The q-Onsager algebra and the quantum torus: 2025 Edition

TerwilligerFest 2025 —— Combinatorics around the q-Onsager Algebra

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- Throughout this paper, t is an indeterminate.
- For any integer m, we define $\delta_{m,ev}$ to equal 1 if m is even and 0 if m is odd.

What is the q-Onsager Algebra?

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Definition 1 (See (Terwilliger 1993, Lemma 5.4).)

The q**-Onsager algebra**, denoted O_q, is the algebra defined by generators W_0, W_1 and relations

$$[W_0, [W_0, [W_0, W_1]_q]_{q^{-1}}] = -(q^2 - q^{-2})^2 [W_0, W_1], \qquad (1)$$

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We call (1) and (2) the *q-Dolan-Grady relations.*

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Sidenote

In their paper, Baseilhac and Kolb describe the next set of elements as being contained in the algebra $\mathcal{B}_c$. This is another version of the O_q, where $O_q \cong \mathcal{B}_c/(c - q^{-1}(q-q^{-1})^2)$.

(1) The Baseilhac-Kolb Elements of O_q

Definition 2 (See (Pascal Baseilhac and Kolb 2020, Section 3).)

In the algebra O_q, we define the elements

$$\{B_{n\delta+\alpha_0}\}_{n=0}^{\infty}, \qquad \{B_{n\delta+\alpha_1}\}_{n=0}^{\infty}, \qquad \{B_{n\delta}\}_{n=1}^{\infty} \tag{3}$$

in the following way:

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in the following way:

$$B_\delta = q^{-2} W_1 W_0 - W_0 W_1,$$
$$B_{\alpha_0} = W_0,$$
$$B_{\delta+\alpha_0} = W_1 + \frac{q[B_\delta, W_0]}{(q-q^{-1})(q^2-q^{-2})},$$
$$B_{n\delta+\alpha_0} = B_{(n-2)\delta+\alpha_0} + \frac{q[B_\delta, B_{(n-1)\delta+\alpha_0}]}{(q-q^{-1})(q^2-q^{-2})}, \qquad n \geq 2,$$
$$B_{\alpha_1} = W_1,$$

The Baseilhac-Kolb Elements of O_q, continued

Definition 2, continued

$$B_{\delta+\alpha_1} = W_0 - \frac{q[B_\delta, W_1]}{(q-q^{-1})(q^2-q^{-2})},$$

$$B_{n\delta+\alpha_1} = B_{(n-2)\delta+\alpha_1} - \frac{q[B_\delta, B_{(n-1)\delta+\alpha_1}]}{(q-q^{-1})(q^2-q^{-2})}, \qquad n \geq 2,$$

$$B_{n\delta} = q^{-2} B_{(n-1)\delta+\alpha_1} W_0 - W_0 B_{(n-1)\delta+\alpha_1} + (q^{-2}-1)\sum_{\ell=0}^{n-2} B_{\ell\delta+\alpha_1} B_{(n-\ell-2)\delta+\alpha_1}, \qquad n \geq 2.$$

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B_{\delta+\alpha_1} &= W_0 - \frac{q[B_\delta, W_1]}{(q-q^{-1})(q^2-q^{-2})}, \\
B_{n\delta+\alpha_1} &= B_{(n-2)\delta+\alpha_1} - \frac{q[B_\delta, B_{(n-1)\delta+\alpha_1}]}{(q-q^{-1})(q^2-q^{-2})}, \qquad n \geq 2, \\
B_{n\delta} &= q^{-2} B_{(n-1)\delta+\alpha_1} W_0 - W_0 B_{(n-1)\delta+\alpha_1} \\
&\quad + (q^{-2}-1) \sum_{\ell=0}^{n-2} B_{\ell\delta+\alpha_1} B_{(n-\ell-2)\delta+\alpha_1}, \qquad n \geq 2.
\end{aligned}
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FACT: The $B_{n\delta}$ elements mutually commute.

