Extending the 2-homogeneous property to distance-biregular graphs

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Combinatorics around the q-Onsager algebra

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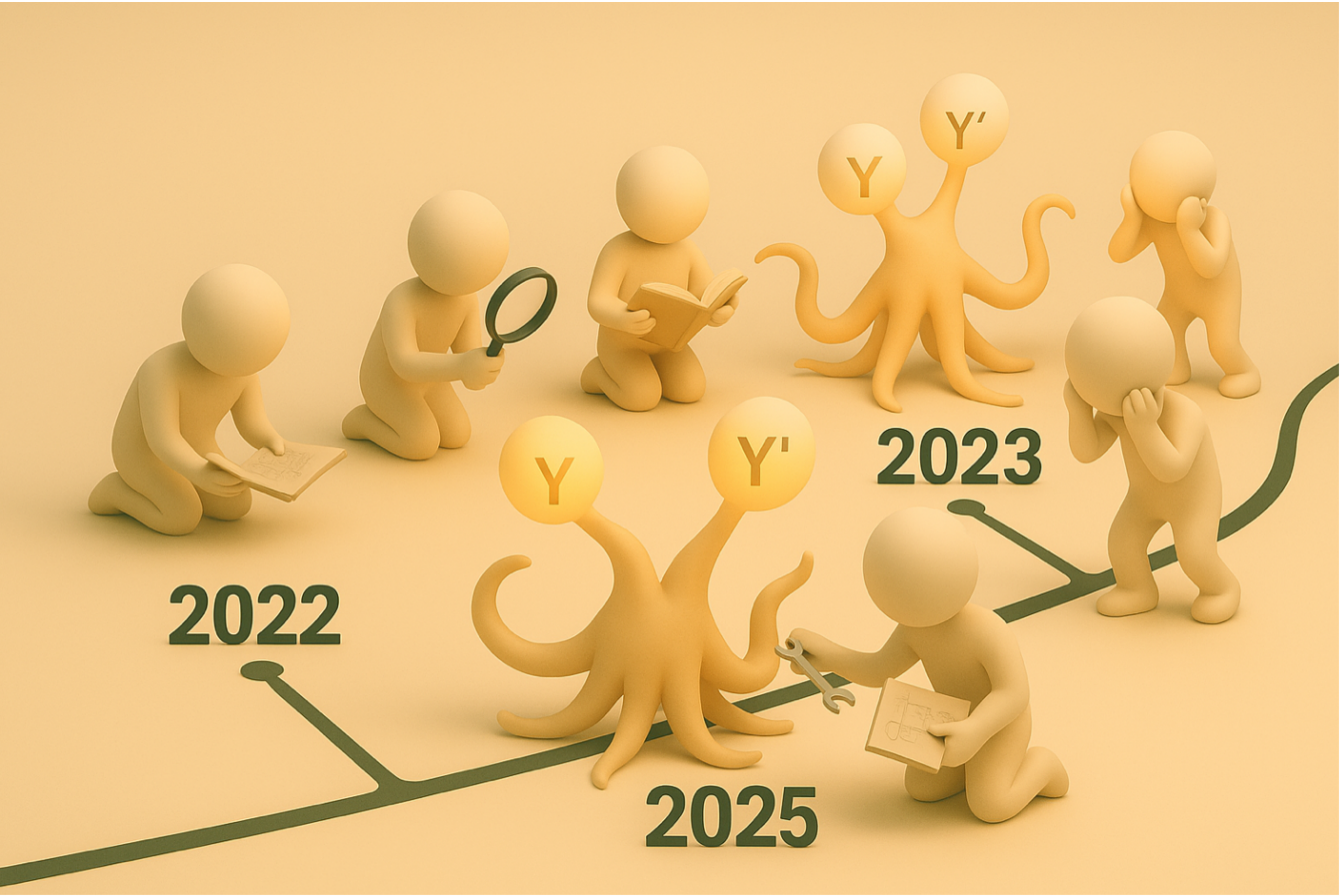
A picture is worth a thousand words

