Metathin Table Algebras and Association Schemes

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This includes joint work with Caroline Wroblewski and Bangteng Xu.

It extends results of Bang and Hirasaka (2005) on association schemes.

Definition: Table Algebra

A *table algebra* (A, B) is a finite dimensional algebra A over the complex numbers $\mathbb{C}$, and a distinguished basis B that contains 1_A, such that the following properties hold:

(1a) The structure constants for B are all non-negative real numbers; that is, for all $b, c \in B$,

$$bc = \sum_{d \in B} \beta_{bcd} d, \quad \text{for some } \beta_{bcd} \in \mathbb{R}_{\geq 0}.$$

(1b) There is an algebra anti-automorphism (denoted by *) of A such that $(a^*)^* = a$ for all $a \in A$; and $B^* = B$.

(1c) For all $b, c \in B$,

$$\beta_{bc1} = 0 \text{ if } c \neq b^*; \text{ and}$$
$$\beta_{bb^*1} (= \beta_{b^*b1}) > 0.$$

Definition: Degree Map, Standard

Let (A, B) be a table algebra. Then (A, B) has a unique *degree map*, which means an algebra homomorphism $\delta : A \to \mathbb{C}$ such that

$$\delta(B) \subseteq \mathbb{R}_{>0}.$$

A degree map δ is called standard if for all $b \in B$,

$$\delta(b) = \beta_{bb^*1}.$$

In this case, (A, B) is called a *standard table algebra* (STA).

Definition: Order

Let (A, B) be a STA.

For any $S \subseteq B$, the *order* of S is defined as

$$o(S) := \sum_{b \in S} \delta(b).$$

Also,

$$S^+ := \sum_{b \in S} b, \quad S^* := \{b^* \mid b \in S\}.$$

Example of STA: Adjacency Algebra of an Association Scheme

An association scheme is a pair $(\mathcal{R}, X)$, where $\mathcal{R}$ is a set of relations on an underlying set X, such that

a) $\mathcal{R}$ forms a partition of $X \times X$.

b) The identity relation $1_X := \{(x,x) \mid x \in X\} \in \mathcal{R}$.

c) If $r \in \mathcal{R}$, then $r^T := \{(y,x) \mid (x,y) \in r\} \in \mathcal{R}$.

d) Let $r, s, t \in \mathcal{R}$ and $x, z \in X$ with $(x,z) \in t$. Then the number of $y \in X$ with $(x,y) \in r$ and $(y,z) \in s$ depends only on t, and not the choice of (x,z).

Adjacency Algebra continued

$B =$ set of $0\backslash 1$ adjacency matrices of the relations in $\mathcal{R}$.

$A =$ span (B), the adjacency algebra.

Then (A, B) is a STA.

For all $b \in B$, $b^* = b^T$, $\delta(b) =$ sum of any row of b, and $o(B) = |X|$.

Set $\text{Adj}(\mathcal{R}) := B$.

Combinatorial Isomorphism of Schemes

A *combinatorial isomorphism* of schemes $(\mathcal{R}, X)$ and $(\mathcal{R}', X')$ is a

bijective map $\phi : \ X \cup \mathcal{R} \ \rightarrow \ X' \cup \mathcal{R}'$ such that

$$\phi(X) = X', \quad \phi(\mathcal{R}) = \mathcal{R}',$$

and $x, y \in X$ and $r \in \mathcal{R}$ with $(x, y) \in r \ \Rightarrow \ (\phi(x), \phi(y)) \in \phi(r)$.

When such a map exists, call the schemes *combinatorially isomorphic*, and write

$$(\mathcal{R}', X') \ \cong_{comb} \ (\mathcal{R}, X).$$

Definition: Closed Subset, Cosets, Quotient Algebra

Let (A, B) be a STA.

(i) For $a = \sum_{b \in B} \alpha_b b$, $\alpha_b \in \mathbb{C}$, $\mathrm{Supp}(a) := \{b \in B | \ \alpha_b \neq 0\}$.

(ii) For $S, T \subseteq B$, $ST := \bigcup_{s \in S, t \in T} \mathrm{Supp}(st)$.

(iii) A *closed subset* of B is $\emptyset \neq C \subseteq B$ such that $CC^* \subseteq C$.

(iv) For a closed subset C, the *right cosets* bC, the *left cosets* Cb, and the *double cosets* CbC each partition B.

(v) For closed subset C and $b \in B$, $b//C := \frac{(CbC)^+}{o(C)}$,

$$B//C := \{b//C | \ b \in B\}, \ A//C := \mathbb{C}(B//C).$$

Quotient algebra $(A//C, B//C)$ is again a standard table algebra, and $o(B) = o(B//C) \cdot o(C)$.

Definition: Isomorphism

Let (A, B), (U, V) be STAs. A *table algebra isomorphism* from (A, B) to (U, V) is an algebra isomorphism

$$\phi : A \to U \quad \text{with} \quad \phi(B) = V.$$

(Thus, the two STAs share the same structure constants.)

