Hau-Wen Huang
Joint work with Dr. Chin-Yen Lee

A skew group ring of $\mathbb{Z}/2\mathbb{Z}$ over $U(\mathfrak{sl}_2)$, Leonard triples and odd graphs

Hau-Wen Huang
Joint work with Dr. Chin-Yen Lee

National Central University, Taiwan

June 24, 2025

Combinatorics around the q-Onsager algebra
Dedicated to Professor Paul Terwilliger on his 70th birthday

Outline

Hau-Wen Huang
Joint work with Dr. Chin-Yen Lee

Section 1: The Skew group rings

The Skew group rings

- Let N be a group.
- Let G be a group.
- Let $\varphi : G \to \mathrm{Aut}(N)$ be a group homomorphism.
- As a set, the **outer semidirect product** $N \rtimes_\varphi G$ of N and G with respect to φ is $N \times G$.
  - The group operation is defined by

$$(n_1, g_1) \cdot (n_2, g_2) = (n_1 \cdot \varphi(g_1)(n_2), g_1 g_2)$$

for all $n_1, n_2 \in N$ and all $g_1, g_2 \in G$.

The Skew group rings

- Let R be a ring with 1.
- Let G be a finite group.
- Let $\varphi : G \to \mathrm{Aut}(R)$ be a group homomorphism.
- As a set, the **skew group ring** $R \rtimes_\varphi G$ of G over R induced by φ consists of all formal sums

$$\sum_{g \in G} a_g g \qquad \text{where } a_g \in R \text{ for all } g \in G.$$

  - The addition operation is component-wise.
  - The multiplication is distributively defined by

$$a_1 g_1 \cdot a_2 g_2 = (a_1 \cdot \varphi(g_1)(a_2))\, g_1 g_2$$

    for all $a_1, a_2 \in R$ and all $g_1, g_2 \in G$.

The Skew group rings

- Every element $a \in R$ can be viewed as the element $a \cdot 1$ of $R \rtimes_\varphi G$.
- Every element $g \in G$ can be viewed as the element $1 \cdot g$ of $R \rtimes_\varphi G$.
- Suppose that R is a unital associative algebra over a field $\mathbb{F}$. Then $R \rtimes_\varphi G$ is a unital associative algebra over $\mathbb{F}$.
  - ◇ The scalar multiplication is defined by

$$c\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} (ca_g)g$$

where $c \in \mathbb{F}$ and $a_g \in R$ for all $g \in G$.

Section 2: Hopf algebras

Hopf algebras

- Let A be a unital associative algebra over $\mathbb{F}$.
- Let $m : A \otimes A \to A$ be the multiplication map given by

$$m(a \otimes b) = ab \qquad \text{for all } a, b \in A.$$

- Let $\iota : \mathbb{F} \to A$ be the unit map given by

$$\iota(c) = c \cdot 1 \qquad \text{for all } c \in \mathbb{F}.$$

Hopf algebras

The algebra A is called a **Hopf algebra** if the following hold:

- There is a unital algebra homomorphism $\Delta : A \to A \otimes A$ such that

  $$(1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta.$$

  The map Δ is called the comultiplication of A.
- There is a unital algebra homomorphism $\varepsilon : A \to \mathbb{F}$ such that

  $$m \circ (1 \otimes (\iota \circ \varepsilon)) \circ \Delta = 1,$$
  $$m \circ ((\iota \circ \varepsilon) \otimes 1) \circ \Delta = 1.$$

  The map ε is called the counit of A.
- There is a linear map $S : A \to A$ such that

  $$m \circ (1 \otimes S) \circ \Delta = \iota \circ \varepsilon,$$
  $$m \circ (S \otimes 1) \circ \Delta = \iota \circ \varepsilon.$$

  The map S is called the antipode of A.

Hau-Wen Huang
Joint work with Dr. Chin-Yen Lee

Section 3: Lie algebras, Hopf algebras and skew group rings

Lie algebras, Hopf algebras and skew group rings

- Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$.
- Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$.
- Each element of $\mathfrak{g}$ can be viewed as an element of $U(\mathfrak{g})$.
- The algebra $U(\mathfrak{g})$ is generated by all $u \in \mathfrak{g}$.
- $U(\mathfrak{g})$ is a Hopf algebra.
  - ◇ The comultiplication of $U(\mathfrak{g})$ is a unital algebra homomorphism $\Delta_1 : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ given by

$$\Delta_1(u) = u \otimes 1 + 1 \otimes u \qquad \text{for all } u \in \mathfrak{g}.$$