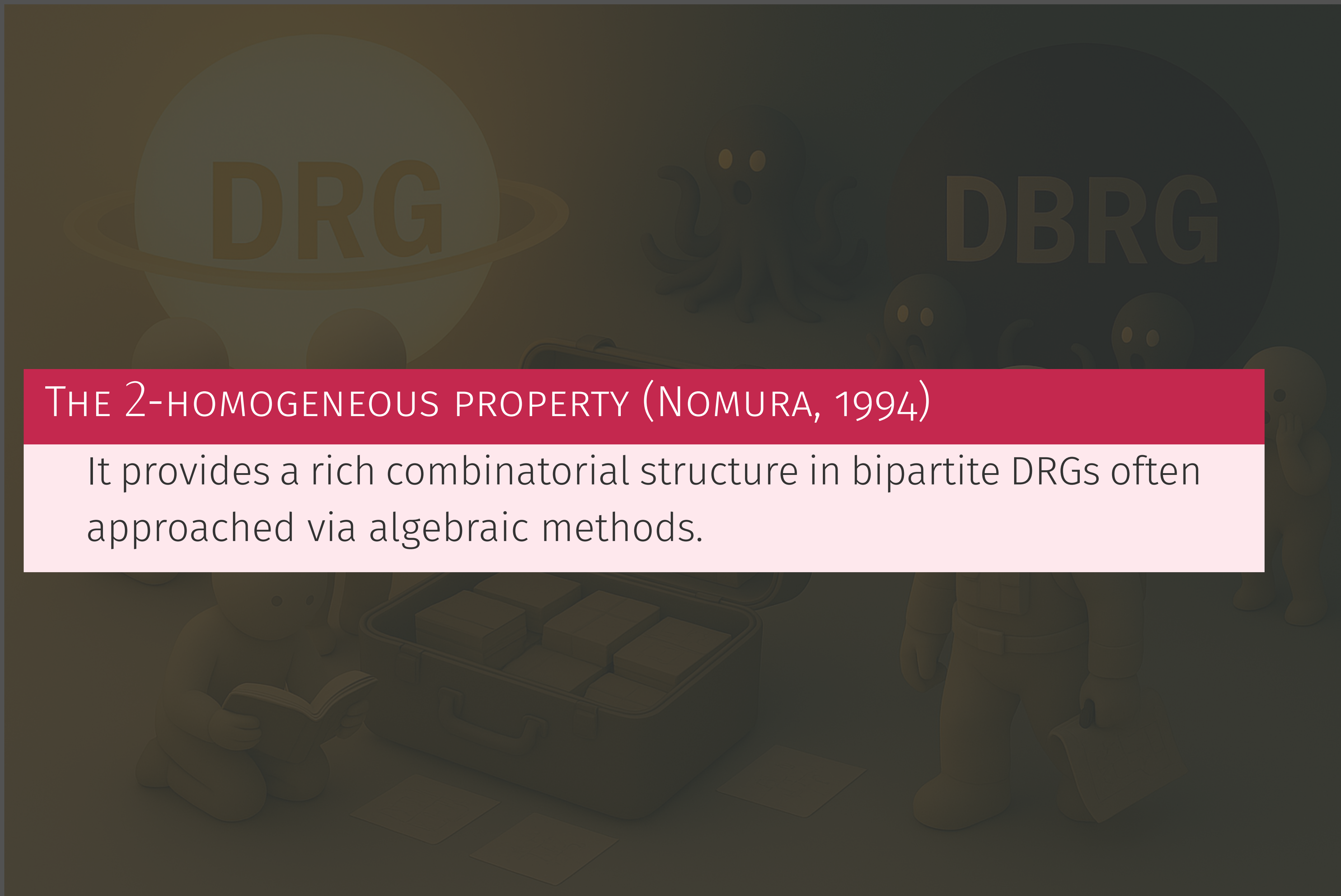



DRG


DBRG
Y
Y'
T(x)


DRG
DBRG


DRG
DBRG



The 2-homogeneous property (Nomura, 1994)

It provides a rich combinatorial structure in bipartite DRGs often approached via algebraic methods.

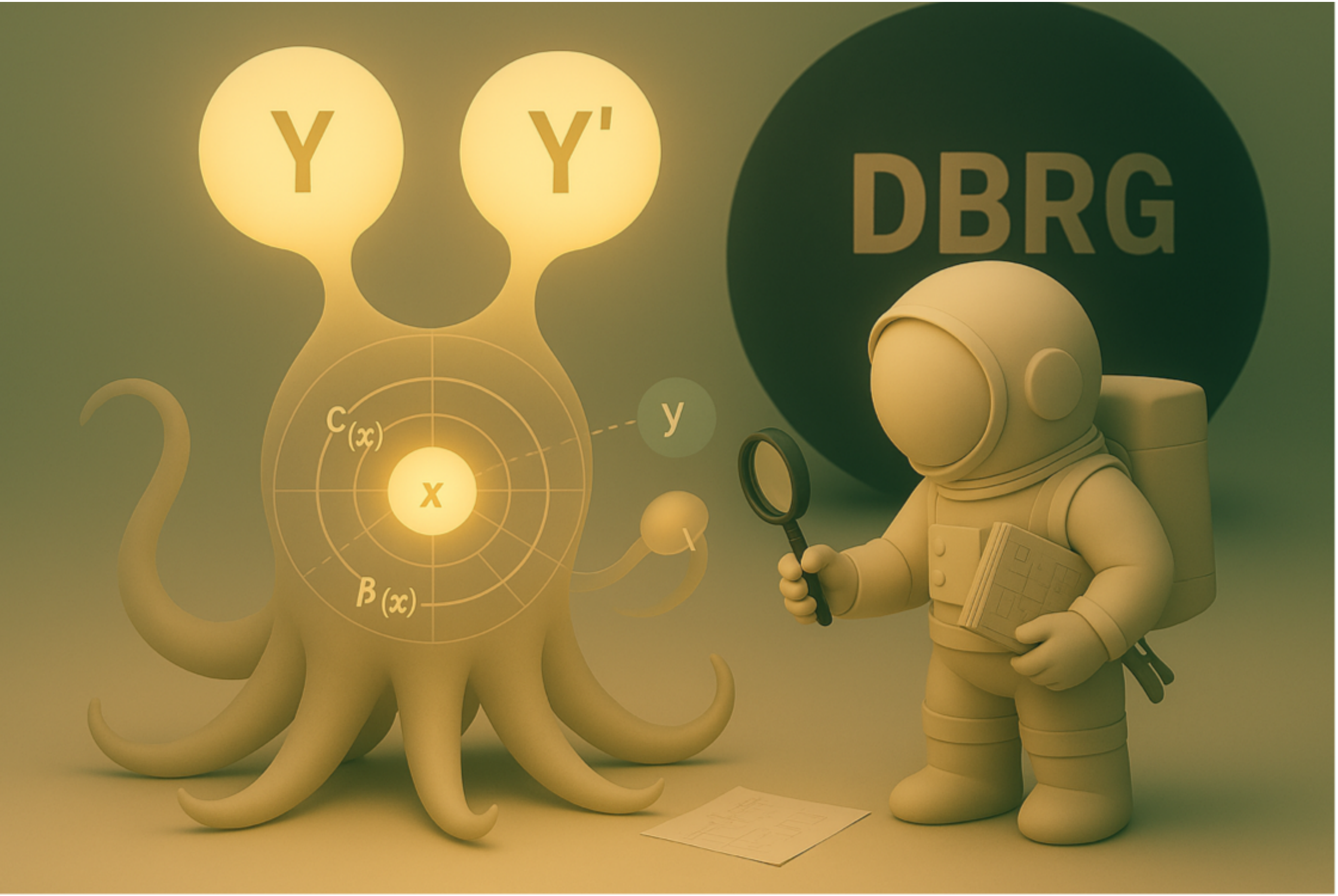
DRG
DBRG

Theorem (Nomura, 1995)

