactly K points, every point lies on exactly R lines, and given a line L and point $x \notin L$ there are exactly T lines containing x and intersecting L (in the literature, one also meets the notation $pg(s, t, \alpha)$, where $K = s + 1, R = t + 1, T = \alpha$).

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- We say a SRG has *geometric parameters* (K, R, T) if it has parameters $(K + K(K-1)(R-1)/T, R(K-1), (R-1)(T-1) + K - 2, RT)$ where K, R are positive integers and T a non-negative integer.

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- Note that $\tau(x) = m$ if and only if x is a Delsarte vertex.

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Theorem

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(1) $(\sigma + 1)(\lambda + 1) - k > (\mu - 1)\binom{\sigma+1}{2}$;

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(2) $\lambda + 1 > (\mu - 1)(2\sigma - 1)$.

Define a line as a maximal clique with at least $\lambda+2-(\mu-1)(\sigma-1)$ vertices. Then $X = (V(\Gamma), \mathcal{L}, \in)$ is a partial linear space, where $\mathcal{L}$ is the set of all lines, Γ is the point graph of X, and the symbol $\in$ means that the relation required for the incidence structure is merely given by inclusion. Moreover, each vertex lies on at most σ lines, and every pair of adjacent vertices in Γ lie in a unique line.

- We optimize the two bounds in the result of Metsch and it turns out that for

$$\lambda > \frac{8}{3}m(\mu - 1) - \frac{2}{3}\mu + 3m - \frac{10}{3}$$

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- This gives that at least $v - \ell$ vertices of the SRG are Delsarte vertices.
- We obtain that there are at least two Delsarte vertices unless $\mu = 1$.
- It is then not so difficult to see that then the SRG has to have geometric parameters.

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- So all lines are Delsarte cliques, and hence our SRG is geometric.

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- Thank you for your attention.