The Dunkl-Watanabe Duality

Tatsuro Ito

Kanazawa University and Anhui University

June 24, 2025

Paul Terwilliger's 70th birthday conference

Ramada Resort, Kranjska Gora, Slovenia

1

<u>q-Johnson scheme</u> $J_q(N, D)$, $N \geq 2D$

$\Omega = \mathbb{F}_q^N$

$X = \binom{\Omega}{D}_q \ni x, y$

$x \underset{i}{\sim} y \iff \dim x \cap y = D - i$

A_i : ith adj. matrix $\in M_X(\mathbb{C})$

$$A_i(x, y) = \begin{cases} 1 & \text{if } x \underset{i}{\sim} y \\ 0 & \text{otherwise} \end{cases}$$

$\mathfrak{A} = \mathrm{Span}\{A_i \mid 0 \leq i \leq D\}$ commutative semi-simple alg.

<u>Bose-Mesner</u> alg.

$= \mathrm{Span}\{E_i \mid 0 \leq i \leq D\}$ primitive idemp

<u>Rm</u> $\mathfrak{A} = M_X^G(\mathbb{C})$, $G = GL(N, q) \curvearrowright X$

Fix $x_0 = \begin{array}{l} \left(\begin{matrix} * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{matrix}\right) \begin{matrix} D \\ \\ N-D \end{matrix} \end{array} \in X = \binom{\Omega}{D}_q$

A_i^* <u>dual adj matrix</u>

$$A_i^*(x,x) = |X| E_i(x_0, x)$$ diagonal matrix

$$\mathcal{T}^* = \mathcal{T}^*(x_0) = \mathrm{Span}\{A_i^* \mid 0 \le i \le D\}$$

<u>dual Bose-Mesner alg</u>

$$= \mathrm{Span}\{E_i^* \mid 0 \le i \le D\}$$ prim. idemp

$$T = T(x_0) = \langle \mathcal{T}, \mathcal{T}^* \rangle \subseteq M_X(\mathbb{C})$$

<u>Terwilliger alg</u> w.r.t. x_0

<u>Rm</u> T is a semi-simple alg.

$A_0 = I,\ A_1,\ A_2,\ \cdots,\ A_D$: <u>P-poly. ordering</u>

$$A = A_1, \qquad A_i = v_i(A), \quad \deg v_i = i$$

$A_0^* = I,\ A_1^*,\ A_2^*,\ \cdots,\ A_D^*$: <u>Q-poly. ordering</u>

$$A^* = A_1^*, \qquad A_i^* = v_i^*(A^*), \quad \deg v_i^* = i$$

$$T = \langle A, A^* \rangle$$

$V = \mathbb{C}X \simeq \mathbb{C}^X$ standard module

$$= \bigoplus_{i=0}^{D} V_i, \qquad V_i = E_i V$$

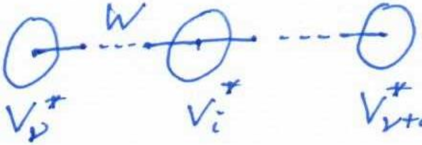
$$= \bigoplus_{i=0}^{D} V_i^*, \qquad V_i^* = E_i^* V$$

$$\begin{cases} A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \\ A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \end{cases}$$ block tridiagonal

$V \supseteq W$: irred. T-module

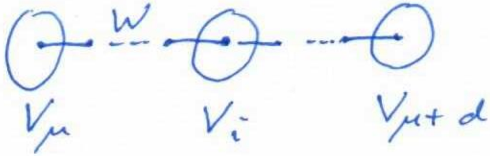
$$\dim E_i^* W = \begin{cases} 1 & \text{if } \nu \le i \le \nu + d \\ 0 & \text{otherwise} \end{cases}$$

ν: <u>end pt</u>



$$\dim E_i W = \begin{cases} 1 & \text{if } \mu \le i \le \mu + d \\ 0 & \text{otherwise} \end{cases}$$

μ: <u>dual end pt</u>



$d = \dim W - 1$

<u>diameter</u>

$W = \mathrm{Span}\{w_0, w_1, \cdots, w_d\}$ <u>standard basis</u>

unique up to scalar mult.