The Baseilhac-Kolb Elements of O_q, continued

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$$B_{\delta+\alpha_1} = W_0 - \frac{q[B_\delta, W_1]}{(q-q^{-1})(q^2-q^{-2})},$$

$$B_{n\delta+\alpha_1} = B_{(n-2)\delta+\alpha_1} - \frac{q[B_\delta, B_{(n-1)\delta+\alpha_1}]}{(q-q^{-1})(q^2-q^{-2})}, \qquad n \geq 2,$$

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FACT: The $B_{n\delta}$ elements mutually commute.
We call the elements in (3) the *Baseilhac-Kolb* elements of O_q.
For notational convenience, we define $B_{0\delta} = q^{-2} - 1$.

(2) Alternating Elements of O_q

Definition 3 (See (P. Baseilhac and Shigechi 2010, Definition 3.1).)

Define the algebra $\mathcal{O}_q$ by the generators

$$\{\mathcal{W}_{-k}\}_{k=0}^{\infty}, \qquad \{\mathcal{W}_{k+1}\}_{k=0}^{\infty}, \qquad \{\mathcal{G}_{k+1}\}_{k=0}^{\infty}, \qquad \{\tilde{\mathcal{G}}_{k+1}\}_{k=0}^{\infty},$$

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and 13 sets of commutator relations.

Lemma 5

There is an algebra homomorphism $\gamma : \mathcal{O}_q \mapsto O_q$ that sends

$$\mathcal{W}_0 \mapsto W_0, \qquad \mathcal{W}_1 \mapsto W_1.$$

The map γ and the alternating elements of O_q

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Definition 4 (See (Terwilliger 2022, Definition 11.5).)

For $k \in \mathbb{N}$, define

$$W_{-k} = \gamma(\mathcal{W}_{-k}), \qquad W_{k+1} = \gamma(\mathcal{W}_{k+1}),$$
$$G_{k+1} = \gamma(\mathcal{G}_{k+1}), \qquad \tilde{G}_{k+1} = \gamma(\tilde{\mathcal{G}}_{k+1}). \tag{4}$$

We call these images the *alternating elements* of O_q. For notational convenience, we define

$$G_0 = \tilde{G}_0 = -\left(q - q^{-1}\right)\left(q + q^{-1}\right)^2.$$

(3) The Lu-Wang Elements

Definition 5 (See (Lu and Wang 2021, Definition 2.1).)

Let $\tilde{\mathbf{U}}^\imath$ denote the algebra defined by generators B_0, B_1, $\mathbb{K}_0^{\pm 1}$, $\mathbb{K}_1^{\pm 1}$ and the following relations:

$$\mathbb{K}_1\mathbb{K}_1^{-1} = 1 = \mathbb{K}_1^{-1}\mathbb{K}_1, \qquad \mathbb{K}_0\mathbb{K}_0^{-1} = 1 = \mathbb{K}_0^{-1}\mathbb{K}_0,$$

$$\mathbb{K}_0, \mathbb{K}_1 \text{ are central,}$$

$$[B_0, [B_0, [B_0, B_1]_q]_{q^{-1}}] = -q^{-1}(q + q^{-1})^2[B_0, B_1]\mathbb{K}_0,$$
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The algebra $\tilde{\mathbf{U}}^\imath$ is known as the *universal q-Onsager algebra.*

The map υ

Lemma 6 (See (Terwilliger 2022, Remark 5.7).)

There exists a surjective algebra homomorphism $\upsilon : \tilde{\mathbf{U}}^\imath \mapsto O_q$ that sends

$$B_0 \mapsto \frac{W_0}{q^{1/2}(q-q^{-1})}, \qquad B_1 \mapsto \frac{W_1}{q^{1/2}(q-q^{-1})}, \qquad \mathbb{K}_0, \mathbb{K}_1 \mapsto 1.$$

Some elements of $\tilde{\mathbf{U}}^\imath$

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For notational convenience, we define $\Theta'_n = \Theta_n = 0$ for $n < 0$.

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Definition 6 (See (Lu and Wang 2021, Section 2).)