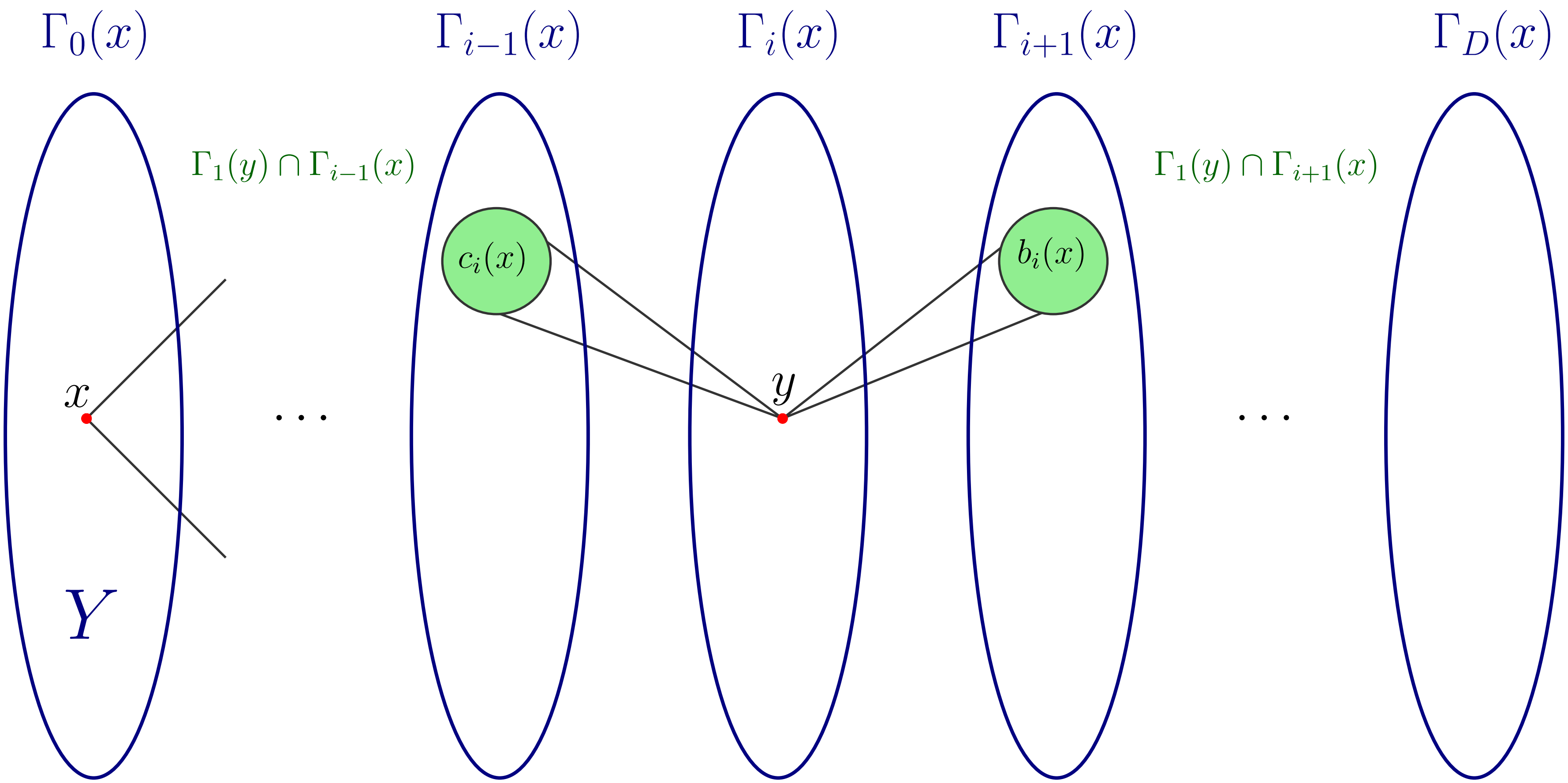
Let Γ be a 2-homogeneous bipartite distance-regular graph with valency k and diameter d. Then, the following hold:

- Γ is the complete bipartite graph $K_{k,k}$;
- Γ is the $2d$-cycle C_{2d};
- Γ is the hypercube $H(d, 2)$;
- Γ is the complement of a $2 \times (k+1)$-grid;
- Γ is a Hadamard graph of valency $k = 4\gamma$;
- Γ has array $\{k, k-1, k-c, c, 1; 1, c, k-c, k-1, k\}$, where $c = \gamma(\gamma+1), \quad k = \gamma(\gamma^2+3\gamma+1)$, and $\gamma > 0$:
  - the 5-dimensional hypercube $H(5, 2)$ ($\gamma = 1$),
  - the double cover of the Higman-Sims graph ($\gamma = 2$).