$E_\mu W \ni w \neq 0, \quad w_i = E^*_{\nu+i} w$

$A|_W$: tridiagonal matrix w.r.t. the standard basis

one more parameter e appears if $d \geq 1$

e : <u>auxiliary parameter</u> $(e \in \mathbb{Z})$

(ν, μ, d, e) : <u>type</u> of W

If $d \geq 1$, the isom. class of W is determined by (ν, μ, d, e)

If $d = 1$, the isom. class of W is determined by (ν, μ)

[T] Paul Terwilliger. The subconstituent Algebra of an Association Scheme (Part III), J. Alg. Combin. 2 (1993). 177–210 6

$$\Delta = \left\{ (\nu, \mu, d, e) \in \mathbb{Z} \;\middle|\; (\#) \right\}$$

$$(\#) \begin{cases} 0 \le \dfrac{D-d}{2} \le \nu \le \mu \le D-d \le D \\ e+d+D \text{ is even}, \quad |e| \le 2\nu - D + d \\ d \in \{ e+D-2\nu, \ \min\{D-\mu, \ e+D-2\nu+2(N-2D)\} \} \end{cases}$$

<u>Thm</u> ([T])

W : irred. T-module of type (ν, μ, d, e), $d \ge 1$

Then $(\nu, \mu, d, e) \in \Delta$

$$x_0 = \begin{array}{c} \\ \left(\begin{array}{c} * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{array}\right) \end{array} \begin{array}{l} D \\ \\ N-D \end{array} \in X = \binom{\Omega}{D}_q$$

$$G = GL(N, q) \curvearrowright X$$

$$H = G_{x_0} = \begin{array}{c} \\ D \\ N-D \end{array} \begin{array}{cc} D & N-D \\ \multicolumn{2}{c}{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}} \end{array}$$

$$T \subseteq M_X^H(\mathbb{C})$$

<u>Question</u> $T = M_X^H(\mathbb{C})$?

<u>Rm</u> In the case of the Johnson scheme $J(N, D)$, $N \geq 2D$

Ω : a set, $|\Omega| = N$

$G = \mathrm{Sym}(\Omega) \curvearrowright X = \binom{\Omega}{D} \ni x_0$

$H = G_{x_0}$

Then $T = M_X^H(\mathbb{C})$ if $N \neq 2D$

$T = M_X^{\langle H, \tau \rangle}(\mathbb{C}) \subsetneq M_X^H(\mathbb{C})$ if $N = 2D$ $\quad \left(\tau : \begin{array}{l} X \longrightarrow X \\ x \longmapsto \Omega - x \end{array} \right)$

$$\tilde{X} = \bigcup_{k=0}^{N} \binom{\Omega}{k}_q$$

$$= \bigcup_{\substack{0 \le i \le D \\ 0 \le j \le N-D}} \tilde{X}_{ij} .$$

H orbit decomp.

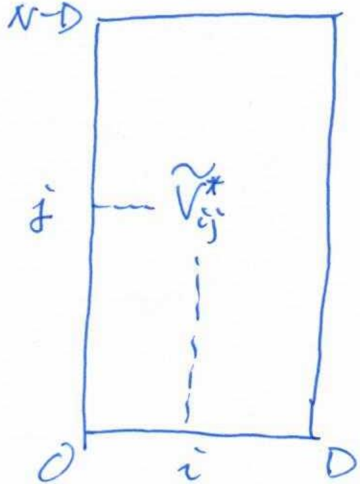
$\tilde{X}_{ij} \ni x$ $\quad$ $\dim x = i+j$

$\dim x \cap x_0 = i$

$$\tilde{V} = \mathbb{C}\tilde{X} \simeq \mathbb{C}^{\tilde{X}}$$

$$= \bigoplus_{\substack{0 \le i \le D \\ 0 \le j \le N-D}} V^*_{ij}$$

$$E^*_{ij} : V \xrightarrow[\text{proj}]{} \tilde{V}^*_{ij} = \mathbb{C}\tilde{X}_{ij}$$



Watanabe alg.

