q-characters for quantum affine symmetric pairs

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Quantum affine algebras

Motivation

- $\mathfrak{g}$ - simple complex Lie algebra (e.g., $\mathfrak{sl}_n(\mathbb{C})$)
- G - corresponding simply-connected Lie group (e.g., $SL_n(\mathbb{C})$)
- $T \subset G$ - maximal torus; W - Weyl group

$$\{ \text{ f.d. simple modules } \} \longleftrightarrow \{ \text{ dominant integral weights } \}$$

- For classification, only information about the HW space is needed.
- What about an invariant, which encodes information about every weight space?

$$\chi\colon [\mathrm{Rep}^{fd}\, G] \;\to\; \mathbb{Z}[T] = \mathbb{Z}[y_1^{\pm 1}, \cdots, y_{n-1}^{\pm 1}]$$
$$V \mapsto \chi_V(t) = \mathrm{Tr}_V(t)$$

(P1) $\operatorname{im} \chi = \mathbb{Z}[T]^W$

(P2) Since all weights of V are of the form $\mu = \lambda_+ - \sum n_i \alpha_i$ ($\lambda_+ = \sum l_i \omega_i$, $n_i, l_i \in \mathbb{Z}_+$),

$$\chi(V) = m_+(1 + \sum M_p), \quad m_+ = \prod y_i^{l_i}, \quad M_p = \prod a_i^{-n_i}$$

Three presentations

Want to generalize: classical $\rightsquigarrow$ quantum and affine

Assumption: $k = \mathbb{C}$, and q is not a root of unity.

Quantum affine algebras admit three distinct presentations

$$U_q(\widehat{\mathfrak{g}})$$

$\swarrow$ Kac–Moody | $\downarrow$ new Drinfeld | $\searrow$ RTT

Kac–Moody	new Drinfeld	RTT
$E_i, F_i, K_i^{\pm}\ (i \in I)$	$x_{i,r}^{\pm}, h_{i,s}\ (i \in I_0, r \in \mathbb{Z}, s \in \mathbb{Z} - \{0\})$	
quant. of $U(\widehat{\mathfrak{g}})$	$0 \to \mathbb{C}\mathbf{c} \to U(\widehat{\mathfrak{g}}) \to U(\mathfrak{g}[t^{\pm}]) \to 0$	

The new Drinfeld presentation

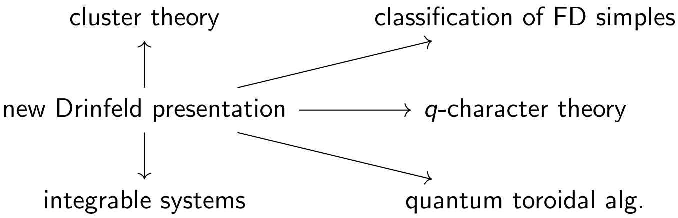
- **Construction** via **braid group action**:

$$x_{i,r}^{\pm} = o(i)^r T_{\omega_i}^{\mp r}(e_i^{\pm}).$$

- Key role played by the almost commutative algebra generated by the $h_{i,s}$.
- **Drinfeld–Cartan operators**:

$$\phi_i^{\pm}(z) = \sum_{k=0}^{\infty} \phi_{i,\pm k}^{\pm} z^{\pm k} = K_i^{\pm 1} \exp\left(\pm(q - q^{-1}) \sum_{k=1}^{\infty} h_{i,\pm k} z^{\pm k} \right).$$

- Applications:



Finite-dimensional representations

FD reps - (non-semisimple) meromorphically braided tensor category.
The following results are due to Chari and Pressley.

- Highest (loop) weight vector:

$$x^+_{i,r} \cdot v_0 = 0, \qquad \phi^\pm_{i,r} \cdot v_0 = \lambda^\pm_{i,r} v_0.$$

- Every FD simple is HW. Every HW simple is a quotient of a Verma.
- Let V be a simple FD $U_q(\widehat{\mathfrak{g}})$-module. The spectrum of Drinfeld–Cartan operators on v_0 can be described in terms of *Drinfeld polynomials*:

$$\sum_{k=0}^{\infty} \lambda^\pm_{i,\pm k} z^{\pm k} = q^{\deg P_i} \frac{P_i(q^{-1}z)}{P_i(qz)}.$$

- There is a bijective correspondence

$$\{\text{ simple FD modules }\} \longleftrightarrow \{\text{ tuples of polynomials with cnst term } 1\}.$$

q-characters & Drinfeld presentation

The following are due to Frenkel, Mukhin and Reshetikhin.