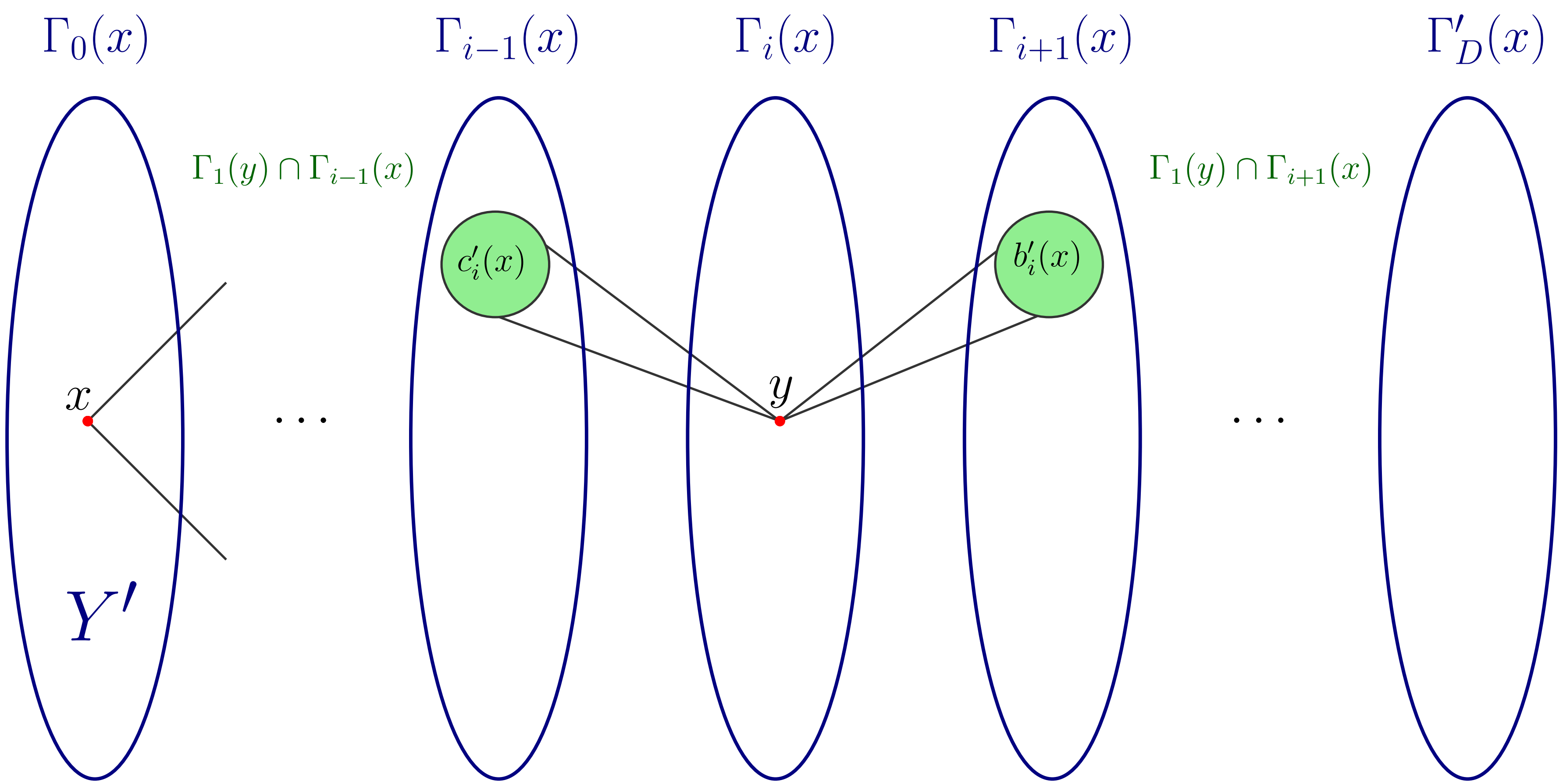

DBRG
Y
Y'


Y
Y'
DBRG
C(x)
x
y
β(x)

Local Distance-Regularity in Y

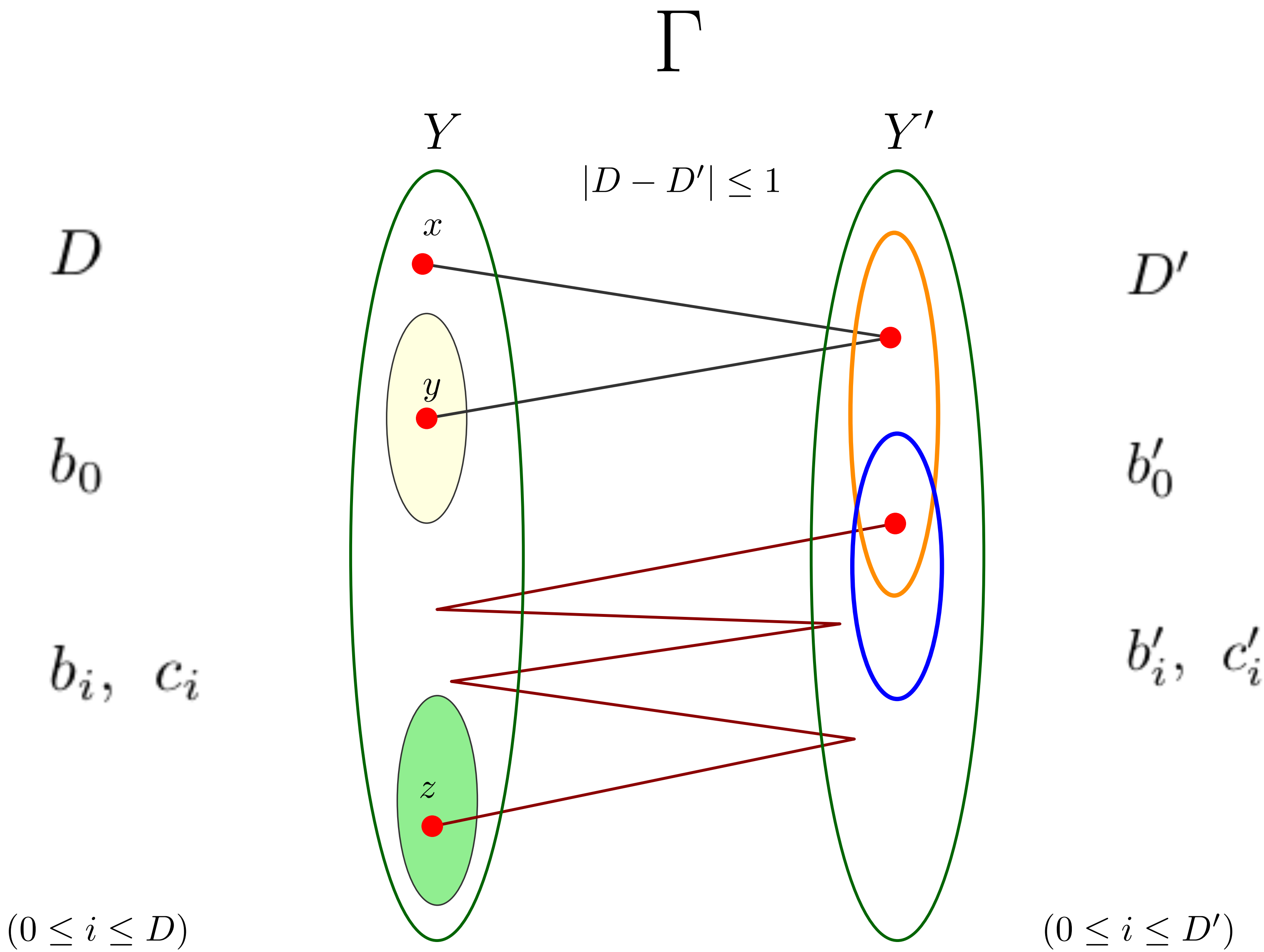


Local Distance-Regularity in Y'



Distance-biregular graphs (DBRGs)

- Let Γ be a (Y, Y')-bipartite **DBRG** (Distance-Biregular Graph).



