Multivariate P-polynomial association schemes and m-distance regular graphs

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Joint work with
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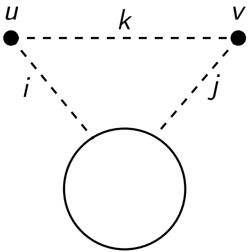
TerwilligerFest - Combinatorics around the q-Onsager algebra
Kranjska gora

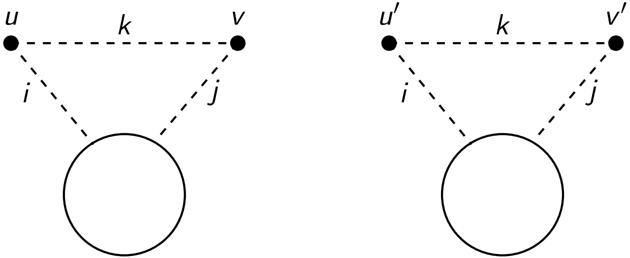
Outline

1. DRG and P-polynomial association scheme
   - Distance regular graphs
   - P-polynomial association scheme
2. m-distance regular graph and m-variate P-polynomial scheme
   - Multivariate P-polynomial scheme
   - m-distance regular graphs

Distance regular graphs



Distance regular graphs



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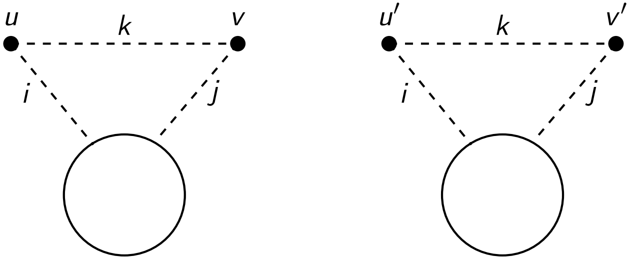
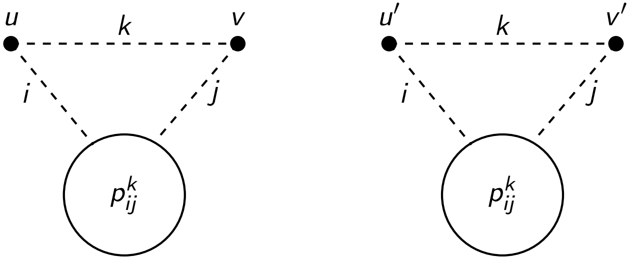


A graph G is distance regular if for any $u, v \in V(G)$ and any $i, j \in \mathbb{Z}$, $|S_i(u) \cap S_j(v)|$ depends only on $\text{dist}(u, v)$.

Distance regular graphs



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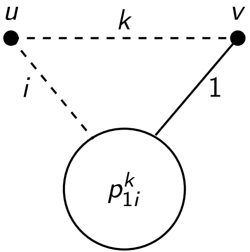
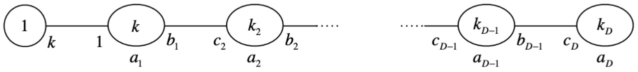


Equivalently, G is distance regular if for any $u, v \in V(G)$ and $i \in \mathbb{Z}$, $|S_i(u) \cap S_1(v)|$ depends only on $\mathrm{dist}(u, v)$.



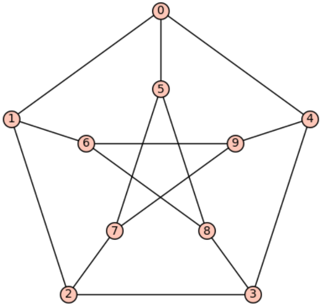
Examples

- C_n
- Petersen graph
- Strongly regular graphs

  Hamming graphs

  Johnson graphs

  Grassmann graphs



$A_0 = I_8$

$$A_1 = \begin{bmatrix} 0&1&1&0&1&0&0&0\\ 1&0&0&1&0&1&0&0\\ 1&0&0&1&0&0&1&0\\ 0&1&1&0&0&0&0&1\\ 1&0&0&0&0&1&1&0\\ 0&1&0&0&1&0&0&1\\ 0&0&1&0&1&0&0&1\\ 0&0&0&1&0&1&1&0 \end{bmatrix}, A_2 = \begin{bmatrix} 0&0&0&1&0&1&1&0\\ 0&0&1&0&1&0&0&1\\ 0&1&0&0&1&0&0&1\\ 1&0&0&0&0&1&1&0\\ 0&1&1&0&0&0&0&1\\ 1&0&0&1&0&0&1&0\\ 1&0&0&1&0&1&0&0\\ 0&1&1&0&1&0&0&0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{bmatrix}$$

