Group algebras and quantum Latin squares

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Outline

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The isomorphism game

R

We need:

- Players: Alice & Bob
- Graphs: X & Y
- Referee: R

Alice Bob

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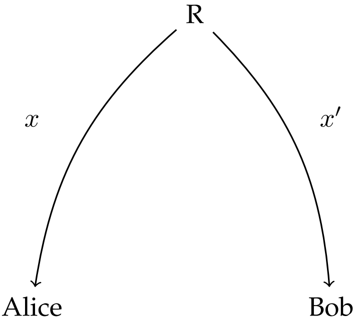

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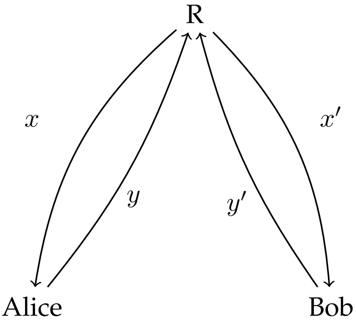

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- if $x \sim x'$, then $y \sim y'$, and
- if $x \neq x'$ and $x \not\sim x'$, then $y \neq y'$ and $y \not\sim y'$

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If they have access to an entangled quantum state, they can win on non-isomorphic graphs.

Definition

Graphs X and Y are *quantum isomorphic*, denoted $X \cong_q Y$, if Alice and Bob can win the isomorphism game using a quantum strategy.

Another definition

Definition

A *quantum permutation matrix* is an $n \times n$ matrix, $U = (u_{ij})_{i,j \in [n]}$ whose entries are elements of a C^*-algebra and satisfy

- $u_{i,j} = u_{i,j}^* = u_{i,j}^2$, for all $i, j \in [n]$ and
- $\sum_{i=1}^{n} u_{i,k} = \sum_{j=1}^{n} u_{\ell,j} = I$, for all $\ell, k \in [n]$.

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Let X and Y be graphs with adjacency matrices A_X and A_Y, respectively. Then X and Y are *quantum isomorphic* if there exits a quantum permutation matrix U satisfying

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Remark

X and Y are isomorphic if and only if there exists a permutation matrix, P satisfying

$$A_X P = P A_Y .$$

Outline

Quantum Latin Squares

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A *quantum Latin square (QLS)*, $\Psi = (\psi_{i,j})_{i,j \in [n]}$, is an $n \times n$ array of vectors from an n-dimensional complex vector space V such that the entries of each row and column form an orthonormal basis of V.

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$(1,1,1)$	$(1,\omega,\omega^2)$	$(1,\omega^2,\omega)$
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A quantum Latin square, where $\omega := e^{2\pi/3}$.

Quantum Latin squares

$\frac{1}{2}$

$(1,1,1,1)$	$(1,-1,1,-1)$	$(1,1,-1,-1)$	$(1,-1,-1,1)$
$(1,-i,-1,i)$	$(1,i,-1,-i)$	$(1,-i,1,-i)$	$(1,i,1,i)$
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Remark

Given a quantum Latin square, we can form a quantum permutation matrix by taking $u_{i,j}$ to be the projection onto the i, j-entry.

Outline

Group-invariant QLS

Definition

For finite groups G and H, we say that a quantum Latin square $\Psi = (\psi_{a,b})$ is (G, H)-*invariant* if its rows are indexed by G and its columns by H and the inner product $\langle \psi_{a,b} \mid \psi_{c,d} \rangle$ depends only on the values of $a^{-1}c \in G$ and $b^{-1}d \in H$.

Again this example

Let $G = \mathbb{Z}_4 = \langle g \rangle$ and $H = \mathbb{Z}_2^2 = \langle x, y \rangle$.

	e	x	y	xy
e	$(1,1,1,1)$	$(1,-1,1,-1)$	$(1,1,-1,-1)$	$(1,-1,-1,1)$
g	$(1,-i,-1,i)$	$(1,i,-1,-i)$	$(1,-i,1,-i)$	$(1,i,1,i)$
g^2	$(1,-1,1,-1)$	$(1,1,1,1)$	$(1,-1,-1,1)$	$(1,1,-1,-1)$
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Indeed we have

$$\langle \psi_{g,e} | \psi_{e,y} \rangle = \frac{1-i}{2} = \langle \psi_{g^2,x} | \psi_{g,xy} \rangle.$$

Transformation matrices

Definition

Suppose that $\Psi = (\psi_{a,b})$ is a (G, H)-invariant quantum Latin square. Define its *transformation matrix*, denoted U^{Ψ}, to be the $|G| \times |H|$ matrix defined entrywise as

$$U^{\Psi}_{g,h} = \langle \psi_{a,b} \mid \psi_{c,d} \rangle$$

for some $a, c \in G,\ b, d \in H$ such that $a^{-1}c = g$ and $b^{-1}d = h$.