  - ◇ The counit of $U(\mathfrak{g})$ is a unital algebra homomorphism $\varepsilon_1 : U(\mathfrak{g}) \to \mathbb{C}$ given by

$$\varepsilon_1(u) = 0 \qquad \text{for all } u \in \mathfrak{g}.$$

  - ◇ The antipode of $U(\mathfrak{g})$ is a unital algebra anti-automorphism $S_1 : U(\mathfrak{g}) \to U(\mathfrak{g})$ given by

$$S_1(u) = -u \qquad \text{for all } u \in \mathfrak{g}.$$

Lie algebras, Hopf algebras and skew group rings

- Let G be a group.
- Let $\mathbb{C}[G]$ be the group algebra of G over $\mathbb{C}$.
- $\mathbb{C}[G]$ is a Hopf algebra.
  - The comultiplication of $\mathbb{C}[G]$ is a unital algebra homomorphism $\Delta_2 : \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G]$ given by
  $$\Delta_2(g) = g \otimes g \qquad \text{for all } g \in G.$$
  - The counit of $\mathbb{C}[G]$ is a unital algebra homomorphism $\varepsilon_2 : \mathbb{C}[G] \to \mathbb{C}$ given by
  $$\varepsilon_2(g) = 1 \qquad \text{for all } g \in G.$$
  - The antipode of $\mathbb{C}[G]$ is a unital algebra anti-automorphism $S_2 : \mathbb{C}[G] \to \mathbb{C}[G]$ given by
  $$S_2(g) = g^{-1} \qquad \text{for all } g \in G.$$

- Assume that G is a finite subgroup of $\mathrm{Aut}(\mathfrak{g})$.
- By the universal property of $U(\mathfrak{g})$ every $g \in G$ can be lifted to an algebra automorphism of $U(\mathfrak{g})$. The group G can be viewed as a finite subgroup of $\mathrm{Aut}(U(\mathfrak{g}))$.
- This induces a skew group ring $U(\mathfrak{g}) \rtimes G$ of G over $U(\mathfrak{g})$.
- We simply write $U(\mathfrak{g})_G = U(\mathfrak{g}) \rtimes G$.

Lie algebras, Hopf algebras and skew group rings

$U(\mathfrak{g})_G$ has the following Hopf algebra structure:

- The comultiplication of $U(\mathfrak{g})_G$ is a unital algebra homomorphism $\Delta : U(\mathfrak{g})_G \to U(\mathfrak{g})_G \otimes U(\mathfrak{g})_G$ defined by

$$\Delta(a) = \Delta_1(a) \quad \text{for all } a \in U(\mathfrak{g}),$$
$$\Delta(g) = \Delta_2(g) \quad \text{for all } g \in G.$$

- The counit of $U(\mathfrak{g})_G$ is a unital algebra homomorphism $\varepsilon : U(\mathfrak{g})_G \to \mathbb{C}$ defined by

$$\varepsilon(a) = \varepsilon_1(a) \quad \text{for all } a \in U(\mathfrak{g}),$$
$$\varepsilon(g) = \varepsilon_2(g) \quad \text{for all } g \in G.$$

- The antipode of $U(\mathfrak{g})_G$ is a unital algebra anti-automorphism $S : U(\mathfrak{g})_G \to U(\mathfrak{g})_G$ defined by

$$S(a) = S_1(a) \quad \text{for all } a \in U(\mathfrak{g}),$$
$$S(g) = S_2(g) \quad \text{for all } g \in G.$$

Lie algebras, Hopf algebras and skew group rings

- The Lie algebra $\mathfrak{sl}_2$ over $\mathbb{C}$ consists of all 2×2 matrices with entries in $\mathbb{C}$ with trace zero. The Lie bracket $[\,,]$ is defined by $[X, Y] = XY - YX$.
- The Lie algebra $\mathfrak{sl}_2$ has the basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- There is a Lie algebra automorphism $\rho : \mathfrak{sl}_2 \to \mathfrak{sl}_2$ that sends

$$E \mapsto F, \qquad F \mapsto E, \qquad H \mapsto -H.$$

- Note that $\rho^2 = 1$.

Lie algebras, Hopf algebras and skew group rings

- Let $G = \langle \rho \rangle \subseteq \mathrm{Aut}(\mathfrak{sl}_2)$ and G is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.
- $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$ is a Hopf algebra.
- From now on, let Δ denote the comultiplication of $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$.

