Codes and Designs in Classical Association Schemes

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Coding theory

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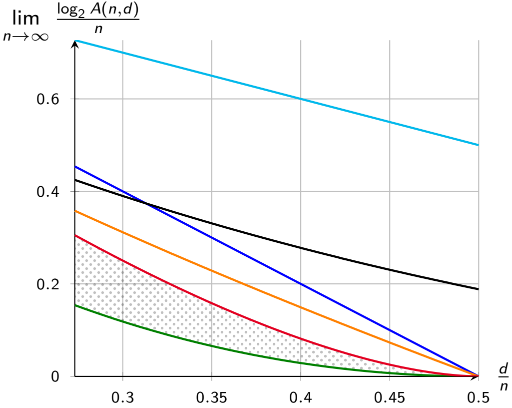

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MRRW bound (1977)

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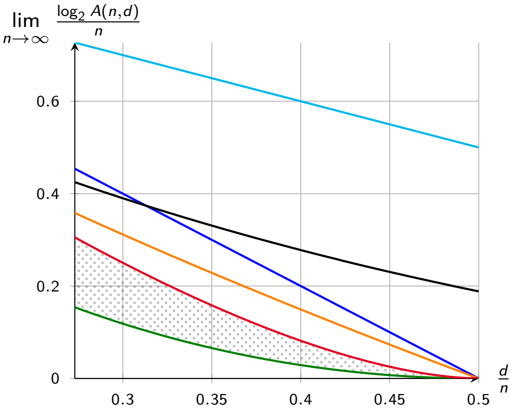

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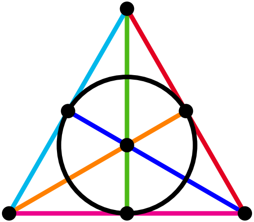
All these upper bounds come from a linear program whose optimal solution is unknown.

Design theory



2-$(7, 3, 1)$ design

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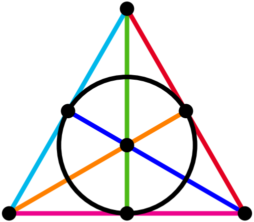

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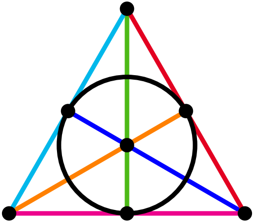

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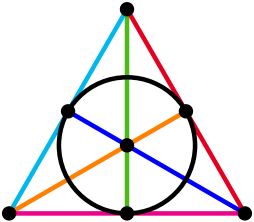

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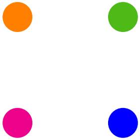
Teirlinck 1987: A t-design of $(t+1)$-sets exists for all t.

Extremal combinatorics



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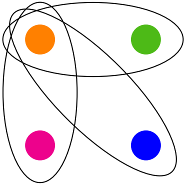

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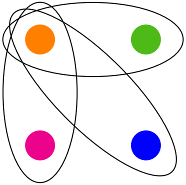

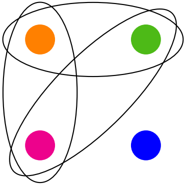

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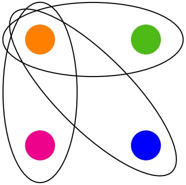

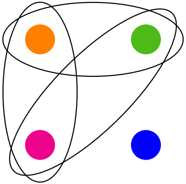

How large can a t-intersecting family of n-subsets of a v-set be?

Erdős-Ko-Rado 1961

For v sufficiently large compared to t, the size of a t-intersecting family of n-subsets of a v-set is at most $\binom{v-t}{n-t}$.

Metric association schemes

Take a finite metric space (X, ρ) and the $X \times X$ matrices A_i with

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } \rho(x,y) = i \\ 0 & \text{otherwise.} \end{cases}$$

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Then $(X, (A_i))$ is a metric association scheme with n classes if the matrices $A_0, A_1, \ldots, A_n$ generate a commutative matrix algebra over $\mathbb{R}$ with “nice” properties.

Polynomial structures

There is a second basis $E_0, E_1, \ldots, E_n$.

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A metric association scheme is always P-polynomial:

$$P_i(k) = P_i(x_k)$$

for some $P_i \in \mathbb{R}[z]$ of degree i and some $x_k \in \mathbb{R}$.