Write $B \cong V$.

Denote the group of automorphisms of (A, B) by $Aut(A, B)$.

Definition: Thin Radical, Thin Residue

Let (A, B) be a STA.

$O_\vartheta(B)$, the *thin radical* of B is the set of all $l \in B$ such that

$$l^* l = 1_A,$$

the *thin elements* of B. $O_\vartheta(B)$ is a group under the algebra multiplication; and lb and bl are in B for all $l \in O_\vartheta(B)$ and $b \in B$.

$O^\vartheta(B)$, the *thin residue* of B, is the smallest closed subset C of B such that $B//C$ is a group (i.e. such that $B//C = O_\vartheta(B//C)$).

Hypothesis: Metathin STA

We assume from now on that (A, B) is a metathin STA. That is,

$$O^{\vartheta}(B) \subseteq O_{\vartheta}(B),$$

so that $O^{\vartheta(B)}$ is itself a group. Let L be any closed subset of B such that

$$O^{\vartheta}(B) \subseteq L \subseteq O_{\vartheta}(B), \quad \text{and} \quad G := B//L \ \text{ is a group.}$$

Now L and G are both groups, and for all $b \in B$, $bL = Lb = LbL$, so there is no distinction among these right, left, and double cosets.

Transversal

Choose a transversal (set of coset reps) $\mathcal{T} := \mathcal{T}(L, B)$ of L in B, with $b_1 = 1_A$. Thus,

$$\mathcal{T} = \{b_g \mid g \in G\}, \qquad B = \dot{\bigcup}_{g \in G} b_g L.$$

Some Normal Subgroups of L

For all $g \in G$, define

$$S_g \ := \ \mathrm{Supp}(b_g b_g^*)$$

Then

$$S_g \ = \ \{x \in L \mid xb_g = b_g\},$$

S_g is a normal subgroup of L, and

S_g is independent of the choice of $\mathcal{T}$.

Also,

$$S_{g^{-1}} \ = \{x \in L \mid b_g x = b_g\}.$$

Normal Subgroups and Isomorphisms

Because $b_g L = L b_g$ for each $g \in G$, there exists an isomorphism

$$\iota_g \ := \ L/S_g \ \to \ L/S_{g^{-1}},$$

where

$$l b_g \ = \ b_g l^{\iota_g} \text{ for all } l \in L.$$

Furthermore, for all $g, h \in G$,

$$(S_g S_h)^{\iota_g} \ = \ S_{g^{-1}} S_{g^{-1}h}. \qquad \text{(I)}$$

Set $\mathcal{S} \ := \ \{S_g \mid g \in G\}, \quad \mathcal{M} \ := \ \{\iota_g \mid g \in G\}.$

Factor Set

There exists a function $\alpha : \; G \times G \; \rightarrow \; L$ such that for all $g, h, k \in G$ and all $l \in L$,

$$\alpha(1,1) \; = \; 1 \quad \text{(II)};$$

$$\alpha(g,h) l^{\iota_g \iota_h} \; \equiv \; l^{\iota_{gh}} \alpha(g,h) \quad \text{mod } S_{(gh)^{-1}} S_{h^{-1}} \quad \text{(III)};$$

$$\alpha(gh,k)\alpha(g,h)^{\iota_k} \; \equiv \; \alpha(g,hk)\alpha(h^{-1},hk)^{-1}\alpha(h^{-1},h)^{\iota_k}$$
$$\text{mod } S_{(ghk)^{-1}} S_{(hk)^{-1}} S_{k^{-1}} \quad \text{(IV)}.$$

Multiplication in (A, B)

For all $g, h \in G$ and all $l_1, l_2 \in L$,

$$(b_g l_1)(b_h l_2) = |S_{g^{-1}} \cap S_h| \sum_{l \in \mathcal{T}(S_{(gh)^{-1}}, S_{(gh)^{-1}} S_{h^{-1}})} b_{gh} \alpha(g,h) l l_1^{\iota_h} l_2. \quad \text{(V)}$$

So we have the following

Results

THEOREM A Any metathin STA (A, B) is determined to isomorphism by L, G, $\mathcal{S}$, $\mathcal{M}$, and α.

COROLLARY The structure constants of any metathin STA are integers, and they are constant (when nonzero) over any product of two distinguished basis elements.

Cohort

Let *cohort* $\mathcal{A} :=$

$$\{\beta : G \times G \to L \mid \beta(g,h) \equiv \alpha(g,h) \ mod \ S_{(gh)^{-1}} S_{h^{-1}}, \ \ \text{all } g,h \in G\}$$

Substitution of β for α leaves (I) - (V) unchanged.

We say that (A, B) is of *type* $(L, G, \mathcal{S}, \mathcal{M}, \mathcal{A})$.

Converse

THEOREM B. Given L, G, $\mathcal{S}$ (with $S_1 = \{1_L\}$), $\mathcal{M}$, $\mathcal{A}$ such that (I)-(IV) hold,

there exists a unique (up to table algebra isomorphism) metathin

STA (A, B) whose multiplication is given by (V), and which is of

type $(L, G, \mathcal{S}, \mathcal{M}, \mathcal{A})$.