$$A_0 + A_1 + A_2 + A_3 = J$$
$$A_1^2 = 2A_2 + 3A_0,$$
$$A_1A_2 = A_2A_1 = 2A_1 + 3A_3$$
$$A_1A_3 = A_3A_1 = A_2$$
$$A_2^2 = 3A_0 + 2A_2$$
$$A_2A_3 = A_3A_2 = A_1$$
$$A_3^2 = A_0$$

Association scheme

A commutative association scheme with d classes on n vertices is a set of $n \times n$ 01-matrices $\mathcal{A} = \{A_0, \ldots, A_d\}$ such that:

- $A_0 = I$
- $\sum_r A_r = J$
- $A_r^T \in \mathcal{A}$ for all r
- For all i, j, $A_i A_j = A_j A_i$ lies in $\mathbb{C}[\mathcal{A}]$, the span of $\mathcal{A}$ over $\mathbb{C}$.

- $v_0(x) = 1, v_1(x) = x, v_2(x) = \frac{1}{2}x^2 - \frac{3}{2}, v_3(x) = \frac{1}{6}x^2 - \frac{7}{6}x$

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An association scheme $\mathcal{A} = \{A_0, \ldots, A_d\}$ is P-polynomial if we can relabel the matrices such that $A_i = v_i(A_1)$ for some polynomial v_i of degree i.

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Examples: Hamming scheme, Johnson scheme...

Non-examples: nonbinary Johnson scheme, association scheme based on attenuated spaces, association scheme based on isotropic spaces...

Equivalence conditions

$$A_iA_j = A_jA_i = \sum_k p_{ij}^k A_k$$

$\mathcal{A} = \{A_0, \ldots, A_d\}$

- $\mathcal{A}$ is a P-polynomial ordering

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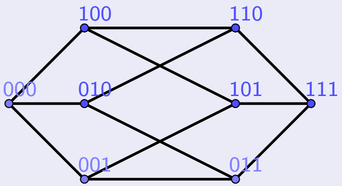

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$\mathcal{A} = \{A_0, \ldots, A_d\}$

- $\mathcal{A}$ is a P-polynomial ordering $(A_i = v_i(A_1))$
- The graph associated to A_1 is distance regular
- $p_{1i}^k \neq 0 \implies i-1 \leq k \leq i+1$, and $p_{1i}^{i+1} \neq 0$, $p_{1i}^{i-1} \neq 0$

Outline

Monomial orders

A monomial order $\leq$ on $\mathbb{C}[x_1, x_2, \dots, x_m]$ is a relation on the set of monomials $x_1^{n_1} x_2^{n_2} \dots x_m^{n_m}$ satisfying:

(i) $\leq$ is a total order

(ii) for monomials u, v, w, if $u \leq v$, then $wu \leq wv$

(iii) $\leq$ is a well-ordering (any non-empty subset of the set of monomials has a minimum element under $\leq$)

An order on $\mathbb{N}^m$ as well

Examples of monomial order

- $\alpha \leq_{\text{lex}} \beta$: the leftmost nonzero entry of $\alpha - \beta$ is negative

  $(0, 2) \leq_{\text{lex}} (1, 0)$

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- $\alpha \leq_{\text{lex}} \beta$: the leftmost nonzero entry of $\alpha - \beta$ is negative
  $(0,2) \leq_{\text{lex}} (1,0)$
- $\alpha \leq_{\text{grlex}} \beta$: $\alpha_1 + \cdots + \alpha_m < \beta_1 + \cdots + \beta_m$ or
  $(\alpha_1 + \cdots + \alpha_m = \beta_1 + \cdots + \beta_m$ and $\alpha \leq_{\text{lex}} \beta)$
  $(1,0) \leq_{\text{grlex}} (0,2) \leq_{\text{grlex}} (1,1)$

