Elliptic hypergeometric functions and the Ruijsenaars model

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What is the point of this talk?

- Philosophy:
  Everything should be made elliptic!
- Example:
  Elliptic hypergeometric integrals
  and the Ruijsenaars model.

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Philosophy

Hierarchy:
Rational – trigonometric (q) – elliptic (p and q)

- Algebra
  Lie groups – Quantum groups – Elliptic quantum groups
- Special functions
  Jacobi polynomials – Askey–Wilson polynomials – Spiridonov–Zhedanov functions
- Physics
- Combinatorics (should be studied more!)

Elliptic curves

Realize elliptic curve (complex torus) multiplicatively

$$(\mathbb{C} \setminus 0)/(x \sim px),$$

where $|p| < 1$.

Equivalently, if $x = e^{2\pi \mathrm{i} z}$,

$$z \in \mathbb{C}/(z = z + 1 = z + \tau),$$

where $p = e^{2\pi \mathrm{i} \tau}$.

Elliptic functions

An elliptic function is a meromorphic function on $\mathbb{C} \setminus 0$ such that

$$f(x) = f(px).$$

The (multiplicative) theta function is defined by

$$\theta(x;p) = \prod_{j=0}^{\infty}(1-xp^j)(1-p^{j+1}/x).$$

It satisfies

$$\theta(px;p) = -x^{-1}\theta(x;p).$$

Any elliptic function can be factored as

$$f(x) = C\frac{\theta(a_1x;p)\cdots\theta(a_nx;p)}{\theta(b_1x;p)\cdots\theta(b_nx;p)},$$

where

$$a_1\cdots a_n = b_1\cdots b_n.$$

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Elliptic hypergeometric integrals

A one-variable elliptic hypergeometric integral has the form

$$\oint g(x)\,\frac{dx}{x},$$

where $f(x) = g(qx)/g(x)$ is elliptic. Equivalently,

$$g(x)g(pqx) = g(px)g(qx).$$

Factoring f as above, we are reduced to solving

$$\frac{g(qx)}{g(x)} = \theta(x;p).$$

This is solved by Ruijsenaars's elliptic gamma function

$$g(x) = \Gamma(x;p,q) = \prod_{j,k=0}^{\infty} \frac{1-p^{j+1}q^{k+1}/x}{1-p^j q^k x}.$$

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Elliptic hypergeometric integrals

Elliptic hypergeometric integrals have the form (Spiridonov, 2001)

$$\oint \frac{\Gamma(a_1 x; p, q) \cdots \Gamma(a_n x; p, q)}{\Gamma(b_1 x; p, q) \cdots \Gamma(b_n x; p, q)} \frac{dx}{x},$$

where

$$\Gamma(x; p, q) = \prod_{j,k=0}^{\infty} \frac{1 - p^{j+1} q^{k+1}/x}{1 - p^j q^k x}.$$

We will encounter multivariable versions of such integrals.

Ruijsenaars operators

Ruijsenaars operators (depending on p, q, t)

$$(D^{(k)}f)(x_1,\dots,x_n) = \sum_{\substack{I\subseteq\{1,\dots,n\},\\ |I|=k}} \prod_{i\in I,\, j\in I^c} \frac{\theta(tx_i/x_j;p)}{\theta(x_i/x_j;p)} \cdot f(x_1,\dots,\underbrace{qx_i}_{i\in I},\dots,x_n).$$

Commutativity (Ruijsenaars 1987)

$$[D^{(k)},D^{(l)}]=0, \qquad k,l=0,1,\dots,n.$$

Define integrable system of relativistic quantum particles.
Generalizes various Calogero–Moser–Sutherland-type models.

When $p=0$, $\theta(x;0)=1-x$, $D^{(k)}$ are the Macdonald operators, having Macdonald polynomials as joint eigenfunctions.

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Q-operators

We will define a commuting family of integral operators Q_c, which also commute with all the operators $D^{(k)}$.

Belousov, Derkachov, Kharchev & Khoroshkin (2024) studied and applied such operators in the hyperbolic limit case. Following those authors, we call them Q-operators.

The terminology originates in Baxter's solution of the eight-vertex model (1972, 1973).