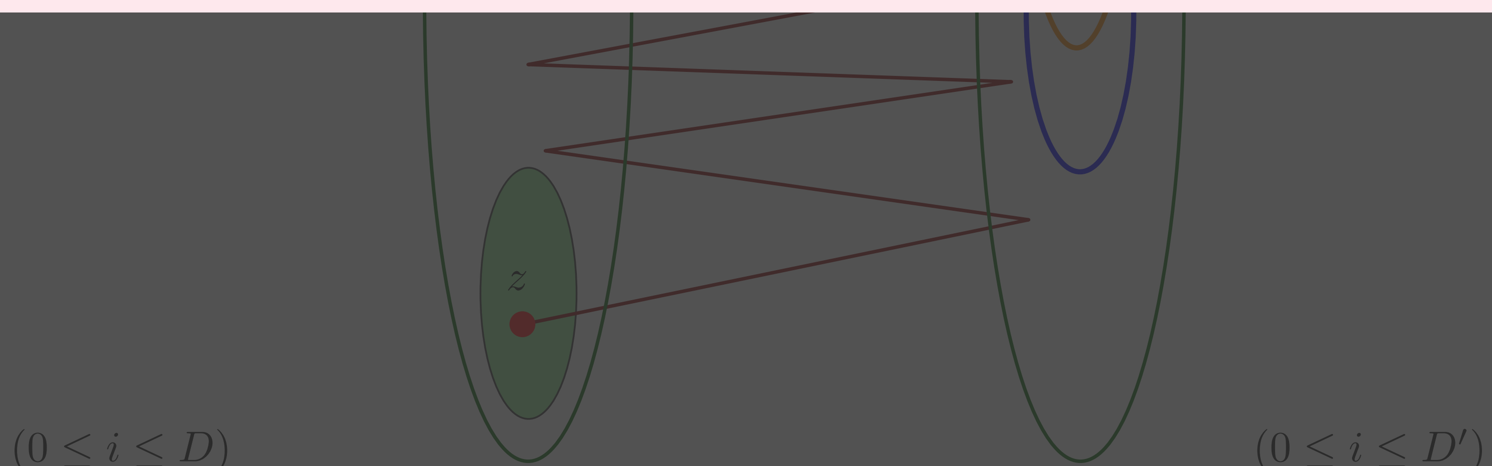
- Let Γ be a (Y, Y')-bipartite **DBRG** (Distance-Biregular Graph).