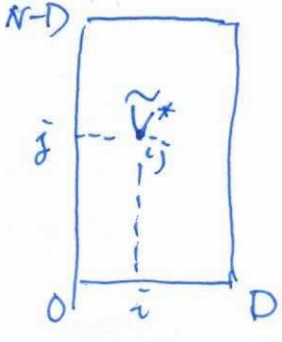
$$\mathcal{H} = \langle L_1, L_2, R_1, R_2, E^*_{ij} \mid 0 \le i \le D,\ 0 \le j \le N-D \rangle$$
$$\subseteq M_{\tilde{X}}^{H}(\mathbb{C})$$

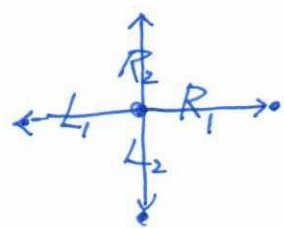
$L_1 : \tilde{X}_{ij} \ni x \longmapsto \sum y \in \tilde{V}^*_{i-1,j}$ over $y < x$, $y \in \tilde{X}_{i-1,j}$

$L_2 : \tilde{X}_{ij} \ni x \longmapsto \sum y \in \tilde{V}^*_{i,j-1}$ over $y < x$, $y \in \tilde{X}_{i,j-1}$

$R_1 : \tilde{X}_{ij} \ni x \longmapsto \sum y \in \tilde{V}^*_{i+1,j}$ over $x < y$, $y \in \tilde{X}_{i+1,j}$

$R_2 : \tilde{X}_{ij} \ni x \longmapsto \sum y \in \tilde{V}^*_{i,j+1}$ over $x < y$, $y \in \tilde{X}_{i,j+1}$





[W] Yuta Watanabe, An algebra associated with a subspace lattice over a finite field and its relation to the quantum affine algebra $U_q(\hat{sl}_2)$, J of Alg. 489 (2017) 475–505

<u>Thm</u> ([W]) $\tilde{V} \supseteq \tilde{W}$ irred. $\mathcal{H}$-module

then $\dim \tilde{E}^*_{ij} \tilde{W} = \begin{cases} 1 & \text{if } \alpha \le i \le D-\alpha-\rho,\ \beta+\rho \le j \le N-D-\beta \\ 0 & \text{otherwise} \end{cases}$

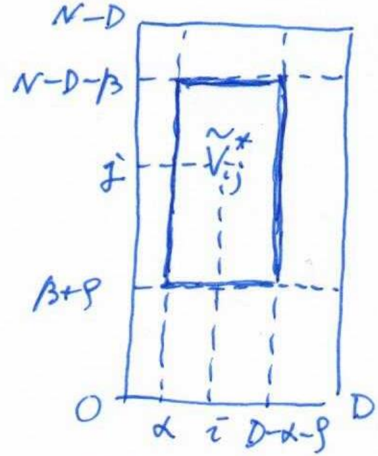
$\lambda = (\alpha, \beta, \rho)$: <u>type</u> of $\tilde{W}$

and

$\{ \tilde{W} : \text{irred } \mathcal{H}\text{-module} \} / \underset{\text{isom}}{\sim}$

$\underset{1:1}{\longleftrightarrow}$

$\tilde{\Lambda} = \left\{ \lambda = (\alpha, \beta, \rho) \in \mathbb{Z}_{\ge 0}^3 \;\middle|\; \begin{array}{l} 0 \le \alpha \le \frac{D-\rho}{2} \\ 0 \le \beta \le \frac{N-D-\rho}{2} \end{array} \right\}$



$\tilde{W}$: type $\lambda = (\alpha, \beta, \rho)$

$\dim \tilde{W} \cap \tilde{V}^*_{ij} = 1$

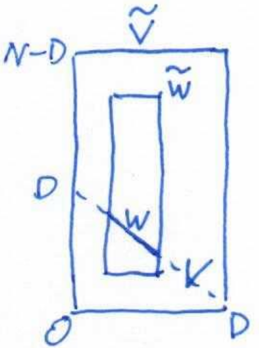
[D] Charles F. Dunkl, An Addition Theorem for some q-Hahn Polynomials, Mh. Math. 85, 5 − 37 (1977)

<u>Thm</u> ([W] + [D])