Region $\mathcal{D}$

- Bivariate P-polynomial association schemes

  Bernard, Crampe, Poulain d'Andecy, Vinet, Zaimi (2022)
- Multivariate P- and/or Q-polynomial association schemes

  Bannai, Kurihara, Zhao, Zhu (2023)

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$\mathcal{D} \subset \mathbb{N}^m$

(i)

- $\epsilon_1, \epsilon_2, \ldots, \epsilon_m \in \mathcal{D}$

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$\mathcal{D} \subset \mathbb{N}^m$

(i)

- $\epsilon_1, \epsilon_2, \ldots, \epsilon_m \in \mathcal{D}$
- $\leq$ monomial order on $\mathbb{N}^m$

$$\left.\begin{array}{l} \mathbf{n} = (n_1, n_2, \ldots, n_m) \in \mathcal{D} \\ 0 \leq n_i' \leq n_i \text{ for all } i \end{array}\right\} \Longrightarrow \mathbf{n}' = (n_1', n_2', \ldots, n_m') \in \mathcal{D}$$

m-variate P-polynomial association scheme

$\leq$ monomial order on $\mathbb{N}^m$

$\mathcal{D} \subset \mathbb{N}^m$ satisfying (i) ($\epsilon_i \in \mathcal{D}$ and 'boxing property')

A commutative association scheme $\mathcal{A}$ is called *m-variate P-polynomial* on $\mathcal{D}$ with respect to $\leq$ if

(ii) There exists a relabeling of the elements of $\mathcal{A} = \{A_{\mathbf{n}} \mid \mathbf{n} \in \mathcal{D}\}$ such that for all $\mathbf{n} \in \mathcal{D}$ we have

$$A_{\mathbf{n}} = v_{\mathbf{n}}(A_{\epsilon_1}, A_{\epsilon_2}, \ldots, A_{\epsilon_m}),$$

where $v_{\mathbf{n}}(\boldsymbol{x})$ is an m-variate polynomial of degree $\mathbf{n}$

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where $v_{\mathbf{n}}(\boldsymbol{x})$ is an m-variate polynomial of degree $\mathbf{n}$ and all monomials $\boldsymbol{x}^\beta$ in $v_n(\boldsymbol{x})$ satisfy $\beta \in \mathcal{D}$

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(iii) For $i = 1, 2, \ldots, m$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathcal{D}$, the product $A_{\epsilon_i} A_{\epsilon_1}^{\alpha_1} A_{\epsilon_2}^{\alpha_2} \ldots A_{\epsilon_m}^{\alpha_m}$ is a linear combination of

$$\{A_{\epsilon_1}^{\beta_1} A_{\epsilon_2}^{\beta_2} \ldots A_{\epsilon_m}^{\beta_m} \mid \beta = (\beta_1, \ldots, \beta_m) \in \mathcal{D},\ \beta \leq \alpha + \epsilon_i\}.$$

An equivalent condition

$\mathcal{A} = \{A_{\mathbf{i}} \mid \mathbf{i} \in \mathcal{D}\}$ is a commutative association scheme

Then the following two statements are equivalent:

(i) $\mathcal{A}$ is an m-variate P-polynomial association scheme on $\mathcal{D}$ with respect to a monomial order $\leq$

(ii) for $i = 1, 2, \ldots, m$ and $\alpha \in \mathcal{D}$,
$p^{\beta}_{\epsilon_i,\alpha} \neq 0 \implies (\beta \leq \alpha + \epsilon_i \text{ and } \alpha \leq \beta + \epsilon_i)$
$p^{\alpha+\epsilon_i}_{\epsilon_i,\alpha} \neq 0$, $p^{\alpha-\epsilon_i}_{\epsilon_i,\alpha} \neq 0$

Examples

- Direct product of P-polynomial association schemes
- Extensions of an association scheme
- Nonbinary Johnson scheme
- Association scheme based on attenuated spaces
- Association scheme based on isotropic spaces
- Generalized 24-cell

m-length of walks

$G = (X, E_1 \sqcup \cdots \sqcup E_m)$ a connected graph ($E_i \cap E_j = \emptyset$ if $i \neq j$)

$\xi = (e_1, e_2, \ldots, e_L)$ a walk on G.