$$\Gamma$$

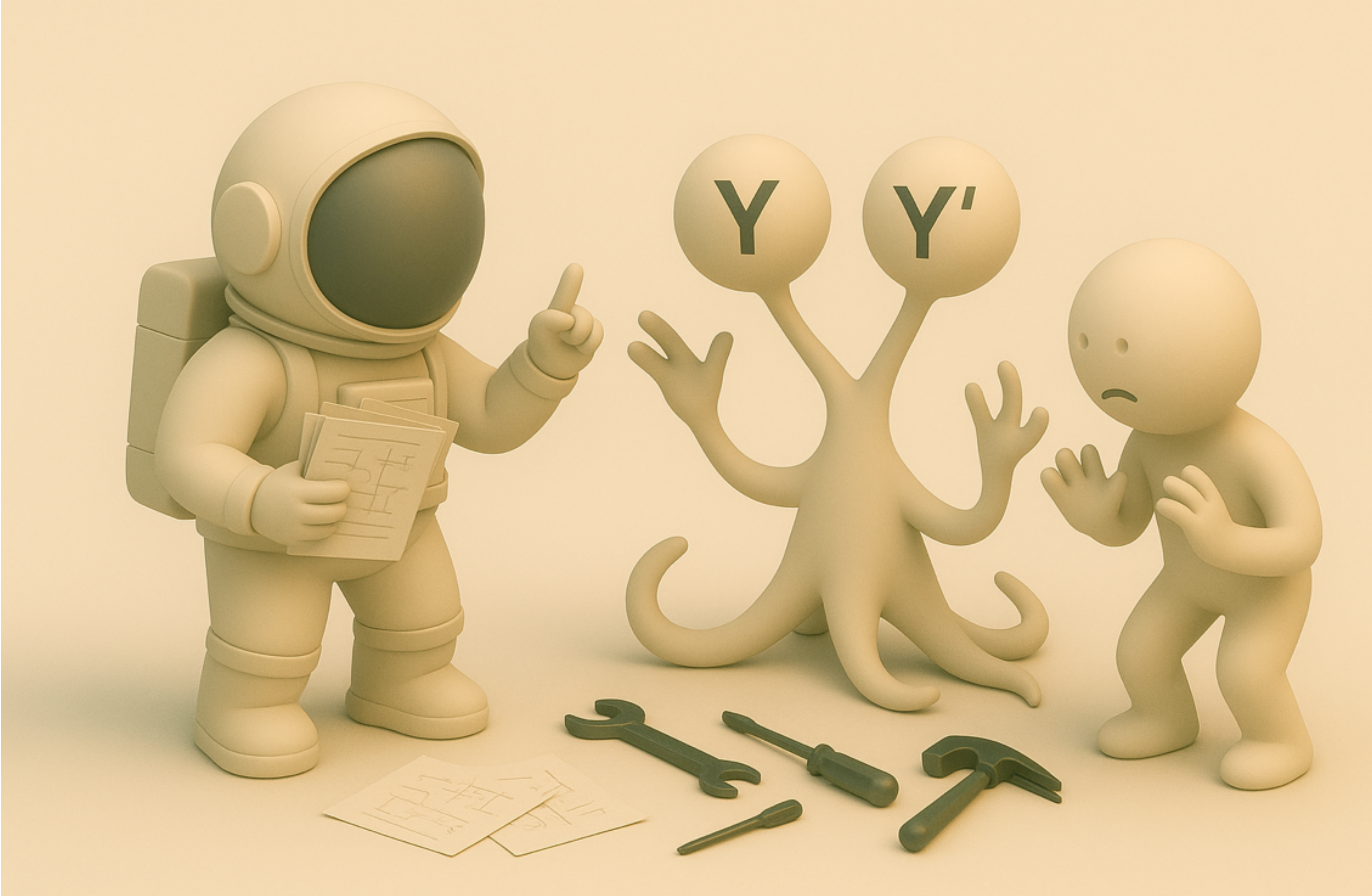
Remark

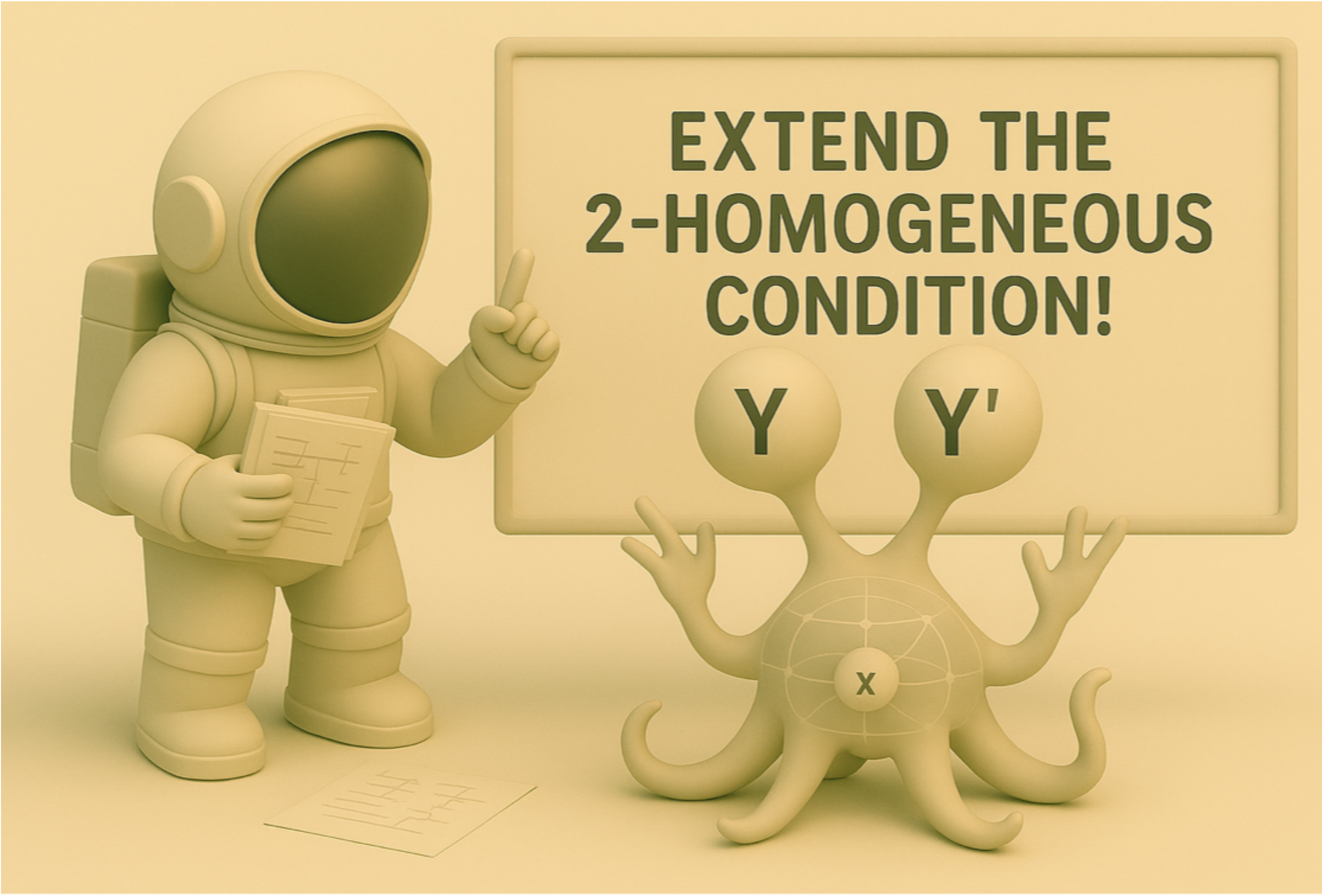
We can express the intersection numbers of a (Y, Y') DBRG in an alternative way, represented as the following two-line array:

$$\begin{bmatrix} b_0; & c_1, & c_2, & \dots & c_{D-1}, & c_D \\ b'_0; & c'_1, & c'_2, & \dots & c_{D'-1}, & c'_D \end{bmatrix}$$