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Our previous example is $(\mathbb{Z}_4, \mathbb{Z}_2^2)$-invariant with the following transformation matrix where $\alpha = \frac{1+i}{2}$ and $\beta = \frac{1-i}{2}$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \beta & \alpha \end{pmatrix}$$

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In fact, these three conditions are sufficient.

Theorem

A matrix, $U \in \mathbb{C}^{G\times H}$ is a transformation matrix for some (G,H)-invariant quantum Latin square if and only if

- *It is unitary,*
- $\overline{U_{a,b}} = U_{a^{-1},b^{-1}}$*, for all* $a \in G$*,* $b \in H$*, and*
- $U_{ab,c} = \sum_{\substack{x,y\in H\\ xy=c}} U_{a,x}U_{b,y}$ *for all* $a,b \in G$ *and* $c \in H$.

Outline

Cayley graphs

Definition

Let G be a group and $D \subseteq G \backslash \{e\}$ a subset with $D^{-1} = D$. The *Cayley graph*, $X := \mathrm{Cay}(G, D)$, has vertex set $V(X) := G$, and

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A vague and mostly useless statement

We can use a (G, H)-transformation matrix, to construct pairs, (X, Y) of quantum isomorphic graphs where $X = \mathrm{Cay}(G, D)$ and $Y = \mathrm{Cay}(H, D')$.

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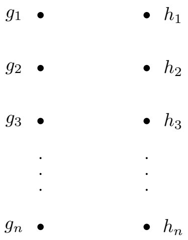

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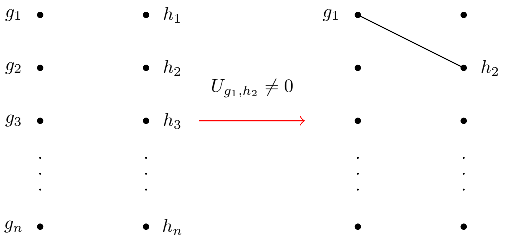

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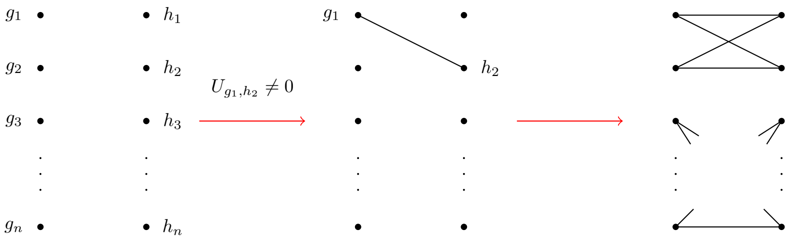

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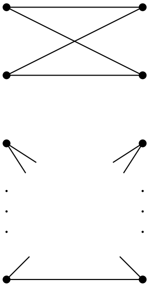

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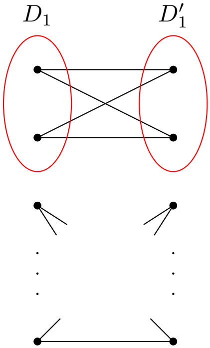

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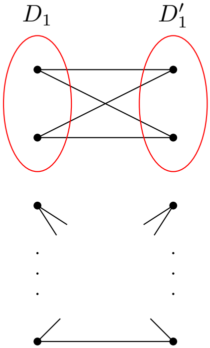

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Then the Cayley graphs $\mathrm{Cay}(G, D_i)$ and $\mathrm{Cay}(H, D'_i)$ are quantum isomorphic for all $i = 1, \ldots, k$.

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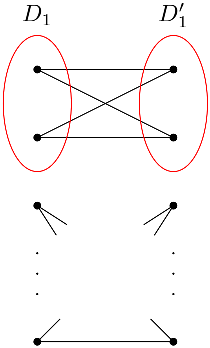

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Then the Cayley graphs $\mathrm{Cay}(G, D_i)$ and $\mathrm{Cay}(H, D'_i)$ are quantum isomorphic for all $i = 1, \dots, k$.

More generally, for any subset I of $[k]$, $\mathrm{Cay}\left(G, \sum_{i \in I} D_i\right)$ and $\mathrm{Cay}\left(H, \sum_{i \in I} D'_i\right)$ are quantum isomorphic.

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Existence

For which groups G, H does a (G, H)-transformation matrix exist?