The elements $\{B_{1,r}\}_{r\in\mathbb{Z}}$ of $\tilde{\mathbf{U}}^\imath$ satisfy $B_{1,0} = B_1$, $B_{1,-1} = B_0\mathbb{K}_0^{-1}$, and for $\ell \in \mathbb{Z}$,

$$[\Theta_1, B_{1,\ell}] = (q + q^{-1})\left(B_{1,\ell+1} - B_{1,\ell-1}\mathbb{K}_\delta\right) \qquad (\mathbb{K}_\delta = \mathbb{K}_0\mathbb{K}_1).$$

The B and Θ elements

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Definition 7

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$$\Theta_n = \Theta'_n - \delta_{n,ev} q^{1-n}\mathbb{K}_\delta^{n/2} - \sum_{\ell=1}^{\lfloor \frac{n-1}{2} \rfloor} (q^2-1) q^{-2\ell} \Theta'_{n-2\ell}\mathbb{K}_\delta^{\ell}.$$

The H' and H elements

The elements $\{H'_n\}_{n=1}^{\infty}$ and $\{H_n\}_{n=1}^{\infty}$ have applications to $\imath$Hall algebras and are defined using generating functions.

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Definition 8

We define the following generating functions for $\tilde{\mathbf{U}}^{\imath}$:

$$\Theta'(t) = (q - q^{-1}) \sum_{n=0}^{\infty} \Theta'_n t^n, \qquad \Theta(t) = (q - q^{-1}) \sum_{n=0}^{\infty} \Theta_n t^n.$$

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Next, we define the generating functions $H'(t)$ and $H(t)$ by

$$\exp\Big((q - q^{-1}) H'(t)\Big) = \Theta'(t), \qquad \exp\Big((q - q^{-1}) H(t)\Big) = \Theta(t).$$

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Since the constant term of $\Theta'(t)$ is 1, the constant term of $H'(t)$ is 0. By a similar argument, the constant term of $H(t)$ is 0.

The H and H' elements, continued.

Definition 9

For $n \geq 1$, we define H'_n and H_n by

$$H'(t) = \sum_{n=1}^{\infty} H'_n t^n, \qquad H(t) = \sum_{n=1}^{\infty} H_n t^n. \tag{5}$$

(Recall that the constant term of each generating function is 0.)

The H and H' elements, continued.

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For $n \geq 1$, we define H'_n and H_n by

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(Recall that the constant term of each generating function is 0.)

Woof. We've now defined all of the elements of $\tilde{\mathbf{U}}^{\imath}$ with images that we care about.

The image of $B_{1,r}$ in O_q

Proposition 2

For $n \geq 0$, the map υ sends

$$B_{1,-n-1} \mapsto \frac{B_{n\delta+\alpha_0}}{q^{1/2}(q-q^{-1})},$$

$$B_{1,n} \mapsto \frac{B_{n\delta+\alpha_1}}{q^{1/2}(q-q^{-1})}.$$

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For $n \geq 0$, the map υ sends

$$\Theta'_n \mapsto -\frac{qB_{n\delta}}{(q-q^{-1})^2}.$$

Analyzing the algebra $\tilde{\mathbf{U}}^\imath$.

For notational convenience, for the elements

$$\{\Theta'_n\}_{n=0}^{\infty}, \qquad \{\Theta_n\}_{n=0}^{\infty}, \qquad \{H'_n\}_{n=1}^{\infty}, \qquad \{H_n\}_{n=1}^{\infty}, \qquad \{B_{1,r}\}_{r\in\mathbb{Z}}$$

of $\tilde{\mathbf{U}}^\imath$, we retain the same notation for their υ-images in O_q.

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of $\tilde{\mathbf{U}}^\imath$, we retain the same notation for their v-images in O_q.
These images are called the **Lu-Wang elements** of O_q.

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For instance, in O_q, we have

$$\Theta'_n = -\frac{qB_{n\delta}}{(q-q^{-1})^2}, \qquad n \geq 0.$$

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For instance, in O_q, we have

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In a similar fashion, for the generating functions

$$\Theta'(t), \qquad \Theta(t), \qquad H'(t), \qquad H(t)$$

of $\tilde{\mathbf{U}}^\imath$, we retain the same notation for their υ-images in O_q.

The Sad Truth

The three classes of elements of O_q we have mentioned so far (Baseilhac-Kolb, Alternating, Lu-Wang) all suffer from the same affliction.

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These elements are defined by recursive formulas and generating functions, and there are no known formulas for them as polynomials in the generators W_0 and W_1 of O_q.

Our primary goal is to make these elements understandable. We introduce a simple but deep algebra called the *quantum torus* (denoted T_q) and introduce a homomorphism $p : O_q \mapsto T_q$. It so happens that the p-images of these elements of O_q have aesthetically pleasing forms in T_q.