- The usual character χ encodes the eigenvalues of the $K_i^{\pm}$.
- There exists q-character χ_q, encoding the eigenvalues of the $\phi_i^{\pm}(z)$, which lifts χ to the affine case:

$$\begin{array}{ccc} [\,\mathrm{Rep}\,U_q(L\mathfrak{g})] & \xrightarrow{\ \chi_q\ } & \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I_0, a\in\mathbb{C}^\times} \\ \Big\downarrow{\scriptstyle res} & & \Big\downarrow \\ [\,\mathrm{Rep}\,U_q(\mathfrak{g})] & \xrightarrow{\ \ \chi\ \ } & \mathbb{Z}[y_i^{\pm 1}]_{i\in I_0} \end{array}$$

- The eigenvalues are expansions of rational functions of the form

$$\gamma_i^{\pm}(z) = q^{\deg Q_i - \deg R_i}\frac{Q_i(q^{-1}z)R_i(qz)}{Q_i(qz)R_i(q^{-1}z)}.$$

- Writing $Q_i(z) = \prod_{r=1}^{k_i}(1 - za_{i,r})$, $R_i(z) = \prod_{s=1}^{l_i}(1 - zb_{i,s})$, the q-character of V can now be expressed as

$$\chi_q(V) = \sum_{\gamma} \dim(V_\gamma) M_\gamma, \qquad M_\gamma = \prod_{i\in I_0}\prod_{r=1}^{k_i} Y_{i,a_{i,r}} \prod_{s=1}^{l_i} Y_{i,b_{i,s}}^{-1}.$$

q-characters & universal R-matrix

- Let $\mathcal{R} \in U_q(\mathfrak{b}_+)\widehat{\otimes} U_q(\mathfrak{b}_-)$ be the universal R-matrix, and (V, π_V) a finite-dimensional representation. The associated L-operator and transfer matrix are

$$L_V(z) = (\pi_{V(z)} \otimes \mathrm{id})(\mathcal{R}), \qquad t_v = \mathrm{Tr}\, q^{2\rho} L_V(z).$$

- The composition

$$\chi_q : [\mathrm{Rep}\, U_q(L\mathfrak{g})] \xrightarrow{t_v} U_q(\mathfrak{b}_-)[[z]] \xrightarrow{HC} U_q(\widetilde{\mathfrak{h}})[[z]]$$

is a ring homomorphism.

- The image is contained in the polynomial ring generated by

$$Y_{i,a} = K_{\omega_i}^{-1} \exp\left(\sum_{k>0} \tilde{h}_{i,-k} a^k z^k \right), \quad \tilde{h}_{i,-k} = \sum_{j \in \mathbb{I}_{fin}} \widetilde{C}_{ji}(q^k) h_{j,-k}.$$

The coproduct

The coproduct plays a key role in proving the **multiplicativity** of Drinfeld polynomials and q-characters:

$$P_i(V \otimes W) = P_i(V) \cdot P_i(W), \qquad \chi_q(V \otimes W) = \chi_q(V) \cdot \chi_q(W).$$

Chari-Pressley and Damiani proved that

- $\boldsymbol{\phi}_i^{\pm}(z)$ are *approximately group-like*:

$$\Delta(\boldsymbol{\phi}_i^{\pm}(z)) \equiv \boldsymbol{\phi}_i^{\pm}(z) \otimes \boldsymbol{\phi}_i^{\pm}(z) \quad \mod \quad U_+ \otimes U_-,$$

- $\boldsymbol{x}_{i,\geq 0}^{+}(z)$, etc., are *approximately twisted-primitive*:

$$\Delta(\boldsymbol{x}_{i,\geq 0}^{+}(z)) \equiv \boldsymbol{x}_{i,\geq 0}^{+}(z) \otimes \boldsymbol{\phi}_i^{\pm}(z) + 1 \otimes \boldsymbol{x}_{i,\geq 0}^{+}(z) \quad \mod \quad U_{\geq 2} \otimes U_-,$$

in the Drinfeld gradation ($\deg x_{i,r}^{+} = \alpha_i$, $\deg x_{i,r}^{-} = -\alpha_i$, $\deg h_{i,r} = 0$).

