b-Coloring Rooted Products of Graphs

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b-Coloring (Irving and Manlove, 1999)

Definition

Consider a graph, G, whose vertices are properly colored.
A *color-dominating vertex* (CDV)
is one that is adjacent to at least one vertex of each other color class.
We have a *b-coloring* when each color class includes a CDV.
If there are k colors, we might say it is a k-b-coloring.
The *b-chromatic number* of G, denoted by $\varphi(G)$,
is the largest k such that G admits a k-b-coloring.

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Theorem

Determining $\varphi(G)$ is NP-hard in general but polynomial for trees.

Motivation and Application

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Poset

Start with a proper coloring of a graph.
If there is a color class that does not contain a CDV,
then (properly) recolor every vertex from this class
and remove this color from further consideration.
Iterating this process eventually yields a b-coloring,
which can be considered a minimal element.

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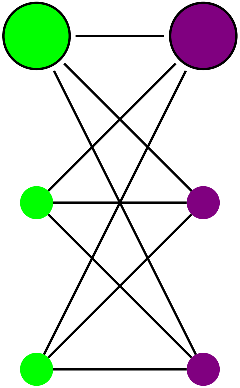
Conflict Avoidance

The b-chromatic number can be thought of as indicating
how many communities can be placed in an area
so that every community has a representative
that is able to communicate with all of the other communities,
reducing misunderstandings.

Some Basic Examples



$\varphi(K_{m,n}) = 2$

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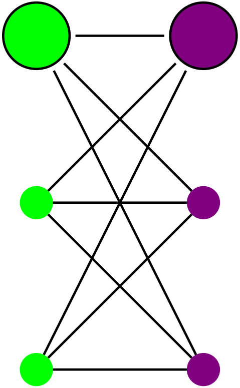

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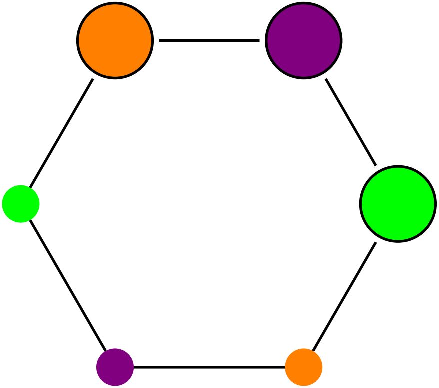

$\varphi(P_n) = \varphi(C_n) = 3$ for $n \geq 5$

Easy Bounds

Observation

$$\chi(G) \leq \varphi(G) \leq \Delta(G) + 1$$

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Comparing φ and χ

The difference between χ and φ can be arbitrarily large.
For example, if G is $K_{n,n}$ minus a matching, then $\chi(G) = 2$ but $\varphi(G) = n$.

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For example, if G is $K_{n,n}$ minus a matching, then $\chi(G) = 2$ but $\varphi(G) = n$.

Comparing φ and $\Delta + 1$

The difference between φ and $\Delta + 1$ can also be arbitrarily large.
For example, if G is $K_{1,n}$, then $\varphi(G) = 2$ but $\Delta(G) = n$.

A Better Upper Bound

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A k-b-coloring needs at least k vertices with degree at least $k-1$.
These are the CDV candidates.

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Definition (Irving and Manlove, 1999)

Label the vertices of G so $d_G(v_1) \geq d_G(v_2) \geq \cdots \geq d_G(v_n)$. We call

$$m(G) = \max\{i : d_G(v_i) \geq i - 1\}$$

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An Improvement

$$\varphi(G) \leq m(G) \leq \Delta(G) + 1$$

Some Results for Regular Graphs

Agreement of Upper Bounds

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Cubic with Four Exceptions (Jakovac and Klavžar, 2010)

Suppose G is connected with $d = 3$ but is not $K_3 \square K_2$, $K_{3,3}$, Petersen, or another 10-vertex sporadic example.