Definition of Q-operators

Our Q-operators are (from now on $\Gamma(x) = \Gamma(x; p, q)$)

$$(Q_c f)(y_1, \dots, y_n)$$
$$= \int_{\mathbf{x} \in \mathbb{T}^{n-1}_{y_1 \cdots y_n}} f(x_1, \dots, x_n) \prod_{1 \leq i \neq j \leq n} \frac{\Gamma(t x_i / x_j)}{\Gamma(x_i / x_j)} \prod_{i,j=1}^{n} \frac{\Gamma(c y_j / x_i)}{\Gamma(c t y_j / x_i)} |d\mathbf{x}|.$$

Similar operators were considered already by Ruijsenaars (2005). Here,

$$\mathbb{T}^{n-1}_{y_1 \cdots y_n} = \{x \in \mathbb{C}^n; |x_1| = |x_2| = \cdots = |x_n|, x_1 \cdots x_n = y_1 \cdots y_n\},$$

$$|d\mathbf{x}| = \frac{dx_1}{2\pi \mathrm{i} x_1} \cdots \frac{dx_{n-1}}{2\pi \mathrm{i} x_{n-1}}.$$

We will not go into details about parameter conditions.

Commutativity

It is not hard to show that

$$[Q_c, D^{(k)}] = 0, \qquad 0 \leq k \leq n.$$

However, the relation

$$[Q_c, Q_d] = 0$$

is not obvious!

GRRY identity

The integral operator identity $[Q_c, Q_d] = 0$ is equivalent to (after a change of parameters)

$$\int_{x\in\mathbb{T}^{n-1}} \frac{\prod_{i=1}^{n}\prod_{j=1}^{2n}\Gamma(ay_jx_i)\Gamma(b/y_jx_i)}{\prod_{1\leq i\neq j\leq n}\Gamma(x_i/x_j)\Gamma(abx_i/x_j)}\,|d\mathbf{x}| = \int_{x\in\mathbb{T}^{n-1}} \frac{\prod_{i=1}^{n}\prod_{j=1}^{2n}\Gamma(by_jx_i)\Gamma(a/y_jx_i)}{\prod_{1\leq i\neq j\leq n}\Gamma(x_i/x_j)\Gamma(abx_i/x_j)}\,|d\mathbf{x}|,$$

where

$$x_1\cdots x_n = y_1\cdots y_{2n} = 1.$$

Conjectured by Gadde, Rastelli, Razamat and Yan (2010). Appeared from quantum field theory.

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Proof of commutativity

Together with Rains we found two proofs of $[Q_c, Q_d] = 0$. I will sketch one of them, based on the elliptic Macdonald polynomials of Langmann, Noumi and Shiraishi (2022).

Work in space of formal power series

$$V = \mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]^{\mathrm{S}_n}[[p]].$$

Elements are series

$$\sum_{k=0}^{\infty} f_k(x_1, \dots, x_n) p^k,$$

where f_k are symmetric Laurent polynomials.

The Ruijsenaars operators $D^{(k)}$ act on this space.

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Elliptic Macdonald polynomials

Monomial symmetric Laurent polynomials

$$m_\lambda(\mathbf{x}) = x_1^{\lambda_1} \cdots x_n^{\lambda_n} + \text{distinct permutations},$$

where $\lambda_1 \geq \cdots \geq \lambda_n$. Note that λ_j may be negative.
Dominance order

$$\lambda \leq \mu \quad \Longleftrightarrow \quad \lambda_1 + \cdots + \lambda_j \leq \mu_1 + \cdots + \mu_j, \quad 1 \leq j \leq n.$$

Elliptic Macdonald polynomials have the form

$$\mathbf{P}_\lambda(\mathbf{x}; p) = \sum_{k=0}^{\infty} P_\lambda^{(k)}(\mathbf{x}) p^k,$$

where

$$P_\lambda^{(k)} \in \operatorname{span}_{\mu \leq \lambda + (k,0,\dots,0,-k)} m_\mu.$$

The constant term $P_\lambda^{(0)}$ is a standard Macdonald polynomial.