Section 4: The finite-dimensional $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules

The finite-dimensional $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules

- For each integer $n \geq 0$ there exists an $(n+1)$-dimensional $U(\mathfrak{sl}_2)$-module L_n that has a basis $\{v_i\}_{i=0}^n$ such that

$$
\begin{aligned}
Ev_i &= iv_{i-1} \quad (1 \leq i \leq n), \qquad Ev_0 = 0, \\
Fv_i &= (n-i)v_{i+1} \quad (0 \leq i \leq n-1), \qquad Fv_n = 0, \\
Hv_i &= (n-2i)v_i \quad (0 \leq i \leq n).
\end{aligned}
$$

- The $U(\mathfrak{sl}_2)$-module L_n is irreducible.
- Every $(n+1)$-dimensional irreducible $U(\mathfrak{sl}_2)$-module is isomorphic to L_n.
- Any finite-dimensional $U(\mathfrak{sl}_2)$-module is completely reducible.

The finite-dimensional $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules

Theorem 6.1

The algebra $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$ has a presentation generated by E, F, H, ρ subject to the relations

$$\begin{aligned} [H, E] &= 2E, \\ [H, F] &= -2F, \\ [E, F] &= H, \\ \rho H &= -H\rho, \\ \rho E &= F\rho, \\ \rho^2 &= 1. \end{aligned}$$

The finite-dimensional $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules

- The $U(\mathfrak{sl}_2)$-module L_n can be extended to a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module by setting

$$\rho v_i = v_{n-i} \qquad (0 \leq i \leq n).$$

We denote this $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module by L_n^+.

- The $U(\mathfrak{sl}_2)$-module L_n can be extended to a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module by setting

$$\rho v_i = -v_{n-i} \qquad (0 \leq i \leq n).$$

We denote this $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module by L_n^-.

The finite-dimensional $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules

- The $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules L_n^+ and L_n^- are irreducible.
- The $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules L_n^+ and L_n^- are not isomorphic.
- Every $(n+1)$-dimensional irreducible $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module is isomorphic to either L_n^+ or L_n^-.
- Any finite-dimensional $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module is completely reducible.

Section 5: The universal Bannai–Ito algebra, the skew group ring $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$ and Leonard triples

The universal Bannai–Ito algebra, the skew group ring $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$ and Leonard triples

- In an algebra the curl bracket

$$\{X, Y\} = XY + YX.$$

- The **universal Bannai–Ito algebra** $\mathfrak{BI}$ is a unital associative algebra over $\mathbb{C}$ generated by X, Y, Z and the relations assert that the three elements

$$\kappa = \{X, Y\} - Z,$$
$$\lambda = \{Y, Z\} - X,$$
$$\mu = \{Z, X\} - Y$$

are central in $\mathfrak{BI}$.

- Let V be a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}} \otimes U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module.
- For any scalar $\theta \in \mathbb{C}$ we define

$$V(\theta) = \{v \in V \mid \Delta(H)v = \theta v\}.$$

The universal Bannai–Ito algebra, the skew group ring $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$ and Leonard triples

Theorem 7.1

Suppose that V is a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}} \otimes U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module. Then $V(1)$ is a $\mathfrak{BI}$-module given by

$$X = \Delta(E\rho),$$
$$Y = \frac{H \otimes 1 - 1 \otimes H}{2},$$
$$Z = (E \otimes 1 - 1 \otimes E)\Delta(\rho).$$

The universal Bannai–Ito algebra, the skew group ring $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$ and Leonard triples

Definition 7.2

Let V denote a nonzero finite-dimensional vector space over a field $\mathbb{F}$. A **Leonard pair** on V consists of an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following conditions:

1. There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal.
2. There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrix representing A is irreducible tridiagonal.

Prof. Terwilliger introduced Leonard pairs and established the theory of Leonard pairs.

Definition 7.3

Let V denote a nonzero finite-dimensional vector space over a field $\mathbb{F}$. A **Leonard triple** on V consists of an ordered triple of linear transformations $A : V \to V, A^* : V \to V, A^\varepsilon : V \to V$ that satisfy the following conditions:

1. There exists a basis for V with respect to which the matrix representing A is diagonal and the matrices representing A^* and A^ε are irreducible tridiagonal.
2. There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrices representing A^ε and A are irreducible tridiagonal.
3. There exists a basis for V with respect to which the matrix representing A^ε is diagonal and the matrices representing A and A^* are irreducible tridiagonal.