The algebra T_q

Definition 10 (See (Gupta 2011).)

Define the algebra T_q by generators

$$x, y, x^{-1}, y^{-1}$$

and relations

$$xx^{-1} = 1 = x^{-1}x, \qquad yy^{-1} = 1 = y^{-1}y, \qquad xy = q^2yx.$$

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The algebra T_q is called the *quantum torus*.
By (Gupta 2011, p. 3), the vector space T_q has a basis consisting of $\{x^a y^b | a, b \in \mathbb{Z}\}$.

An algebra homomorphism from O_q to T_q

Definition 11

For the algebra T_q, define

$$w_0 = x + x^{-1}, \qquad w_1 = y + y^{-1}.$$

In the algebra T_q, we have

$$[w_0, [w_0, [w_0, w_1]_q]_{q^{-1}}] = -(q^2 - q^{-2})^2 [w_0, w_1],$$
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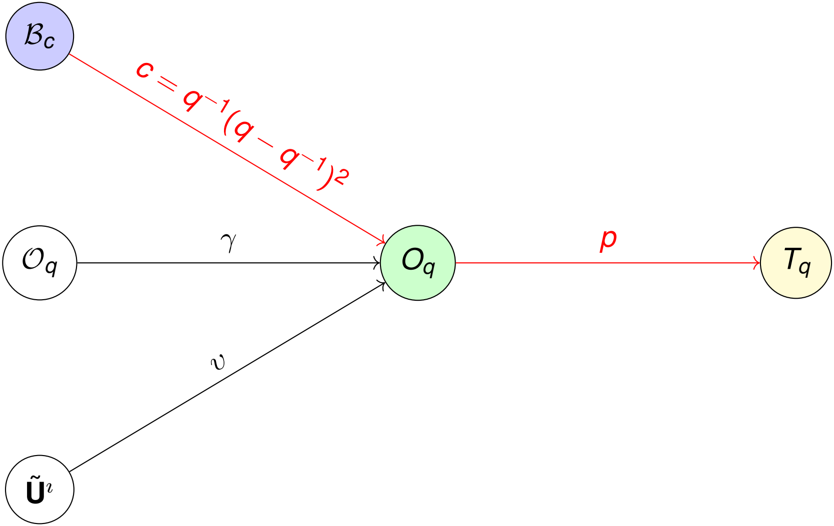
Proposition 4

There exists an algebra homomorphism $p : O_q \mapsto T_q$ that sends $W_0 \mapsto w_0$ and $W_1 \mapsto w_1$.

The Images of the Baseilhac-Kolb Elements



The elements $B_{n\delta+\alpha_0}$ and $B_{n\delta+\alpha_1}$

Theorem 1

For $n \geq 0$, the map p sends

$$B_{n\delta+\alpha_0} \mapsto x(yx)^n + x^{-1}(y^{-1}x^{-1})^n,$$
$$B_{n\delta+\alpha_1} \mapsto y(xy)^n + y^{-1}(x^{-1}y^{-1})^n.$$

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This follows from the recurrences by which these elements are defined.

The $B_{n\delta}$ elements

Next, we apply the map p to the elements $\{B_{n\delta}\}_{n=1}^{\infty}$ of O_q.

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Theorem 2

For $n \geq 1$, the map p sends

$$B_{n\delta} \mapsto (q^{-2} - 1)\Bigg(q^{-n}[n+1]_q (xy)^n + q^n [n+1]_q (xy)^{-n} + \sum_{\ell=1}^{n-1} (1 + q^{4\ell - 2n})(xy)^{n-2\ell} \Bigg).$$

The $B_{n\delta}$ elements

Next, we apply the map p to the elements $\{B_{n\delta}\}_{n=1}^{\infty}$ of O_q.

Theorem 2

For $n \geq 1$, the map p sends

$$B_{n\delta} \mapsto (q^{-2} - 1)\Big(q^{-n}[n+1]_q (xy)^n + q^n [n+1]_q (xy)^{-n} + \sum_{\ell=1}^{n-1} (1 + q^{4\ell - 2n})(xy)^{n-2\ell} \Big).$$

Alternatively (if you prefer fractions to sums),...