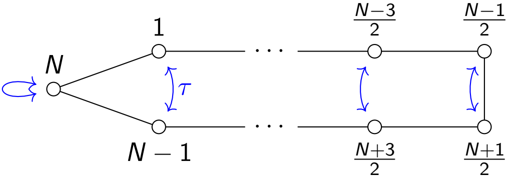
Quantum symmetric pairs

Quantum symmetric pairs

- Classical symmetric pair: ss. Lie algebra $\mathfrak{g}$ + involution σ, e.g.,
  - $\mathfrak{g} = \mathfrak{sl}_n$, $\sigma(x) = -x^t \quad \rightsquigarrow \quad \mathfrak{g}^\sigma = \mathfrak{so}_n$,
  - $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, $\sigma(x) = -J^{-1}x^t J \quad \rightsquigarrow \quad \mathfrak{g}^\sigma = \mathfrak{sp}_n$.
- **Motivation** in the classical case: real forms of Lie groups, classification of symmetric spaces
- **Problem**: $U_q(\mathfrak{g}^\sigma) \not\subset U_q(\mathfrak{g})$. Correct compatible quantization - coideal subalgebra $\mathcal{B}_{c,s} \subset U_q(\mathfrak{g})$:

$$\Delta(\mathcal{B}_{c,s}) \subset \mathcal{B}_{c,s} \otimes U_q(\mathfrak{g}), \qquad \mathcal{B}_{c,s} \xrightarrow{q \to 1} U(\mathfrak{g}^\sigma)$$

- There is a monoidal action: Rep $\mathcal{B}_{c,s} \curvearrowleft$ Rep $U_q(\mathfrak{g})$.
- Quantum symmetric pairs are classified by **Satake diagrams**, e.g.,



- Generated by "$B_i = F_i - E_{\tau(i)}$"

QSP - motivation

1. Quantum Schur–Weyl duality

$$\begin{array}{lccccc} \text{type A:} & U_q(\mathfrak{sl}_n) & \curvearrowright & V_q^{\otimes m} & \curvearrowleft & \mathcal{H}(\mathrm{A}_{m-1}) \\ & \cup & & & & \cap \\ \text{type B:} & \mathcal{B}_{c,s}(\mathrm{AIII}) & \curvearrowright & V_q^{\otimes m} & \curvearrowleft & \mathcal{H}(\mathrm{B}_m). \end{array}$$

2. Gelfand–Zetlin patterns

$$\mathfrak{sl}_n \subset \mathfrak{sl}_{n+1} \subset \mathfrak{sl}_{n+2} \subset \dots$$
$$U_q'(\mathfrak{so}_n) \subset U_q'(\mathfrak{so}_{n+1}) \subset U_q'(\mathfrak{so}_{n+2}) \subset \dots$$

3. Evaluation homomorphisms

$$U_q(L\mathfrak{sl}_n) \to U_q(\mathfrak{sl}_n)$$
$$Y_q^{tw}(\mathfrak{so}_n) \to U_q'(\mathfrak{so}_n)$$
$$Y_q^{tw}(\mathfrak{sp}_{2n}) \to U_q'(\mathfrak{sp}_{2n})$$

QSP - history & motivation

- Early examples appear in math. physics and integrable systems (with boundaries).
- Systematic study originated by G. Letzter in late 1990s - early 2000s.
- S. Kolb: Kac–Moody quantum symmetric pairs (2012).
- Three important developments around 2012-15:
  - M. Balagovic & S. Kolb: bar involution, universal K-matrix,
  - H. Bao & W. Wang: $\imath$canonical basis, SW duality, KL theory,
  - M. Ehrig, C. Stroppel: categorification, relation to cat. $\mathcal{O}$, Nazarov-Wenzl algebras.
- W. Wang's $\imath$programme: "QSP constitute a vast generalization of quantum groups", with most algebraic/geometric/categorical features from quantum group theory generalizing to QSP.
- Recent developments: $\imath$Hall algebra constructions, new Drinfeld presentation, super versions.