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This imposes an ordering on $E_0, E_1, \dots, E_n$.

Classical association schemes

Hamming scheme

$X = \{0, 1\}^n$

Johnson scheme

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Conjecture (Bannai 19??)

(P and Q)-polynomial schemes with sufficiently many classes are either classical or "relatives" of the classical ones.

Polar spaces

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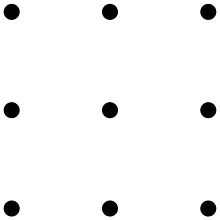

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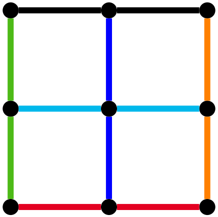

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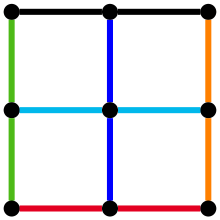

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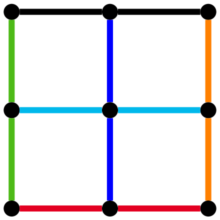

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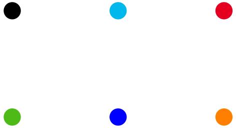

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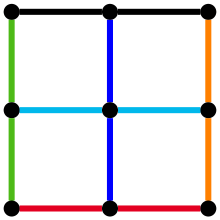

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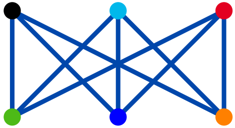

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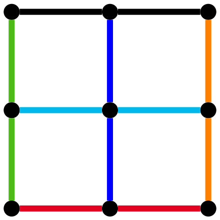

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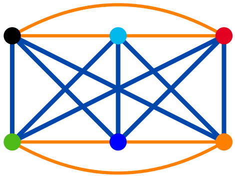

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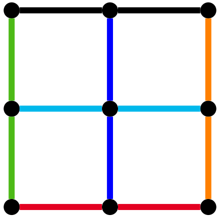

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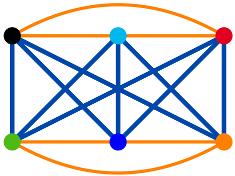
Polar space scheme

X is the set of n-spaces in a polar space of rank n and $\rho(x, y) = n - \dim(x \cap y)$.



The six families of polar spaces

Up to isomorphism, there are six polar spaces of rank n.

form	name	type
Hermitian	Hermitian	${}^2A_{2n-1}$
Hermitian	Hermitian	${}^2A_{2n}$
alternating	symplectic	C_n
quadratic	hyperbolic	D_n
quadratic	parabolic	B_n
quadratic	elliptic	${}^2D_{n+1}$

Codes and designs

Inner distribution $(a_0, a_1, \ldots, a_n)^T$ of a subset Y of X:

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t-design

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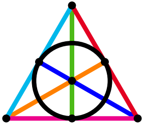
Examples of t-designs

The t-designs in the

- Johnson scheme are combinatorial t-designs



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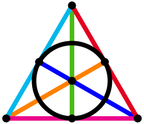

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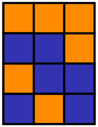

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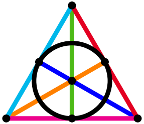

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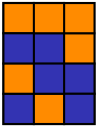

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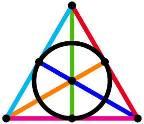

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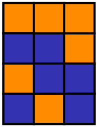

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The magic of linear programming

$$a_0 = 1, \qquad a_i \geq 0, \qquad \sum_{i=0}^{n} a_i = |Y|, \qquad a'_k = \sum_{i=0}^{n} Q_k(i) a_i$$

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Linear program (Delsarte 1973)

Find $a_0, a_1, \ldots, a_n$ that maximize $a_0 + a_1 + \cdots + a_n$ subject to the above constraints.

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Linear program (Delsarte 1973)

Find $a_0, a_1, \ldots, a_n$ that maximize $a_0 + a_1 + \cdots + a_n$ subject to the above constraints.

The smallest bound that can be obtained in this way is called the linear programming (LP) optimum, denoted by $LP(d)$.