The $B_{n\delta}$ elements – alternate version

Theorem 2A

For $n \geq 1$, the map p sends

$$B_{n\delta} \mapsto (q^{-2} - 1)\Bigg(q^{-n}[n+1]_q(xy)^n + q^n[n+1]_q(xy)^{-n} + \frac{(xy)^{n-1} - (xy)^{1-n}}{xy - (xy)^{-1}} + \frac{(yx)^{n-1} - (yx)^{1-n}}{yx - (yx)^{-1}} \Bigg).$$

The $B_{n\delta}$ elements – alternate version

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For $n \geq 1$, the map p sends

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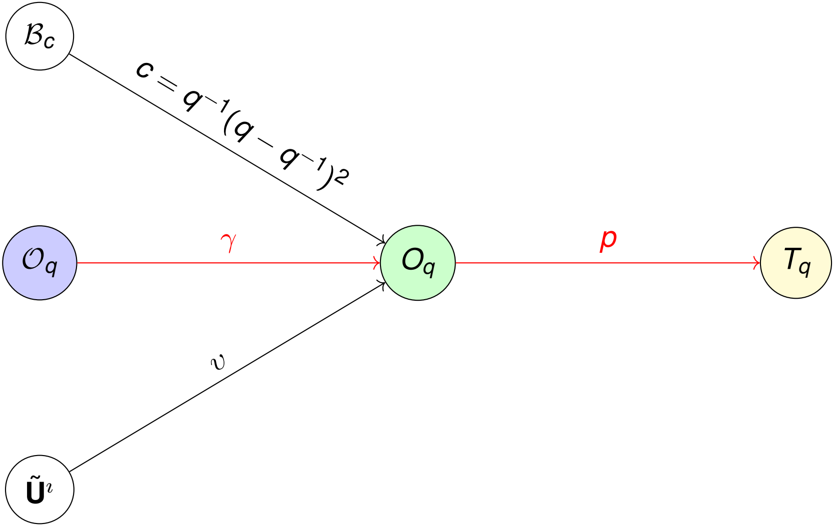
$$\left. + \frac{(xy)^{n-1} - (xy)^{1-n}}{xy - (xy)^{-1}} + \frac{(yx)^{n-1} - (yx)^{1-n}}{yx - (yx)^{-1}} \right).$$

Sidenote

Now you may be wondering – T_q is not a commutative algebra . . . are fractions even allowed? You may notice that the image of $B_{n\delta}$ is contained in the subalgebra of T_q generated by (xy) (recall that $yx = q^{-2}xy$). We denote this subalgebra by $\widehat{T_q}$.

Alternating Elements



The Alternating Elements of O_q

We switch gears now and consider the alternating elements of O_q, denoted by

$$\{W_{-k}\}_{k=0}^{\infty}, \qquad \{W_{k+1}\}_{k=0}^{\infty}, \qquad \{G_{k+1}\}_{k=0}^{\infty}, \qquad \{\tilde{G}_{k+1}\}_{k=0}^{\infty},$$

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We begin by defining notation for their images in T_q. Our goal is to express these images in closed form in terms of x and y.

Definition 12

For $k \geq 0$, we define the following elements of T_q:

$$w_{-k} = p(W_{-k}), \quad w_{k+1} = p(W_{k+1}),$$
$$g_{k+1} = p(G_{k+1}), \quad \tilde{g}_{k+1} = p(\tilde{G}_{k+1}).$$

Generating Functions

Some of the results in this section are more succinctly written in terms of generating functions.

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Definition 13 (See (Terwilliger 2022, Definition 12.1))

For the algebra O_q, define the generating functions

$$W^-(t) = \sum_{i=0}^{\infty} W_{-i} t^i, \qquad W^+(t) = \sum_{i=0}^{\infty} W_{i+1} t^i,$$

$$G(t) = \sum_{i=0}^{\infty} G_i t^i, \qquad \tilde{G}(t) = \sum_{i=0}^{\infty} \tilde{G}_i t^i.$$

Let $w^-(t), w^+(t), g(t), \tilde{g}(t)$ denote their p-images in T_q.