Metathin STAs as Schemes

Let (A, B) be a metathin STA of type $(L, G, \mathcal{S}, \mathcal{M}, \mathcal{A})$.

Fix $\alpha \in \mathcal{A}$, define $\hat{\alpha}(g, h) := \alpha(g, h)\alpha(h^{-1}, h)^{-1}$.

Let $X := \{(g, l) \mid g \in G, l \in L\}$.

Each $e \in G$, $p \in L$, define relation $\mathrm{rel}_\alpha(b_e p)$ on X:

$$((g, l), (h, l_1)) \in \mathrm{rel}(b_e p)$$

$$\iff e = gh^{-1} \text{ and } p \equiv l^{i_{gh^{-1}}} \hat{\alpha}(g, h^{-1}) l_1^{-1} \; mod \; S_{hg^{-1}}.$$

Denote the set of these relations as

$$\mathcal{R}_\alpha := \{ \mathrm{rel}_\alpha(b_e p) \mid e \in G, p \in L\}.$$

Metathin STAs as Schemes continued

Define $\mathcal{A}'$ as the set of all $\alpha \in \mathcal{A}$ such that

$$\alpha(gh,k)\alpha(g,h)^{\iota_k} \equiv \alpha(g,hk)\alpha(h^{-1},hk)^{-1}\alpha(h^{-1},h)^{\iota_k}$$

$$\bmod\ S_{(ghk)^{-1}}S_{k^{-1}}.$$

THEOREM C. (i) If (A,B) is isomorphic to the adjacency algebra of some association scheme, then for some $\alpha \in \mathcal{A}'$, the association scheme is combinatorially isomorphic to the set of relations $\mathcal{R}_\alpha$ on set $X = G \times L$.

(ii) For any $\alpha \in \mathcal{A}'$, $(\mathcal{R}_\alpha, X)$ is an association scheme such that $B \cong \text{Adj}(\mathcal{R}_\alpha)$ via $b \leftrightarrow \text{rel}_\alpha(b)$ for all $b \in B$.

Combinatorial Isomorphism of Schemes from a Metathin STA

Assume for the next (and last) two theorems that (A, B) is a metathin STA of type $(L, G, \mathcal{S}, \mathcal{M}, \mathcal{A})$ with $L = O^{\vartheta}(B)$.

If $\psi \in Aut(A, B)$, we have $\psi(B) = B$, $\psi(L) = L$, and ψ induces a group automorphism of G such that for all $g \in G$,

$$\psi(b_g) \ = \ b_{\psi(g)} u_{\psi,g} \text{ for some } u_{\psi,g} \in L \text{ (defined mod } S_{\psi(g^{-1})}).$$

DEFINITION. Fix $\alpha \in \mathcal{A}'$, $\psi \in Aut(A, B)$, $w \in G$. A set

$$\{z_g \mid g \in G\} \subseteq L$$

is called α-ψ-*w-admissible* if, for all $g \in G$,

$$z_g \equiv \alpha(\psi(g), \psi(g)^{-1})(u_{\psi,g} z_1)^{\iota_{\psi(g^{-1})}} \hat{\alpha}(w, w^{-1}\psi(g^{-1})) \mod S_{\psi(g)}.$$

Combinatorial Isomorphism of Schemes continued

THEOREM D. Let $\alpha, \beta \in \mathcal{A}'$. Then $(\mathcal{R}_\alpha, X) \cong_{comb} (\mathcal{R}_\beta, X)$ if and only if for some $\psi \in Aut(A, B)$, $w \in G$, and an α-ψ-w-admissible set $\{z_g \mid g \in G\} \subseteq L$, then for all $g, h \in G$,

$$\hat{\beta}(g,h) \equiv \psi^{-1}\left(u_{\psi,gh}^{-1} z_g^{\iota_{\psi(gh)}} \hat{\alpha}(\psi(g)w, w^{-1}\psi(h)) z_{h^{-1}}^{-1}\right) \mod S_{(gh)^{-1}}.$$

THEOREM E. Assume that L is abelian. Let $\alpha \in \mathcal{A}'$, $\psi \in Aut(A, B)$, $w \in G$; and let $\{z_g \mid g \in G\}$ be an α-ψ-w-admissible subset of L. Define $\beta : G \times G \to L$ by

$$\beta(g,h) :\equiv \psi^{-1}\left(u_{\psi,gh}^{-1} z_g^{\iota_{\psi(gh)}} \hat{\alpha}(\psi(g)w, w^{-1}\psi(h)) z_{h^{-1}}^{-1}\right)\alpha(h^{-1}, h)$$

mod $S_{(gh)^{-1}}$, for all $g, h \in G$.

Then $\beta \in \mathcal{A}'$, and $(\mathcal{R}_\beta, \mathrm{X}) \cong_{comb} (\mathcal{R}_\alpha, X)$.