New Drinfeld presentation for affine QSP

- Rank one case (q-Onsager algebra) - Kolb–Baseilhac (2017).
- General split simply-laced case (Satake = Dynkin diagram, AI) - Lu–Wang (2020).
- Non-simply laced, quasi-split cases ($AIII$) - Lu, Wang, Zhang (2021-24).

The algebra $\mathcal{B}_{c,s}$ is generated by $A_{i,r}$, $H_{i,s}$ ($r \in \mathbb{Z}$, $s > 0$). The $H_{i,s}$ generate a *commutative* subalgebra.

General principle

Current presentations are constructed from Kac–Moody-type presentations via affine braid group actions.

$$E_i \rightsquigarrow x_{i,r}^{+} = T_{\omega_i}^{-r}(E_i), \quad F_i \rightsquigarrow x_{i,r}^{-} = T_{\omega_i}^{r}(F_i),$$

$$B_i \rightsquigarrow A_{i,r} = \mathbf{T}_{\omega_i}^{-r}(B_i)$$

It appears research on $\imath$Hall algebras facilitated finding the relevant braid group symmetries.

Braid group action - type AI

The two braid group actions are different and incompatible. One is not the restriction of the other!

Lusztig braid group action

$$T_i(E_i) = -F_iK_i, \qquad T_i(E_j) = \sum_{r+s=-a_{ji}} (-1)^r q_i^{-r} E_i^{(s)} E_j E_i^{(r)},$$
$$T_i(F_i) = -K_i^{-1}E_i, \quad T_i(F_j) = \sum_{r+s=-a_{ji}} (-1)^r q_i^{r} F_i^{(r)} F_j F_i^{(s)},$$

QSP braid group action

$$\mathbf{T}_i(B_j) = \begin{cases} \mathbb{K}_j^{-1} B_j & \text{if } i=j, \\ B_j & \text{if } a_{ji}=0, \\ B_jB_i - q_iB_iB_j & \text{if } a_{ji}=-1, \\ [2]_{q_i}^{-1}(B_jB_i^2 - q_i[2]_{q_i}B_iB_jB_i + q_i^2B_i^2B_j) + B_j\mathbb{K}_i & \text{if } a_{ji}=-2. \end{cases}$$

Source of difficulty

Example

Take type A. Then:

$$T_2(E_1) = -q^{-1}[E_1, E_2]_q, \quad T_2(F_1) = [F_1, F_2]_q, \quad T_2(K_1^{-1}) = (K_1 K_2)^{-1}.$$

Hence

$$T_2(B_1) = T_2(F_1 - q^{-2}\mathbb{K}_1 E_1 K_1^{-1}) = [F_1, F_2]_q + q^{-3}\mathbb{K}_1\mathbb{K}_2[E_1, E_2]_q (K_1 K_2)^{-1}.$$

On the other hand

$$\mathbf{T}_2(B_1) = [B_1, B_2]_q = T_2(B_1) - q^{-2}\mathbb{K}_1[E_1 K_1^{-1}, F_2]_q.$$

Example: q-Onsager algebra

- Originally defined by P. Terwilliger (coming from combinatorics, tridiagonal pairs).
- $\mathcal{O}_q$ is a coideal subalgebra of $U_q(\widehat{\mathfrak{sl}}_2)$.
- Generated by

$$B_i = F_i - c_i E_i K_i^{-1} + s_i K_i^{-1} \quad (i = 0, 1).$$

- Relations:

$$B_1^3 B_0 - [3] B_1^2 B_0 B_1 + [3] B_1 B_0 B_1^2 - B_0 B_1^3 = q[2]^2 c_1 (B_0 B_1 - B_1 B_0)$$
$$B_1^3 B_0 - [3] B_1^2 B_0 B_1 + [3] B_1 B_0 B_1^2 - B_0 B_1^3 = q[2]^2 c_1 (B_0 B_1 - B_1 B_0)$$

- The parameters s_i do not change the algebra structure but affect the coideal structure.