Q-numbers

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Polar space schemes	q-Krawtchouk polynomials
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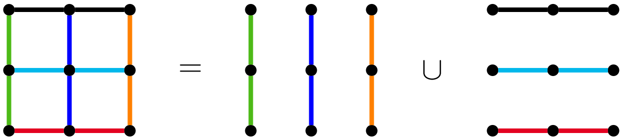
Bipartite halves



The polar space D_2

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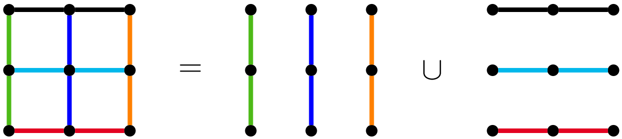

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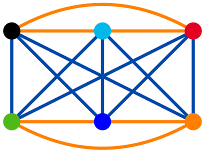

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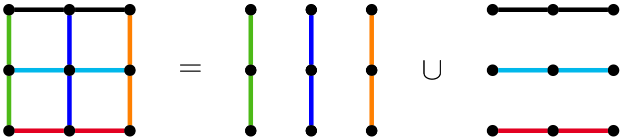

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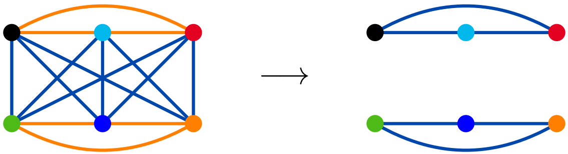

Each $\frac{1}{2}D_m$ gives a (P and Q)-polynomial association scheme with $n = \lfloor \frac{m}{2} \rfloor$ classes.

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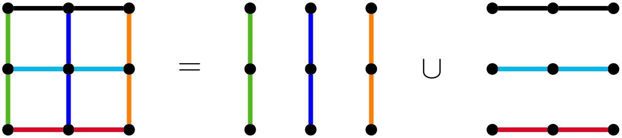

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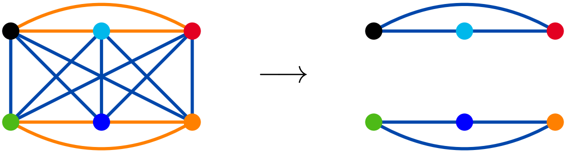

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The numbers $Q_k(i)$ come from q-Hahn polynomials.

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The polynomials can be written in a unified way in terms of q-hypergeometric series of type ${}_3\phi_2$ with parameters b and c.

The linear programming bound

Theorem (Schmidt-W. 2023)

Ordinary q-analogs $J_q(n, v)$, $\frac{1}{2}D_m$, ${}^2A_{2n-1}$

$$\mathsf{LP}(d) = |X| \prod_{\ell=0}^{d-2} \frac{qb^\ell - 1}{qcb^{n+\ell} - 1}$$

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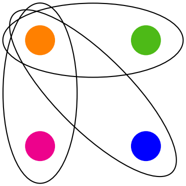
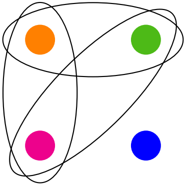
This solves the coding problem in nearly all classical association schemes asymptotically, except for the Hamming and Johnson schemes.

Intersecting sets

A family Y consisting of n-subsets of a v-set is t-intersecting if $|x \cap y| \geq t$

Intersecting sets





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How large can a t-intersecting set be?

Erdős-Ko-Rado-type results

Corollary (Schmidt-W. 2025+)

A t-intersecting set Y in an affine or ordinary q-analog scheme satisfies

$$|Y| \leq \frac{|X|}{\mathrm{LP}(n-t+1)},$$

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In particular, we obtain new bounds on t-intersecting sets in the polar space

- ${}^2A_{2n-1}$
- B_n and C_n for $n-t$ even
- D_n for $n-t$ odd.

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Fazeli-Lovett-Vardy 2014

A t-(v, n, λ) design over $\mathbb{F}_q$ exists, provided that v is large enough and $n > 12(t + 1)$.

Designs in polar spaces

A t-(v, n, λ) design in a polar space $\mathcal{P}$ of rank v is a collection Y of n-spaces in $\mathcal{P}$ such that each t-space of $\mathcal{P}$ lies in exactly λ members of Y.

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Theorem (W. 2025)

Let $\mathcal{P}$ be a polar space of rank v. For all positive integers t and n with $n > 10.5\,t$ and for v large enough with $v > n^2$, there exists a t-(v, n, λ) design in $\mathcal{P}$ whose size is at most q^{21vt}.