Absence of 2-Homogeneity in DBRGs




EXTEND THE
2-HOMOGENEOUS
CONDITION!
Y
Y'
x

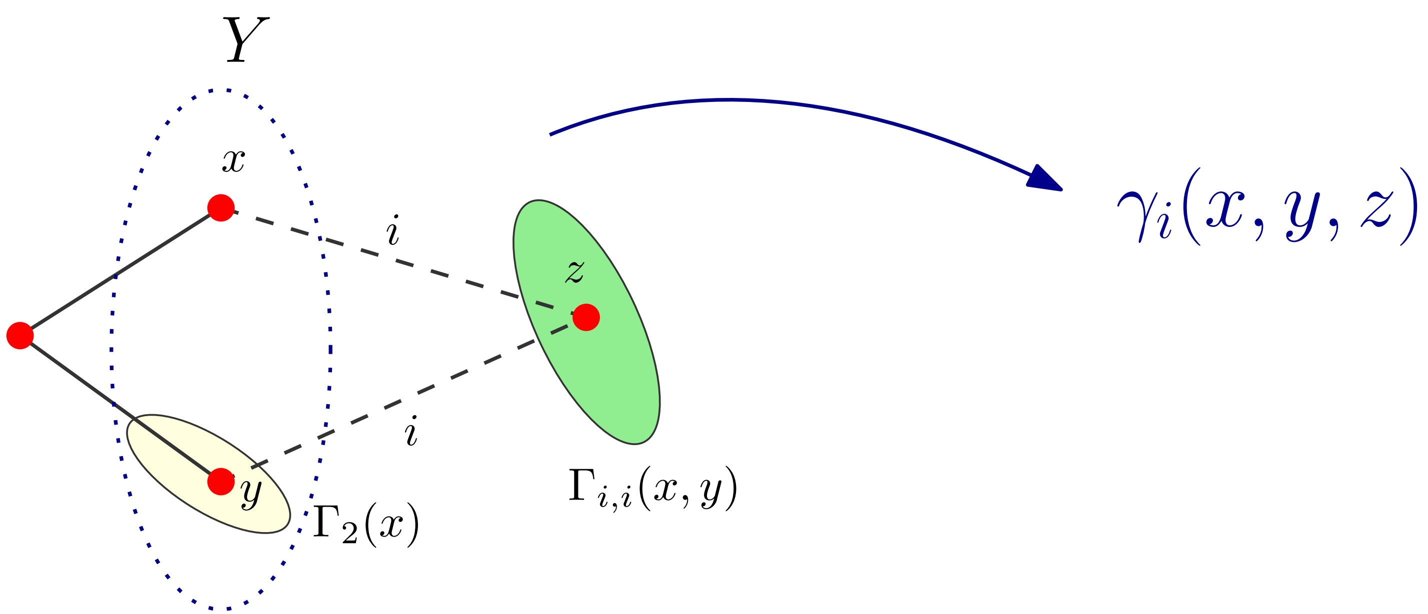
The 2-Y-homogeneous condition

An important scalar

The scalar $\gamma_i(x, y, z)$

Let Γ be a (Y, Y')-DBRG.

- Suppose that every vertex in Y has eccentricity $D \geq 3$.
- Fix an integer i with $1 \leq i \leq D$.
- Choose a pair of vertices (x, y) such that $x \in Y$ and $y \in \Gamma_2(x) \subseteq Y$.
- Consider all vertices $z \in \Gamma_{i,i}(x, y)$.
- For each such triple (x, y, z), define a quantity $\gamma_i(x, y, z)$.



The scalar $\gamma_i(x, y, z)$

Let Γ be a (Y, Y')-DBRG.

- Suppose that every vertex in Y has eccentricity $D \geq 3$.
- Fix an integer i with $1 \leq i \leq D$.

Definition

For i $(1 \leq i \leq D)$ and for $x \in Y$, $y \in \Gamma_2(x)$ and $z \in \Gamma_{i,i}(x, y)$, define:

$$\gamma_i(x, y, z) = \begin{cases} |\Gamma_{1,1}(x, y) \cap \Gamma_{i-1}(z)|, & \text{if } \Gamma_{i,i}(x, y) \neq \emptyset, \\ 0, & \text{if } \Gamma_{i,i}(x, y) = \emptyset. \end{cases}$$

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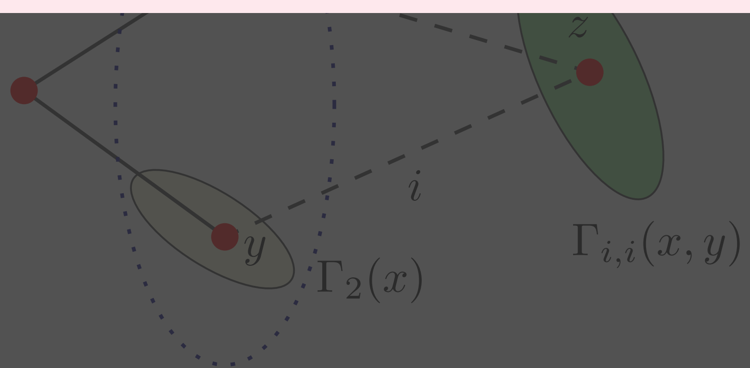
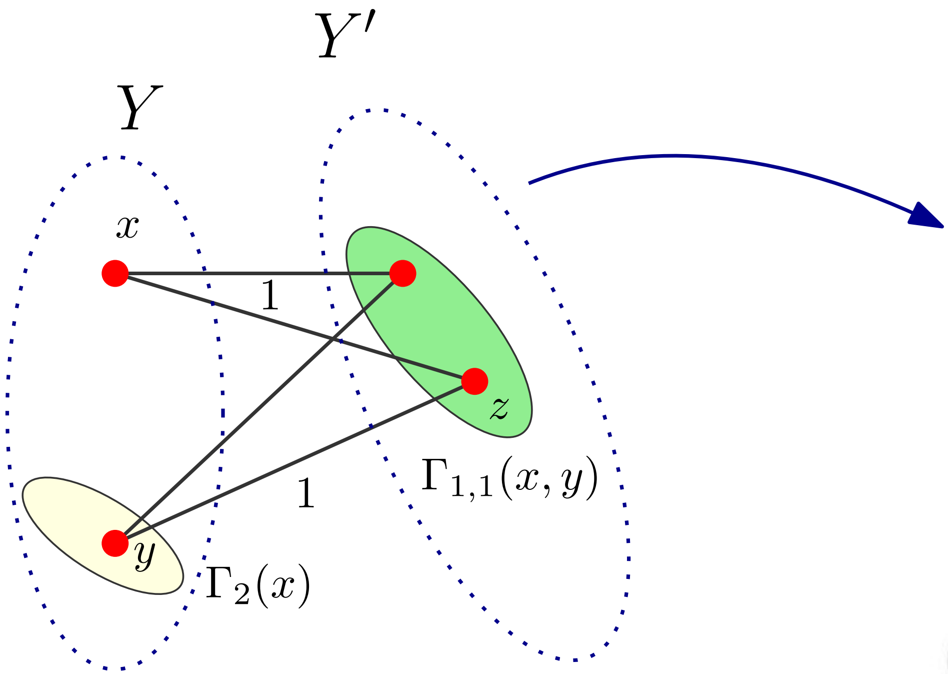
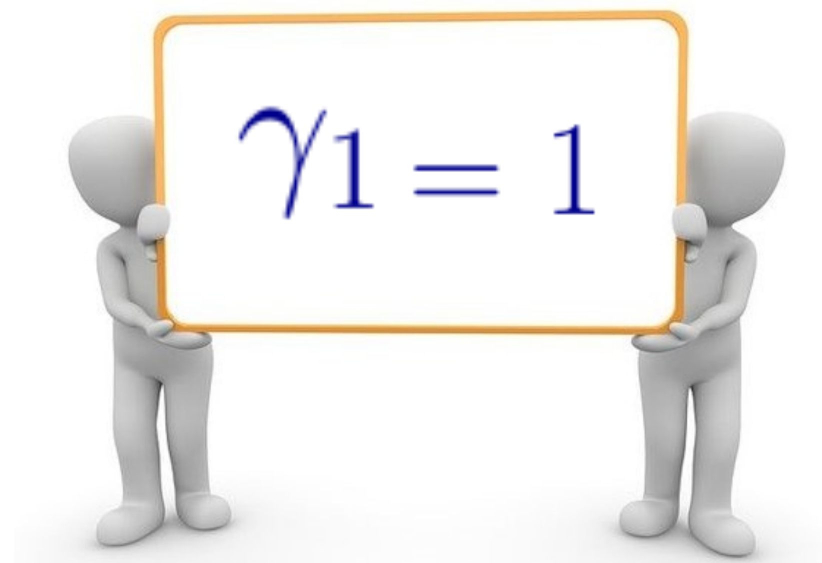


