Codes over Finite Ring $\mathbb{Z}_k$, MacWilliams Identity and Theta Function

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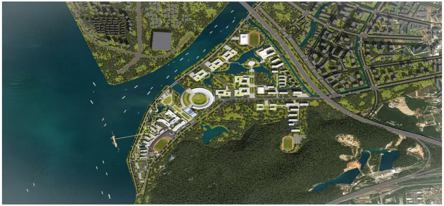

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Introduction

We study linear codes over $\mathbb{Z}_k$ based on lattices and theta functions:

- We obtain the complete weight enumerators MacWilliams identity and the symmetrized weight enumerators MacWilliams identity based on the theory of theta function.
- We extend the main work by Bannai, Dougherty, Harada and Oura to the finite ring $\mathbb{Z}_k$ for any positive integer k and present the complete weight enumerators MacWilliams identity in genus g.
- When $k = p$ is a prime number, we establish the relationship between the theta function of associated lattices over a cyclotomic field and the complete weight enumerators with Hamming weight of codes, which is an analogy of the results by G. Van der Geer and F. Hirzebruch since they showed the identity with the Lee weight enumerators.

Related work

- Conway et al.(1988) presented the foundational work about the relationship between binary codes and lattices in 1988.
- Bonnecaze et al.(1995) marked a pivotal shift by constructing lattices from linear codes over $\mathbb{Z}_4$, revealing properties analogous to those of binary codes.
- Bannai et al.(1999) established correspondences between Type II codes over $\mathbb{Z}_{2k}$ and even unimodular lattices.
- $\cdots$

Extensions of the MacWilliams Identity

The MacWilliams theorem for linear codes over a finite field $\mathbb{F}_q$ establishes an identity that relates the weight enumerators of a code to the weight enumerators of its dual code. There are numerous generalizations over the MacWilliams identity:

- Code types: Cyclic codes/convolutional codes $\rightarrow$ Codes over finite rings(special rings: Galois ring), [Blake99][Nebe04][Suzuki12]
- Context extension: Finite fields $\mathbb{F}_q$ $\rightarrow$ Finite rings (e.g., $\mathbb{Z}_k$, Frobenius rings) [Klemm87]
- Weight types: Hamming $\rightarrow$ Complete/Symmetric/Lee weight enumerators [Britz02]

Limitations

- Bannai et al. only gave the complete weight enumerators MacWilliams identity in genus g for codes over $\mathbb{Z}_k$ when k is an **even number**.
- Hirzebruch only showed that the MacWilliams identity with the Lee weight enumerator when $k = p$ **is a prime number** based on the ring of algebraic integers over a cyclotomic field (Hirzebruch 1973, 1987). The case of an arbitrary positive integer k remained unresolved.
- $\cdots$

Contributions 1-2

We obtain a unified framework for linear codes over $\mathbb{Z}_k$ with an **arbitrary positive integer** k.

- By defining k auxiliary theta functions, we establish an explicit relation between the theta function of the lattice associated with code $C \subset \mathbb{Z}_k^n$ and the complete weight enumerators of C (Theorem 1), from which the complete MacWilliams identity arises naturally. This is a modification of the result by F. Hirzebruch since he constructed the identity with the Lee weight enumerators.
- Derive the complete weight enumerators and the symmetrized weight enumerators MacWilliams identity for codes over $\mathbb{Z}_k$ using the theory of theta functions (Theorem 2), thereby providing a novel interpretation in comparison with traditional proofs.

Contributions 1-2

The first complete weight enumerators identity was given by MacWilliams in 1972. The complete weight enumerators of C is

$$W_C(X_0, X_1, \cdots, X_{q-1}) = \sum_{c \in C} X_0^{w_0(c)} X_1^{w_1(c)} \cdots X_{q-1}^{w_{q-1}(c)}.$$

Proposition 1 Let $C \subset \mathbb{Z}_k^n$ be a k-ary code and Γ_C be the associated lattice of C. We have the following results:
(1) $C \subset C^{\perp}$ if and only if $\Gamma_C \subset \Gamma_C^*$.
(2) If k is an even number, then C is doubly even if and only if Γ_C is an even lattice.
(3) C is self-dual if and only if Γ_C is unimodular.

Contributions 1-2

- Construction A: [Bannai99][Ebeling12](**even** k), established the correspondence between codes and lattices, which reveals the relationship between theta function, modular form and MacWilliams identity, and presented the correspondence between type II codes over $\mathbb{Z}_{2k}$ and even unimodular lattices, as well as the complete weight enumerators and symmetrized weight enumerators MacWilliams identities in genus g.
- We extend construction A for **any positive integer** k and provide the connections between a k-ary code C and the associated lattice Γ_C.