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Example

$$
\begin{aligned}
\mathbf{P}_{00}(x_1,x_2;p) = 1 \\
&+ \frac{q(1-t)^2(1+t)}{t(1-q)(1-tq)}\left(\frac{x_1}{x_2}+\frac{x_2}{x_1}\right)p \\
&+ \frac{q(1-t^2)^2(1-q^2)(1-t^2q)}{t(1-tq)^3(1-tq^2)}\left(\frac{x_1}{x_2}+\frac{x_2}{x_1}\right)p^2 \\
&+ \frac{q^2(1-t)(1-t^2)(1-t^2q)}{t^2(1-q)(1-q^2)(1-tq^2)}\left(\frac{x_1^2}{x_2^2}+\frac{x_2^2}{x_1^2}\right)p^2 \\
&+ \mathcal{O}(p^3).
\end{aligned}
$$

No explicit formula for $\mathbf{P}_\lambda$ is known, not even for $\lambda = (0,0)$.

Elliptic Macdonald polynomials diagonalize Ruijsenaars operators

$\mathbf{P}_\lambda(\mathbf{x};p)$ are eigenfunctions of $D^{(k)}$.

Schauder basis: any $f \in V$ can be written uniquely

$$f(\mathbf{x};p) = \sum_\lambda A_\lambda(p)\mathbf{P}_\lambda(\mathbf{x};p),$$

$A_\lambda \in \mathbb{C}[[p]]$, with convergence as formal power series.

Langmann et al. show that the series defining $\mathbf{P}_\lambda$ converges for some range of parameters.
Very hard to prove and we don't need it.

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Expansion of integral

We want to show that Q_c act on V.

$$\Gamma(x;0,q) = \prod_{j=0}^{\infty} \frac{1}{1-xq^j} = \frac{1}{(x;q)_\infty}.$$

Integral kernel of Q_c is

$$\prod_{1\leq i\neq j\leq n} \frac{(x_i/x_j;q)_\infty}{(tx_i/x_j;q)_\infty} \prod_{i,j=1}^{n} \frac{(tcy_j/x_i;q)_\infty}{(cy_j/x_i;q)_\infty} (1+\Phi_1 p+\Phi_2 p^2+\cdots),$$

with $\Phi_k(\mathbf{x};\mathbf{y})$ Laurent polynomials.

Expansion of integral

By standard results from Macdonald theory (here P_λ are usual Macdonald polynomials)

$$\int_{\mathbf{x}\in\mathbb{T}^{n-1}_{y_1\cdots y_n}} P_\lambda(\mathbf{x}) \prod_{1\leq i\neq j\leq n} \frac{(x_i/x_j;q)_\infty}{(tx_i/x_j;q)_\infty} \prod_{i,j=1}^{n} \frac{(tcy_j/x_i;q)_\infty}{(cy_j/x_i;q)_\infty} |d\mathbf{x}| = \phi_\lambda(c)P_\lambda(\mathbf{y}),$$

for some ϕ_λ.

Using this fact, is it easy to see that Q_c act on V (formal power series in p with symmetric Laurent polynomial coefficients).

Conclusion of proof

The rest is hand-waving.

Since Q_c acts on V, we can expand

$$Q_c \mathbf{P}_\lambda(\mathbf{x};p) = \sum_\mu A_{\lambda\mu}(p)\mathbf{P}_\mu(\mathbf{x};p).$$

We write

$$D^{(k)}\mathbf{P}_\lambda(\mathbf{x};p) = E_\lambda^{(k)}(p)\mathbf{P}_\lambda(\mathbf{x};p).$$

The equation $[Q_c, D^{(k)}] = 0$ then gives

$$A_{\lambda\mu}(p)(E_\lambda^{(k)}(p) - E_\mu^{(k)}(p)) = 0, \qquad 0 \le k \le n.$$

This implies that $A_{\lambda\mu}$ vanishes for $\lambda \neq \mu$, that is, $\mathbf{P}_\lambda(\mathbf{x};p)$ are eigenvectors of Q_c.

This implies that $[Q_c, Q_d] = 0$ on V, which implies that the integral kernel of the commutator vanishes.

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Concluding remark

We have not been able to extend the applications of Q-operators due to Belousov et al.

Main problem is to relate n-particle and $(n+1)$-particle system in elliptic case.

Macdonald polynomials are stable,

$$P_\lambda(x_1, \dots, x_n, 0) = P_\lambda(x_1, \dots, x_n)$$

but we don't know any similar property of elliptic Macdonald polynomials.

Maybe a better combinatorial understanding of elliptic Macdonald polynomials would be useful.

Happy birthday, Paul!