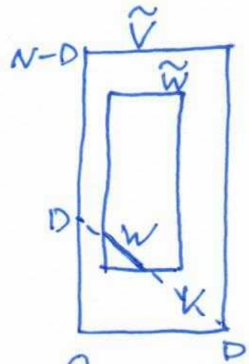
$\mathcal{H} = M^{H}_{\tilde{X}}(\mathbb{C})$ <u>Dunkl - Watanabe duality</u>

<u>Cor</u> $\{ W : \text{irred. } M^{H}_{X}(\mathbb{C})\text{-modules} \} / \underset{\text{isom}}{\cong}$

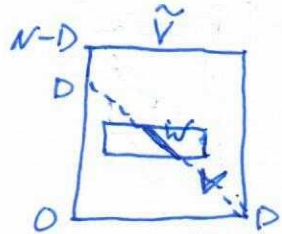
$\underset{1:1}{\longleftrightarrow} \Lambda = \{ \lambda = (\alpha, \beta, \rho) \in \tilde{\Lambda} \mid \alpha + \beta \leq D - \rho \}$



case I $\beta - \alpha \leq 0$



case II $0 \leq \beta - \alpha \leq N - 2D$



case III $N - 2D \leq \beta - \alpha$

☆ $W = \tilde{W} \cap V \neq 0 \iff \lambda \in \Lambda$ for irred $\mathcal{H}$-module $\tilde{W}$ of type $\lambda \in \tilde{\Lambda}$

[LIW] Xiaoye Liang, Tatsuro Ito and Yuta Watanabe,
The Terwilliger algebra of the Grassmann scheme $J_q(N,D)$
revisited from the viewpoint of the quantum affine algebra $U_q(\widehat{sl}_2)$,
Linear Alg. Appl. 596 (2020), 117-144

$$\varphi : \Lambda \longrightarrow \Delta = \{(\nu, \mu, d, e) \in \mathbb{Z} \mid (\#)\}$$

the condition for the types of irred. T-modules, <u>allowing</u> $d=0$

$$\lambda = (\alpha, \beta, \rho) \longmapsto \varphi(\lambda) = (\nu, \mu, d, e)$$

$$\nu = \rho + \max\{\alpha, \beta\}$$
$$\mu = \rho + \alpha + \beta$$

$$d = \begin{cases} D - \rho - 2\alpha & \text{if case I} \quad \beta - \alpha \le 0 \\ D - \rho - \alpha - \beta & \text{if case II} \quad 0 \le \beta - \alpha \le N - 2D \\ N - D - \rho - 2\beta & \text{if case III} \quad N - 2D \le \beta - \alpha \end{cases}$$

$$e = \begin{cases} \rho & \text{if case I} \\ \rho + \alpha - \beta & \text{if case II} \\ \rho - N + 2D & \text{if case III} \end{cases}$$

Thm ([LIW])

(1) φ is $1:1$ if $N \neq 2D$

$2:1$ if $N = 2D$

(If $N=2D$, $\varphi|_{\Lambda_1} : \Lambda_1 \longrightarrow \Delta$ is $1:1$, $\Lambda_1 = \{\lambda \mid \beta - \alpha \leq 0\}$

(2) W: irred M_X^H-module of type $\lambda \in \Lambda$

Then W: irred. T-module of type $\varphi(\lambda) \in \Delta$

Thm (joint with Xiaoye Liang)

$$T \subsetneqq M_X^H(\mathbb{C})$$

Rm This was also obtained recently by Hau-Wen Huang by a different approach.

Proof

$\exists\ \lambda_i = (\alpha_i, \beta_i, \rho_i) \in \Lambda, \quad i = 1, 2$

s.t. $\varphi(\lambda_i) = (\nu_i, \mu_i, d_i, e_i)$

with $\nu_1 = \nu_2, \quad \mu_1 = \mu_2, \quad d_1 = d_2 = 0$

and $e_1 \neq e_2$

Choose irred. $M_X^H(\mathbb{C})$-modules W_i of type λ_i, $i = 1, 2$

Then $W_1 \not\simeq W_2$ as $M_X^H(\mathbb{C})$-modules

$W_1 \simeq W_2$ as T-modules

$\boxed{//}$