The m-length $\ell_m(\xi)$ of ξ with respect to the m-partition $\{E_i | \ i = 1, 2, \ldots, m\}$ of $E(G)$ is

$\ell_m(\xi) = (|\{j \mid e_j \in E_1\}|, |\{j \mid e_j \in E_2\}|, \ldots, |\{j \mid e_j \in E_m\}|)$.

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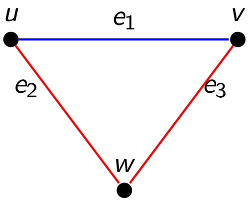

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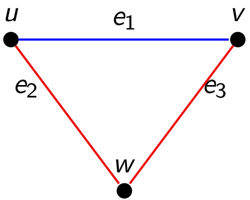


$E_1 = \{e_1\}$

$E_2 = \{e_2, e_3\}$

$\ell_2((e_1)) = (1, 0)$

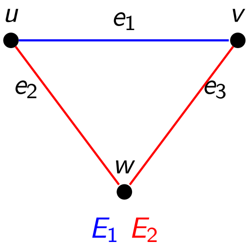
$\ell_2((e_2 e_3)) = (0, 2)$

m-distance between vertices

$\leq$ a monomial order on $\mathbb{C}[x_1, \ldots, x_m]$

The *m*-distance d_m between $x, y \in V(G)$ is

$d_m(x, y) = \min_{\leq}\{\ell_m(\xi) \mid \xi$ is a walk between x and $y\}$.



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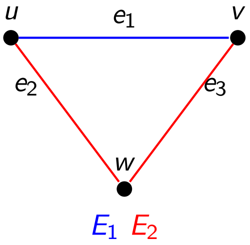

$\leq_{\text{lex}}$: $d_2(u, v) = (0, 2)$

$\leq_{\text{grlex}}$: $d_2(u, v) = (1, 0)$

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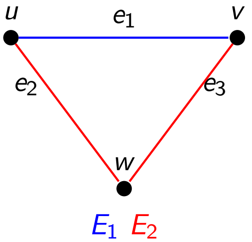


$\leq_{\text{lex}}$: $d_2(u, v) = (0, 2)$

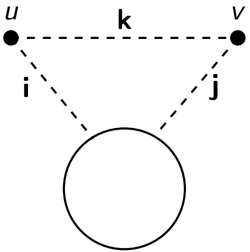
$\leq_{\text{grlex}}$: $d_2(u, v) = (1, 0)$

$d_m(x, y) = \alpha$,

$\beta \leq \alpha$, there may not be vertices at m-distance β

$\gamma_i \leq \alpha_i$, there may not be vertices at m-distance γ.

m-distance regular graphs $G = (V, E_1 \sqcup \cdots \sqcup E_m)$



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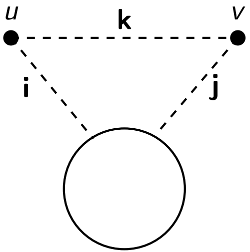

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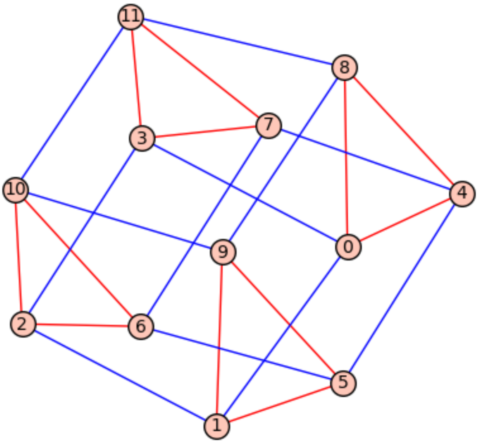
$d_m(x, y) = \alpha$,
$\gamma_i \leq \alpha_i$, there are vertices at m-distance γ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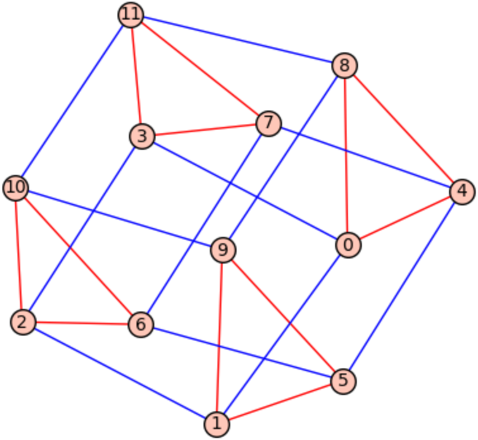