q-Onsager algebra in the current presentation

The q-Onsager algebra is isomorphic to the algebra generated by H_m and A_r, where $m \geq 1$, $r \in \mathbb{Z}$, subject to the following relations:

$$\begin{aligned}
[H_m, H_n] &= 0, \\
[H_m, A_r] &= \frac{[2m]}{m} A_{r+m} - \frac{[2m]}{m} A_{r-m} C^m, \\
[A_r, A_{s+1}]_{q^{-2}} - q^{-2}[A_{r+1}, A_s]_{q^2} &= c_1 C^r \Theta_{s-r+1} - q^{-2} c_1 C^{r+1} \Theta_{s-r-1} \\
&\quad + c_1 C^s \Theta_{r-s+1} - q^{-2} c_1 C^{s+1} \Theta_{r-s-1},
\end{aligned}$$

where $m, n \geq 1$; $r, s \in \mathbb{Z}$; $C = q^4 c_0 c_1$ and

$$1 + \sum_{m=1}^{\infty} (q - q^{-1}) \Theta_m z^m = \exp\left((q - q^{-1}) \sum_{m=1}^{\infty} H_m z^m \right).$$

Representations

Simple modules over the q-Onsager algebra have been classified by Ito and Terwilliger. They have also proposed the notion of a Drinfeld polynomial for such modules. It is unclear how their approach relates to the new Drinfeld presentation.

q-characters for QSP

Questions

Assumption: From now on assume that we are in the split case (Satake = Dynkin diagram), type ABCD.

- What is the relationship between the Drinfeld generators of $\mathcal{B}_{c,s}$ and $U_q(\widehat{\mathfrak{g}})$?

$$\begin{array}{ccc} \mathcal{B}_c & \xhookrightarrow{B_i \mapsto F_i - c_i E_i K_i^{-1}} & U_q(\widehat{\mathfrak{g}}) \\ B_i \rightsquigarrow \mathbf{T}^r_{\omega_i}(B_i) \Big\| & & \Big\| x_i^{\pm} \rightsquigarrow T^r_{\omega_i}(x_i^{\pm}) \\ \mathcal{B}_c^{Dr} & \xhookrightarrow{?} & U_q(\widehat{\mathfrak{g}})^{Dr} \end{array}$$

- How do the Drinfeld generators of $\mathcal{B}_{c,s}$ behave under coproduct?

$$\begin{array}{ccc} \mathcal{B}_c & \xrightarrow{B_i \mapsto 1 \otimes B_i + B_i \otimes K_i^{-1}} & \mathcal{B}_c \otimes U_q(\widehat{\mathfrak{g}}) \\ B_i \rightsquigarrow \mathbf{T}^r_{\omega_i}(B_i) \Big\| & & \Big\| \\ \mathcal{B}_c^{Dr} & \xrightarrow{?} & \mathcal{B}_c^{Dr} \otimes U_q(\widehat{\mathfrak{g}})^{Dr} \end{array}$$

Main results

Factorization theorem (P., Li–P.)

The generating series $\grave{\boldsymbol{\Theta}}_i(z)$ have the following *factorization property*:

$$\grave{\boldsymbol{\Theta}}_i(z) \equiv \phi_i^-(z^{-1})\phi_i^+(Cz) \qquad \mathrm{mod}\ U_q(\widehat{\mathfrak{g}})_+[\![z]\!].$$

Coproduct theorem (P., Li–P.)

The generating series $\grave{\boldsymbol{\Theta}}_i(z)$ are *approximately group-like*:

$$\Delta(\grave{\boldsymbol{\Theta}}_i(z)) \equiv \grave{\boldsymbol{\Theta}}_i(z) \otimes \grave{\boldsymbol{\Theta}}_i(z) \qquad \mathrm{mod}\ U_q(\widehat{\mathfrak{g}}) \otimes U_q(\widehat{\mathfrak{g}})_+[\![z]\!].$$

Applications to q-characters

Idea

Boundary q-characters count the dimensions of $\grave{\mathbf{\Theta}}_i(z)$-eigenspaces.

$$\mathcal{K}^0 = \frac{1 - q_i^{-2} C z^2}{1 - C z^2} \exp\left(-(q - q^{-1}) \sum_{i \in \mathbb{I}_{fin}} \sum_{k>0} \frac{k}{[k]_{q_i}} H_{i,k} \otimes \tilde{h}_{i,-k} z^k \right) \in \mathcal{B}_{\mathbf{c}} \otimes U_q(\widetilde{\mathfrak{h}})[\![z]\!].$$

We define the *boundary q-character map* to be

$$\chi_q^\imath \colon \operatorname{Rep} \mathcal{B}_{\mathbf{c}} \to U_q(\widetilde{\mathfrak{h}})[\![z]\!], \qquad [V] \mapsto \mathrm{Tr}_V(\mathcal{K}^0 \circ (\pi_V \otimes 1)).$$

Applications to q-characters

Consider $U_q(\widetilde{\mathfrak{h}})[\![z]\!]$ as a $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^\times}$-module via the ring homomorphism

$$\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^\times} \to \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^\times} \hookrightarrow U_q(\widetilde{\mathfrak{h}})[\![z]\!], \qquad Y_{i,a} \mapsto Y_{i,C a} Y_{i,a^{-1}}^{-1}.$$

Compatibility theorem (P., Li–P.)