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Lemma 12

Define $T = \frac{q+q^{-1}}{qt+q^{-1}t^{-1}}$ and $S = \frac{q+q^{-1}}{qt^{-1}+q^{-1}t}$.
Then,

$$\eta(T) = \frac{1+q^2t^2}{1-q^2t^2}, \qquad \eta(S) = \frac{1+q^{-2}t^2}{1-q^{-2}t^2}.$$

The Process of Finding Closed Forms

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By (Terwilliger 2021b, Lemma 8.22), there is a unique set of generating functions $w^+(t), w^-(t), g(t), \tilde{g}(t)$ that satisfy this equation

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By (Terwilliger 2021b, Lemma 8.22), there is a unique set of generating functions $w^+(t), w^-(t), g(t), \tilde{g}(t)$ that satisfy this equation as well as the generating function forms (at the level of T_q) of the defining relations for $\mathcal{O}_q$, which will be omitted.

The Alternating Generating Functions in T_q

It can be shown that these are the generating functions that work.

Theorem 3

The generating functions in T_q described above take the following form:

$$w^-(t) = \eta(t)(x + x^{-1})$$

$$w^+(t) = \eta(t)(y + y^{-1})$$

$$\tilde{g}(t) = (q^2 - q^{-2})\eta(t)\left((q^{-1}xy + q^{-1}x^{-1}y^{-1})t - (q + q^{-1})\right)$$

$$g(t) = (q^2 - q^{-2})\eta(t)\left((qx^{-1}y + qxy^{-1})t - (q + q^{-1})\right)$$

The p-Images of the Alternating Elements of O_q

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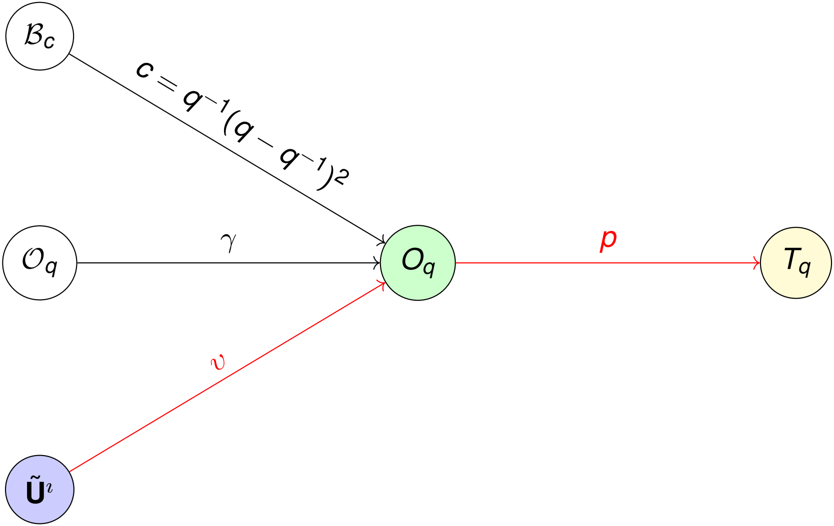
The Lu-Wang elements of O_q



The $B_{1,r}$ elements

Theorem 5

For $r \in \mathbb{Z}$, the map p sends

$$B_{1,r} \mapsto \frac{y(xy)^r + y^{-1}(x^{-1}y^{-1})^r}{q^{1/2}(q - q^{-1})}.$$

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For $r \in \mathbb{Z}$, the map p sends

$$B_{1,r} \mapsto \frac{y(xy)^r + y^{-1}(x^{-1}y^{-1})^r}{q^{1/2}(q - q^{-1})}.$$

This is because $B_{1,r}$ is merely a scalar multiple of $B_{r\delta+\alpha_1}$ (for $r \geq 0$) or $B_{(-r-1)\delta+\alpha_0}$ (for $r < 0$).

Next we have Θ'_n

Recall that Θ'_n (as an element of O_q) is a scalar multiple of $B_{n\delta}$. This gives us the following theorem.

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Theorem 6

For $n \geq 1$, the map p sends

$$
\begin{aligned}
\Theta'_n \mapsto & \frac{1}{q-q^{-1}}\Big(q^{-n}[n+1]_q(xy)^n + q^n[n+1]_q(xy)^{-n} \\
& + \sum_{\ell=1}^{n-1}(1+q^{4\ell-2n})(xy)^{n-2\ell}\Big). \\
= & \frac{1}{q-q^{-1}}\Big(q^{-n}[n+1]_q(xy)^n + q^n[n+1]_q(xy)^{-n} \\
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The Θ′ elements – 2 of 2

Theorem 6A

For $n \geq 1$, the map p sends

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The Θ_n Elements – a slight detour

While the p-images of the Θ_n elements of O_q can be identified using the definition of Θ_n, it is more reasonable to use generating functions.