Contributions 1-2

Theorem 1

Let $C \subset \mathbb{Z}_k^n$ be a k-ary code, $\Gamma_C = \frac{1}{\sqrt{k}}\rho^{-1}(C)$ be the associated lattice of C, then

$$\vartheta_{\Gamma_C}(z) = W_C(A_0(z), A_1(z), \cdots, A_{k-1}(z)).$$

Furthermore, combing with the theta function, we derive the complete weight enumerators and the symmetrized weight enumerators MacWilliams identity for codes over $\mathbb{Z}_k$:

Theorem 2

Let $C \subset \mathbb{Z}_k^n$ be a k-ary code. Then we have

$$W_{C^\perp}(A_0(z), A_1(z), \cdots, A_{k-1}(z)) = \frac{1}{|C|} W_C\Big(\sum_{j=0}^{k-1} A_j(z), \sum_{j=0}^{k-1} e^{\frac{2\pi j}{k} i} A_j(z), \cdots, \sum_{j=0}^{k-1} e^{\frac{2\pi (k-1) j}{k} i} A_j(z)\Big).$$

Contribution 3

- Establish the complete weight enumerator MacWilliams identity in genus g (Theorem 3) which is valid for all positive integers k. This significantly extends Theorem 5.1 of Bannai, which required k to be **even.**

Theorem 3

Let $C \subset \mathbb{Z}_k^n$ be a k-ary code. $\mathfrak{C}_{C,g}(z_a)$ is the complete weight enumerators in genus g. Then we have

$$\mathfrak{C}_{C^\perp,g}(z_a) = \frac{1}{|C|^g} T \mathfrak{C}_{C,g}(z_a).$$

- If $g = 1$, the above result becomes the ordinary complete weight enumerators MacWilliams identity.
- The symmetrized weight enumerators MacWilliams identity in genus g from Theorem 3 directly by treating z_a and $-z_a$ as the equivalent elements in the complete weight enumerators.

Contribution 4

When $k = p$ is an odd prime number, G. van der Geer and F. Hirzebruch considered the Lee weight enumerators $S_C(X_0, X_1, \cdots, X_{\frac{p-1}{2}})$ for codes $C \subset F_p^n$ defined as the following

$$S_C(X_0, X_1, \cdots, X_{\frac{p-1}{2}}) = \sum_{c \in C} X_0^{w_0(c)} X_1^{w_1(c)} \cdots X_{\frac{p-1}{2}}^{w_{\frac{p-1}{2}}(c)},$$

where $w_j(c)$ is the number of elements in the codeword c which are equal to a_j or $p - a_j$, and established the MacWilliams identity between the Lee weight enumerators and theta function,

$$\theta_{\Gamma_C}(z) = S_C(\theta_0(z), \theta_1(z), \cdots, \theta_{\frac{p-1}{2}}(z)).$$

Contribution 4

We generalize their results to complete weight enumerators by defining p theta functions $\vartheta_0(z), \vartheta_1(z), \cdots, \vartheta_{p-1}(z)$, and present the identity to show the relationship between the **complete weight** enumerators with **Hamming weight** of codes in F_p^n and the theta function of associated lattices over a cyclotomic field.

Theorem 4

Let $C \subset F_p^n$ be a linear code such that $C \subset C^{\perp}$. $W_C(X_0, X_1, \cdots, X_{p-1})$ is the complete weight enumerator of C with Hamming weight, then we have

$$\vartheta_{\Gamma_C}(z) = W_C(\vartheta_0(z), \vartheta_1(z), \cdots, \vartheta_{p-1}(z)),$$

where

$$\Gamma_C = \rho^{-1}(C) \subset \mathfrak{D}^n, \quad \vartheta_{\Gamma_C}(z) = \sum_{x \in \Gamma_C} e^{2\pi i z \, Tr_{K+/\mathbb{Q}}(\frac{x\overline{x}}{p})},$$

$$\vartheta_j(z) = \sum_{x \in \mathfrak{B}+j} e^{2\pi i z \, Tr_{K+/\mathbb{Q}}(\frac{x\overline{x}}{p})}, z \in \mathbb{H} \textit{ upper half plane of complex number.}$$

Future Works

- The further questions are to consider the modified theta functions or the nu-function of a lattice associated with a k-ary code.
- To explore whether these functions are a kind of special modular forms, as well as present the MacWilliams identities of these functions based on the theory of theta function and modular form.

Thank you!