The universal Bannai–Ito algebra, the skew group ring $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$ and Leonard triples

Theorem 7.4

Suppose that V is a finite-dimensional irreducible $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}} \otimes U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module. The following conditions are equivalent:

1. *$V(1)$ is nonzero.*
2. *The $\mathfrak{BI}$-module $V(1)$ is irreducible.*
3. *X, Y, Z act on the $\mathfrak{BI}$-module $V(1)$ as a Leonard triple.*

Section 6: Applications to the odd graphs

- Let Ω be a finite set.
- The notation $\mathbb{C}^{\Omega}$ stands for a vector space over $\mathbb{C}$ that has a basis Ω.
- Let 2^{Ω} denote the power set of Ω.
- The notation $\lessdot$ represents the covering relation of the poset $(2^{\Omega}, \subseteq)$.

Applications to the odd graphs

Theorem 8.1

For any finite set Ω, there exists a unique $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module $\mathbb{C}^{2^\Omega}$ such that

$$Ex = \sum_{y \lessdot x} y \qquad \text{for all } x \in 2^\Omega,$$

$$Fx = \sum_{x \lessdot y} y \qquad \text{for all } x \in 2^\Omega,$$

$$Hx = (|\Omega| - 2|x|)x \qquad \text{for all } x \in 2^\Omega,$$

$$\rho x = \Omega \setminus x \qquad \text{for all } x \in 2^\Omega.$$

Applications to the odd graphs

- Fix an element $x_0 \in 2^\Omega$.
- $\mathbb{C}^{2^{\Omega \setminus x_0}} \otimes \mathbb{C}^{2^{x_0}}$ is a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}} \otimes U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module.
- The linear map $\mathbb{C}^{2^\Omega} \to \mathbb{C}^{2^{\Omega \setminus x_0}} \otimes \mathbb{C}^{2^{x_0}}$ given by

$$x \mapsto (x \setminus x_0) \otimes (x \cap x_0) \qquad \text{for all } x \in 2^\Omega$$

  is a linear isomorphism.
- This induces a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}} \otimes U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module structure on $\mathbb{C}^{2^\Omega}$. We denote this $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}} \otimes U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module by

$$\mathbb{C}^{2^\Omega}(x_0).$$

- The space $\mathbb{C}^{2^\Omega}(x_0)(1)$ is a $\mathfrak{BI}$-module.

Applications to the odd graphs

- Now, fix an integer $d \geq 1$ and let Ω denote a $(2d+1)$-element set.
- For any integer k with $0 \leq k \leq |\Omega|$, the notation $\binom{\Omega}{k}$ stands for the set of all k-element subsets of Ω.
- The **odd graph** O_{d+1} is a finite simple connected graph whose vertex set is $\binom{\Omega}{d}$ and two vertices x, y are adjacent if $x \cap y = \emptyset$.

Applications to the odd graphs

- The adjacency operator **A** of O_{d+1} is the linear map $\mathbb{C}^{\binom{\Omega}{d}} \to \mathbb{C}^{\binom{\Omega}{d}}$ given by

$$\mathbf{A}x = \sum_{x \cap y = \emptyset} y \qquad \text{for all } x \in \binom{\Omega}{d}.$$

- Let $x_0 \in \binom{\Omega}{d}$ be given. The dual adjacency operator $\mathbf{A}^*(x_0)$ of O_{d+1} with respect to x_0 is a linear map $\mathbb{C}^{\binom{\Omega}{d}} \to \mathbb{C}^{\binom{\Omega}{d}}$ given by

$$\mathbf{A}^*(x_0)x = \left(2d - \tfrac{2d+1}{d+1}(|x_0 \setminus x| + |x \setminus x_0|)\right)x \quad \text{for all } x \in \binom{\Omega}{d}.$$

- The **Terwilliger algebra** $\mathbf{T}(x_0)$ of O_{d+1} with respect to x_0 is the subalgebra of $\mathrm{End}(\mathbb{C}^{\binom{\Omega}{d}})$ generated by **A** and $\mathbf{A}^*(x_0)$.

Applications to the odd graphs

Theorem 8.2

For any $x_0 \in \binom{\Omega}{d}$ the following statements hold:

1. *There exists a unique algebra homomorphism $\mathfrak{BI} \to \mathbf{T}(x_0)$ given by*

$$\begin{aligned} X &\mapsto \mathbf{A}, \\ Y &\mapsto \tfrac{d+1}{2d+1}\mathbf{A}^*(x_0) + \tfrac{1}{2(2d+1)}, \\ Z &\mapsto \tfrac{d+1}{2d+1}\{\mathbf{A}, \mathbf{A}^*(x_0)\} + \tfrac{1}{2d+1}\mathbf{A}. \end{aligned}$$

   Moreover the algebra homomorphism is surjective.

2. *X, Y, Z act on each irreducible $\mathbf{T}(x_0)$-module as a Leonard triple.*

Thank you for your attention