Lemma 13 (See (Lu and Wang 2021, Lemma 2.9).)

$$\Theta(t) = \frac{1 - t^2}{1 - q^{-2}t^2}\Theta'(t).$$

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$$\Theta(t) = \frac{1 - t^2}{1 - q^{-2}t^2}\Theta'(t).$$

This formula allows us to “traverse three sides of the square”:

$$\begin{array}{ccc} \Theta'_n & \longrightarrow & \Theta'(t) \\ \downarrow & & \downarrow \\ \Theta_n & \longleftarrow & \Theta(t) \end{array}$$

The images of $\Theta'(t)$ and $\Theta(t)$

Using partial fractions and some algebra, we routinely find a nice form for the p-image of the generating function $\Theta'(t)$.

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Theorem 7

The map p sends

$$\Theta'(t) \mapsto \frac{(1-q^2t^2)(1-q^{-2}t^2)}{(1-xyt)(1-yxt)(1-x^{-1}y^{-1}t)(1-y^{-1}x^{-1}t)}.$$

By the formula from the last slide, the map p sends

$$\Theta(t) \mapsto \frac{(1-q^2t^2)(1-t^2)}{(1-xyt)(1-yxt)(1-x^{-1}y^{-1}t)(1-y^{-1}x^{-1}t)}.$$

The image of Θ_n

A similar partial fraction decomposition gives the image of Θ_n.

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Theorem 9

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$$\begin{aligned}\Theta_n \mapsto & [n+1]_q \frac{(qyx)^n + (qyx)^{-n}}{q - q^{-1}} \\ & + \frac{q+q^{-1}}{q-q^{-1}} q^{1-n} \sum_{\ell=1}^{n-1} (qyx)^{n-2\ell} \\ = & \frac{[n+1]_q}{q-q^{-1}} \frac{(qyx)^{n+1} - (qyx)^{-n-1}}{qyx - (qyx)^{-1}} \\ & - \frac{q^2[n-1]_q}{q-q^{-1}} \frac{(qyx)^{n-1} - (qyx)^{1-n}}{qyx - (qyx)^{-1}}.\end{aligned}$$

The Final Stage – The H_n' and H_n elements

Our last goal is to find the p-images of the elements H_n' and H_n of O_q

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$$\exp\Big((q-q^{-1})H'(t)\Big) = \Theta'(t), \qquad \exp\Big((q-q^{-1})H(t)\Big) = \Theta(t),$$

or, alternatively,

$$H'(t) = (q-q^{-1})^{-1}\log(\Theta'(t)), \qquad H(t) = (q-q^{-1})^{-1}\log(\Theta(t))$$

where log is the formal natural logarithm

$$\log(1-z) = -\sum_{n=1}^{\infty}\frac{z^n}{n}.$$

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These formulas routinely give the p-images of the generating functions $H'(t)$ and $H(t)$.

The Images of $H'(t)$ and $H(t)$

Theorem 10

The map p sends

$$H'(t) \mapsto \frac{\log(1 - xy) + \log(1 - yx) + \log(1 - x^{-1}y^{-1})}{q - q^{-1}} + \frac{\log(1 - y^{-1}x^{-1}) - \log(1 - q^2t^2) - \log(1 - q^{-2}t^2)}{q - q^{-1}}.$$

$$H(t) \mapsto \frac{\log(1 - xy) + \log(1 - yx) + \log(1 - x^{-1}y^{-1})}{q - q^{-1}} + \frac{\log(1 - y^{-1}x^{-1}) - \log(1 - q^2t^2) - \log(1 - t^2)}{q - q^{-1}}.$$

The images of H'_n and H_n

If we expand out the logarithm Taylor Series, we obtain the p-images of H'_n and H_n, which have convenient closed forms.

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Theorem 11

For $n \geq 1$, the map p sends

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4. For more details, consider my papers:

   `arxiv:2304.09326` ((Goff 2024)),
   `arxiv:2504.13362`