Cartesian product of distance regular graphs

Connection

$\leq$ a monomial order on $\mathbb{N}^m$

$\mathcal{D} \subset \mathbb{N}^m$ satisfies condition (i)

$\mathcal{A} = \{A_{\mathbf{n}} | \ \mathbf{n} \in \mathcal{D}\}$ a symmetric association scheme on a set X

Define $E_i \subseteq X \times X$ by $(x, y) \in E_i \quad \Longleftrightarrow \quad (A_{\epsilon_i})_{xy} = 1$

The following statements are equivalent:

(i) $\mathcal{A}$ is an m-variate P-polynomial association scheme on $\mathcal{D}$ with respect to $\leq$

(ii) $G = (X, E_1 \sqcup \cdots \sqcup E_m)$ is m-distance-regular with respect to the edge partition $\{E_i \mid i = 1, 2, \ldots, m\}$ and $\leq$
$\mathcal{D}$ is the set of all m-distances in G and the matrix A_ℓ is the ℓ-th m-distance matrix of G for all $\ell \in \mathcal{D}$

More examples

- Direct product of P-polynomial association schemes

More examples

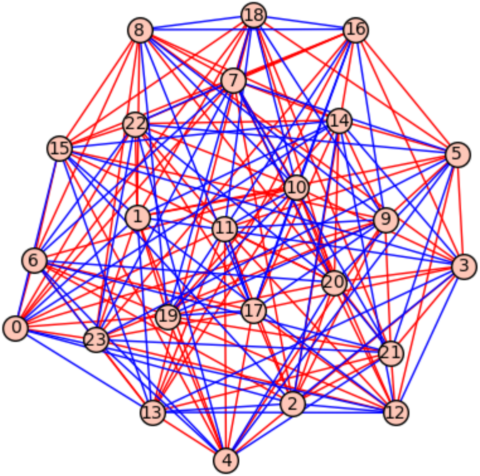
- Direct product of P-polynomial association schemes $\leftrightarrow$ Cartesian product of distance regular graphs
- $\mathcal{A} = \{A_0, \ldots, A_d\}$ an association scheme on q vertices
  Extension (symmetrization) of $\mathcal{A}$ of length n

More examples

- Direct product of P-polynomial association schemes $\leftrightarrow$ Cartesian product of distance regular graphs
- $\mathcal{A} = \{A_0, \ldots, A_d\}$ an association scheme on q vertices
  Extension (symmetrization) of $\mathcal{A}$ of length $n \leftrightarrow$
  Hamming graph $H(n, q)$, d-distance regular

24-cell, $(\pm 1, \pm 1, 0, 0)$

2, $\sqrt{6}$,



There exists at most one m-variate P-polynomial association scheme $\mathcal{Z}$ with order $\leq$ and with fixed generating matrices A_{ϵ_i}, $i = 1, 2 \ldots, m$.

More

- Terwilliger algebra of m-variate P- and Q-polynomial association schemes
- For a given $k = (k_1, k_2, \ldots, k_m)$, are there finitely many m-distance-regular graph $G = (X, \Gamma_1 \sqcup \Gamma_2 \sqcup \cdots \sqcup \Gamma_m)$ such that the graph (X, Γ_i) is k_i-regular?
- Partial order $\preceq$ such that $\alpha \preceq \beta \implies \alpha \leq \beta$
- Generators

Thank you!