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Theorem (Árnadóttir & Roberson, 2025+)

Let G, H be finite groups. A (G, H)-invariant QLS exists if and only if the group algebras of G and H are isomorphic.

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Theorem (Árnadóttir & Roberson, 2025+)

Let G, H be finite groups. A (G, H)-invariant QLS exists if and only if the group algebras of G and H are isomorphic.

Remark

The group algebras of G and H being isomorphic is equivalent to the degrees of their irreducible characters being the same.

Proof - part 1

For a group G with left regular representation λ, the group algebra is given by $\Lambda_G := \mathrm{span}\{\lambda(g) : g \in G\}$.

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Lemma

Let G be a finite group with left regular representation λ and irreducible, unitary representations $\rho_1, \ldots, \rho_r$ with degrees $d_1 \ldots, d_r$. Then the map

$$\Phi_G : \Lambda_G \to \bigoplus_{i=1}^{r} \big(I_{d_i} \otimes M_{d_i}(\mathbb{C})\big), \quad \lambda(g) \mapsto \bigoplus_{i=1}^{r} \big(I_{d_i} \otimes \rho_i(g)\big)$$

(extended linearly) is an algebra isomorphism that preserves trace and commutes with the conjugate transpose.

Proof - part 2

Lemma

Let G and H be groups with isomorphic group algebras, Λ_G and Λ_H. Then there exists an isomorphism, $\Psi : \Lambda_G \to \Lambda_H$ that preserves trace and commutes with the conjugate transpose.

Proof - part 3

Lemma

Let G and H be finite groups with left regular representations λ and λ', respectively. Then $U \in \mathbb{C}^{G\times H}$ is a (G, H)-transformation matrix if and only if the linear map $\Psi_U : \Lambda_H \to \Lambda_G$ given by

$$\Psi_U(\lambda'(b)) = \sum_{a\in G} U_{a,b}\lambda(a)$$

is an algebra isomorphism preserving trace and conjugate transpose. Moreover, every such isomorphism from Λ_H to Λ_G is attained in this way.

Conclusion of proof

Proof. It follows from the previous lemmas that:

- There is one-to-one correspondence between (G, H)-transformation matrices and isomorphisms between the algebras that preserve trace and conjugate transpose.

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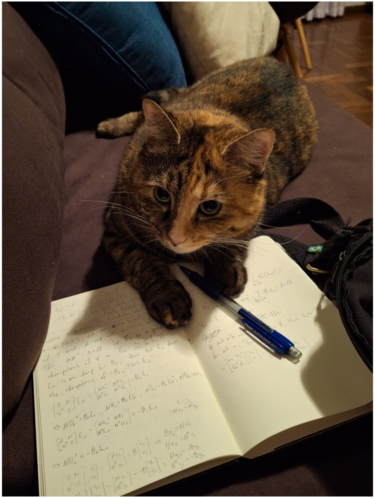
Proof. It follows from the previous lemmas that:

- There is one-to-one correspondence between (G, H)-transformation matrices and isomorphisms between the algebras that preserve trace and conjugate transpose.
- Such isomorphisms exists if and only if the algebras are isomorphic.

Thus the existence theorem follows. □

Thank you

My paper



My cat

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Let G and H be finite abelian groups with $|G| = |H| =: n$. Let $\chi_1, \ldots, \chi_n$ and $\chi'_1, \ldots, \chi'_n$ be the irreducible characters of G and H, respectively.

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Let G and H be finite abelian groups with $|G| = |H| =: n$. Let $\chi_1, \ldots, \chi_n$ and $\chi'_1, \ldots, \chi'_n$ be the irreducible characters of G and H, respectively. For $g \in G$ and $h \in H$, define

$$\psi_{g,h} := \frac{1}{n}\big(\chi_1(g^{-1})\chi'_1(h), \ldots, \chi_n(g^{-1})\chi'_n(h)\big).$$

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$$\psi_{g,h} := \frac{1}{n}\big(\chi_1(g^{-1})\chi'_1(h), \ldots, \chi_n(g^{-1})\chi'_n(h)\big).$$

Then $\big(\psi_{g,h}\big)_{g \in G, h \in H}$ is a (G, H)-invariant quantum Latin square with inner product matrix

$$U = \frac{1}{n^2} M^* M'$$

where M and M' are the character tables of G and H, respectively, and M^* denotes the conjugate transpose of M.

Example

Take $G := \mathbb{Z}_4$ and $H := \mathbb{Z}_2^2$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

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We get the following (G, H)-invariant quantum Latin square:

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