The diagram commutes:

$$\begin{array}{ccc} [\mathrm{Rep}\, U_q(\widehat{\mathfrak{g}})] & \xrightarrow{\chi_q} & \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^\times} \\ \curvearrowright & & \curvearrowright \\ \mathrm{Rep}\, \mathcal{B}_{\mathbf{c}} & \xrightarrow{\chi_q^\imath} & U_q(\widetilde{\mathfrak{h}})[\![z]\!], \end{array} \quad \text{i.e.,}$$

$$\chi_q^\imath(V \otimes W) = \chi_q^\imath(V) \cdot \chi_q(W).$$

Upshot: we can define q-characters for QSP using the new Drinfeld presentation, satisfying the elementary condition of *compatibility* with the usual q-character map. Yet in other words: our map is a homomorphism of modules.

Idea of proof

- Step 0: get the intuition for what the answers should be by looking at restrictions (to the q-Onsager algebra) of evaluation modules of $U_q(\widehat{\mathfrak{sl}}_2)$. This essentially reduces computations to $U_q(\mathfrak{sl}_2)$.
- Step 1: prove the factorization and coproduct theorems in rank 1.
- Step 2: generalize to higher ranks.
  - Find reduced decompositions for the fundamental weights ω_i, e.g.,

    $$\omega_i = \pi^i[n-i+1, n] \cdots [2, i+1][1, i]$$

    in type $\mathsf{A}_n^{(1)}$.
  - Write $\mathbf{T}_{\omega_i'}(B_i)$ as an explicit (recursively defined) polynomial in E_j, F_j, K_j.
  - Show that

    $$\eta(\mathbf{T}_{\omega_i'}(B_i)) \equiv T_{\omega_i'}(\eta(B_i)) \qquad \text{mod } U_q(\widehat{\mathfrak{g}})_{d_i \geq 1, +}.$$

    Essentially need to show that terms with E_0 vanish (since $\deg E_0 = -\theta$).

Idea of proof

- Step 2: generalize to higher ranks.
  - Deduce that

$$\begin{array}{ccc}
\mathcal{O}_q & \xhookrightarrow{\ \eta\ } & U_q(\widehat{\mathfrak{sl}}_2) \\
\wr\downarrow \iota_i & & \wr\downarrow \iota_i \\
\mathcal{O}_{q,[i]} & & U_q(\widehat{\mathfrak{sl}}_2)_{[i]} \\
\downarrow & & \downarrow \\
\mathcal{B}_{\mathbf{c}} & \xhookrightarrow{\ \eta\ } & U_q(\widehat{\mathfrak{g}})
\end{array}$$

    commutes modulo $U_q(\widehat{\mathfrak{g}})_{\neq i,+}$.
  - Using the Drinfeld presentation and the explicit polynomials, show that

$$\eta(A_{i,r}) \equiv \iota_i \eta(A_r) \qquad \text{mod } U_q(\widehat{\mathfrak{g}})_{d_i \geq 1,+},$$

    for all $r \in \mathbb{Z}$ (extending from $r = 0, -1$).

Work in progress

Progress:

- The representation theory of finite type AI QSP is well understood. We are able to compute q-characters of evaluation modules using a Gelfand–Tsetlin approach.
- In type AIII, there appears to be a geometric realization of q-characters a la Nakajima, using the K-theory of isotropic flag varieties. They are also likely related to the characters of orientifold KLR algebras via Schur–Weyl duality.

Problems:

- How to realize QSP q-characters in terms of the universal K-matrix? This would allow us to mimic many methods used by Frenkel, Mukhin, Reshetikhin and others.
- How to connect QSP q-characters to representation theory? Can they be used to obtain a classification in higher ranks?

Happy Birthday Paul!!!