Some Results on Products

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Cartesian

F. A. Chaouche, A. Berrachedi: "Some bounds for the b-chromatic number of a generalized Hamming graphs"; *Far East J. Appl. Math.* 26 (2007), 375–391

F. Maffray, A. Silva: "b-colouring the Cartesian product of trees and some other graphs"; *Discrete Appl. Math.* 161 (2013), 650–669

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Direct, Strong, Lexicographic

M. Jakovac, I. Peterin: "On the b-chromatic number of some graph products"; *Studia Sci. Math. Hungar.* 49 (2012), 156–169

I. Koch, I. Peterin: "The b-chromatic index of direct product of graphs"; *Discrete Appl. Math.* 190 (2015), 109–117

Another Product

Definition (Godsil and McKay, 1978)

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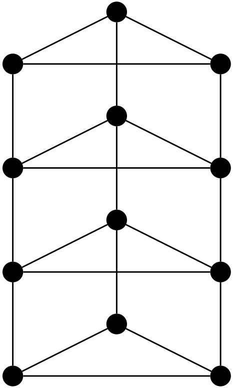
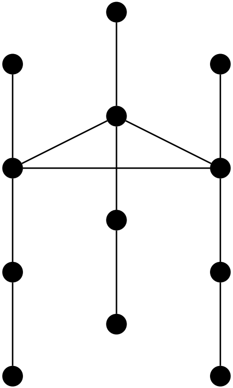
Given graphs G and H, choose a vertex $v \in V(H)$ to be the *root* of H. The *rooted product* of G and H, written $G \circ_v H$, has vertex set $V(G) \times V(H)$ and edge set

$$\{(x,v)(x',v) : xx' \in E(G)\} \cup \bigcup_{x \in V(G)} \{(x,y)(x,y') : yy' \in E(H)\}.$$

Rooted Product as Subgraph of Cartesian Product



$C_3 \,\square\, P_4$



$C_3 \circ_v P_4$, with $d_{P_4}(v) = 2$

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Given graphs G and H with $v \in V(H)$,

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Example

Let n be at least 5, let G be P_n, and let H be $K_{1,n-1}$.
Recall that $\varphi(G) = 3$ and $\varphi(H) = 2$.
If $d_H(v) = 1$, then $\varphi(G \circ_v H) = n$.

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If $d_H(v) = 1$, then $\varphi(G \circ_v H) = n$.

Observation

Without restricting G and H,
no function of $\varphi(G)$ and $\varphi(H)$ can bound $\varphi(G \circ_v H)$ from above.
Even the amount by which $\varphi(G \circ_v H)$ exceeds our lower bound can be arbitrarily large.

A New Parameter

Definition (BJL)

Let G be a graph. We write $n_\varphi(G)$ to denote the maximum number of CDVs in any $\varphi(G)$-b-coloring of G.

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Theorem (BJL)

Given graphs G and H with $v \in V(H)$, if

$$\Delta(H) < \varphi(G) + d_H(v) \leq n_\varphi(G),$$

then

$$\varphi(G \circ_v H) \geq \varphi(G) + d_H(v).$$

Moreover, if $\varphi(G) = m(G)$, then equality holds in this bound.

Another New Parameter

Definition (BJL)

Let H be a graph.
Let D denote the set of all maximum-degree vertices of H.
We will call a subset $D' \subseteq D$ a *far set* of H if for every two distinct vertices y_1 and y_2 in D', the distance between y_1 and y_2 is at least 4.
We define

$$n_f(H) = \max\{|D'| : D' \text{ is a far set of } H\}.$$

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Theorem (BJL)

Given graphs G and H with $v \in V(H)$, if

$$\Delta(G) + d_H(v) \leq \Delta(H) < n(G)n_f(H),$$

then

$$\varphi(G \circ_v H) = \Delta(H) + 1.$$

Relationships Between Theorems

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Hypotheses

In our first theorem, we assume that $\varphi(G) + d_H(v)$ is greater than $\Delta(H)$. In our second, we more or less assume the reverse.

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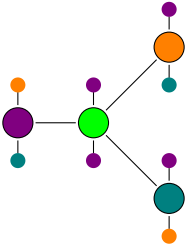
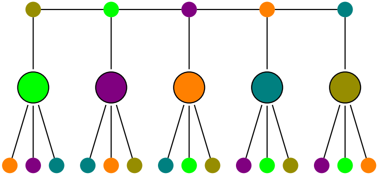
Location of CDVs

In our first theorem, we locate our CDVs in the "G-layer".
In our second, they are in the "H-layers".

Examples of Each Theorem



$K_{1,3} \circ_v P_3$, with $d_{P_3}(v) = 2$



$P_5 \circ_v K_{1,4}$, with $d_{K_{1,4}}(v) = 1$

Fixing Factors: Paths

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Proposition (BJL)

Let $a, b \geq 2$ *and* $v \in V(P_b)$.
If $d_{P_b}(v) = 1$*, then*

$$\varphi(P_a \circ_v P_b) = m(P_a \circ_v P_b) = \begin{cases} 2 & a = b = 2 \\ 3 & (a = 2 \text{ and } b \geq 3) \text{ or } 3 \leq a \leq 5 \\ 4 & a \geq 6 \end{cases}.$$

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Fixing Factors: Path by Cycle

Proposition (BJL)

If $a \geq 2$, $b \geq 3$, and $v \in V(C_b)$, then

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Fixing Factors: Cycle by Path

Proposition (BJL)

Let $a \geq 3$, $b \geq 2$, *and* $v \in V(P_b)$.
If $d_{P_b}(v) = 1$, *then*

$$\varphi(C_a \circ_v P_b) = \begin{cases} 3 & a = 3 \text{ or } a = 5 \\ 4 & a = 4 \text{ or } a \geq 6 \end{cases}$$

and

$$m(C_a \circ_v P_b) = \begin{cases} 3 & a = 3 \\ 4 & a \geq 4 \end{cases}.$$

If $d_{P_b}(v) = 2$, *then*

$$\varphi(C_a \circ_v P_b) = m(C_a \circ_v P_b) = \min\{5, a\}.$$

Fixing Factors: Cycles

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If $a, b \geq 3$ and $v \in V(C_b)$, then

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m-degree of the Rooted Product

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Proposition (BJL)

Suppose G and H are regular. Then

$$m(G \circ_v H) = \begin{cases} \Delta(G) + \Delta(H) + 1 & \Delta(G) + \Delta(H) + 1 \leq n(G) \\ n(G) & \Delta(H) + 1 \leq n(G) < \Delta(G) + \Delta(H) + 1 \\ \Delta(H) + 1 & n(G) < \Delta(H) + 1 \end{cases}.$$

Fixing Factors: Complete Graphs

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Proposition (Kouider and Mahéo, 2002)

Suppose $a \geq b$.
If $a \leq b(b-1)$, *then* $a \leq \varphi(K_a \square K_b) \leq b(b-1)$.
If $a \geq b(b-1)$, *then* $\varphi(K_a \square K_b) = a$.

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Proposition (BJL)

Let H be a graph with $\Delta(H) < n$ and let v be any vertex of H.
Then $\varphi(K_n \circ_v H) = n$.

Comparing m and φ in Factors vs Product, Part 1

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Example

Consider $G = K_{a+1,a+1}$ and $H = K_{a,a}$ with $a \geq 3$ and $v \in V(H)$.

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Color the remaining neighbors of these vertices so they become CDVs.

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The rest of the vertices can be colored greedily.

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Observation

Can have $\varphi(G) < m(G)$ and $\varphi(H) < m(H)$ but $\varphi(G \circ_v H) = m(G \circ_v H)$.

Comparing m and φ in Factors vs Product, Part 2

Example

Take $G = C_5$ and $H = P_b$ with $b \geq 5$ and $d_H(v) = 1$.
Then $\varphi(G) = \varphi(H) = m(G) = m(H) = 3$,
but $\varphi(G \circ_v H) = 3$ and $m(G \circ_v H) = 4$.

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but $\varphi(G \circ_v H) = 3$ and $m(G \circ_v H) = 4$.

Observation

Can have $\varphi(G) = m(G)$ and $\varphi(H) = m(H)$ but $\varphi(G \circ_v H) < m(G \circ_v H)$.

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Can have $\varphi(G) = m(G)$ and $\varphi(H) = m(H)$ but $\varphi(G \circ_v H) < m(G \circ_v H)$.

Note

This example is not as broad as the one on the previous slide.
There may be only two choices for the first factor.
Most other examples we considered with $\varphi(G) = m(G)$ and $\varphi(H) = m(G)$ have $\varphi(G \circ_v H) = m(G \circ_v H)$.

Suggestions for Future Research

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How are equality between φ and m in the factors related to equality in the rooted product?

Question 2

When is $\varphi(G \circ_v H)$ either $\Delta(H) + 1$ or $\varphi(G) + d_H(v)$?

Question 3

When is $\varphi(G \circ_v H)$ simply the larger of $\varphi(G)$ and $\varphi(H)$?

Some Citations

These Results

S. Bockting-Conrad, M. Jakovac, M. S. Lang:
"On the b-chromatic number of rooted product graphs";
Filomat 39 (2025), 3805–3815

A Nice Overview

M. Jakovac, I. Peterin:
"The b-chromatic number and related topics – a survey";
Discrete Appl. Math. 235 (2018), 184–201