A special case: the scalar $\gamma_1(x, y, z)$



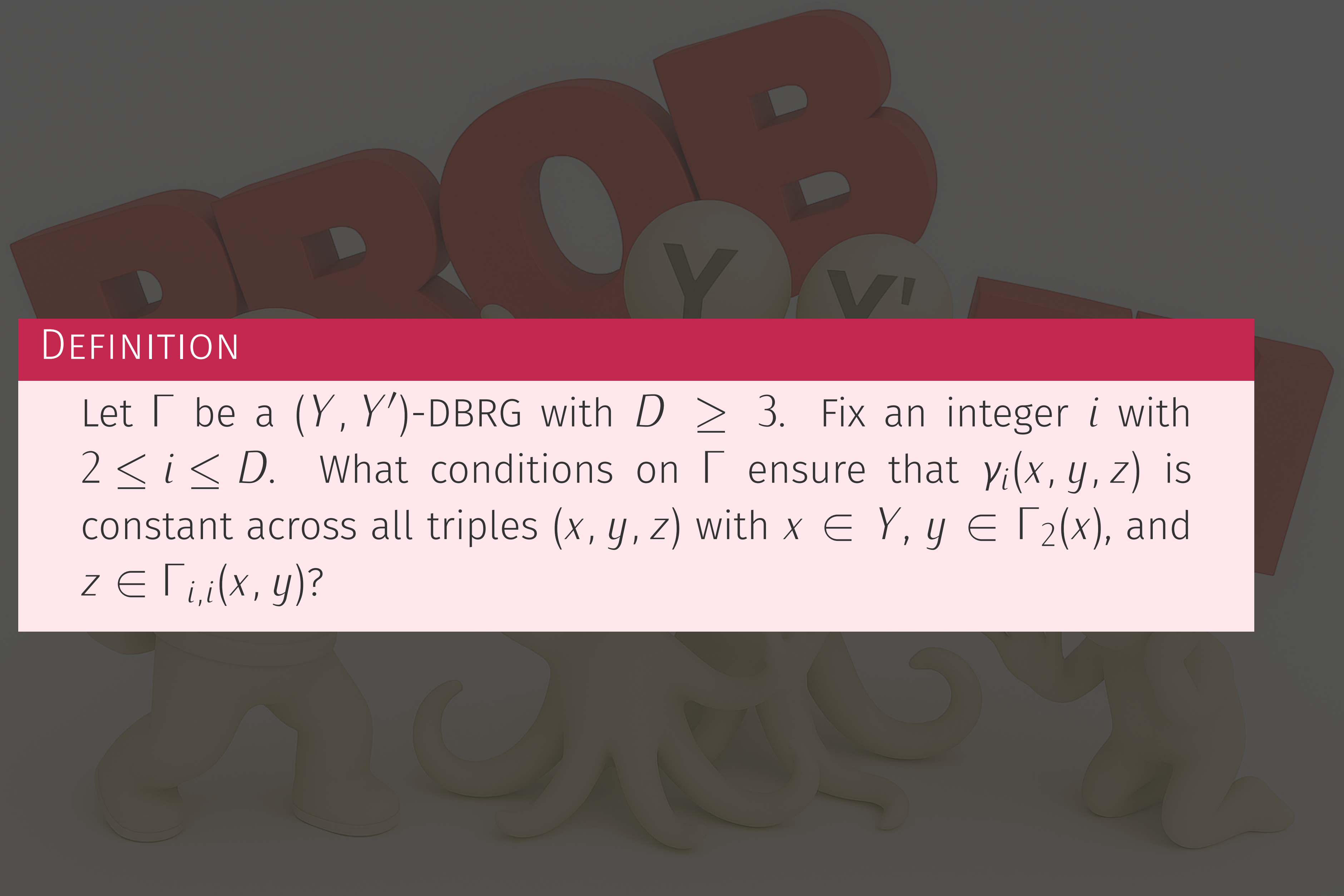
$$\gamma_1(x, y, z) = 1$$

$\forall x \in Y,\ \forall y \in \Gamma_2(x),$
$\forall z \in \Gamma_{1,1}(x, y).$



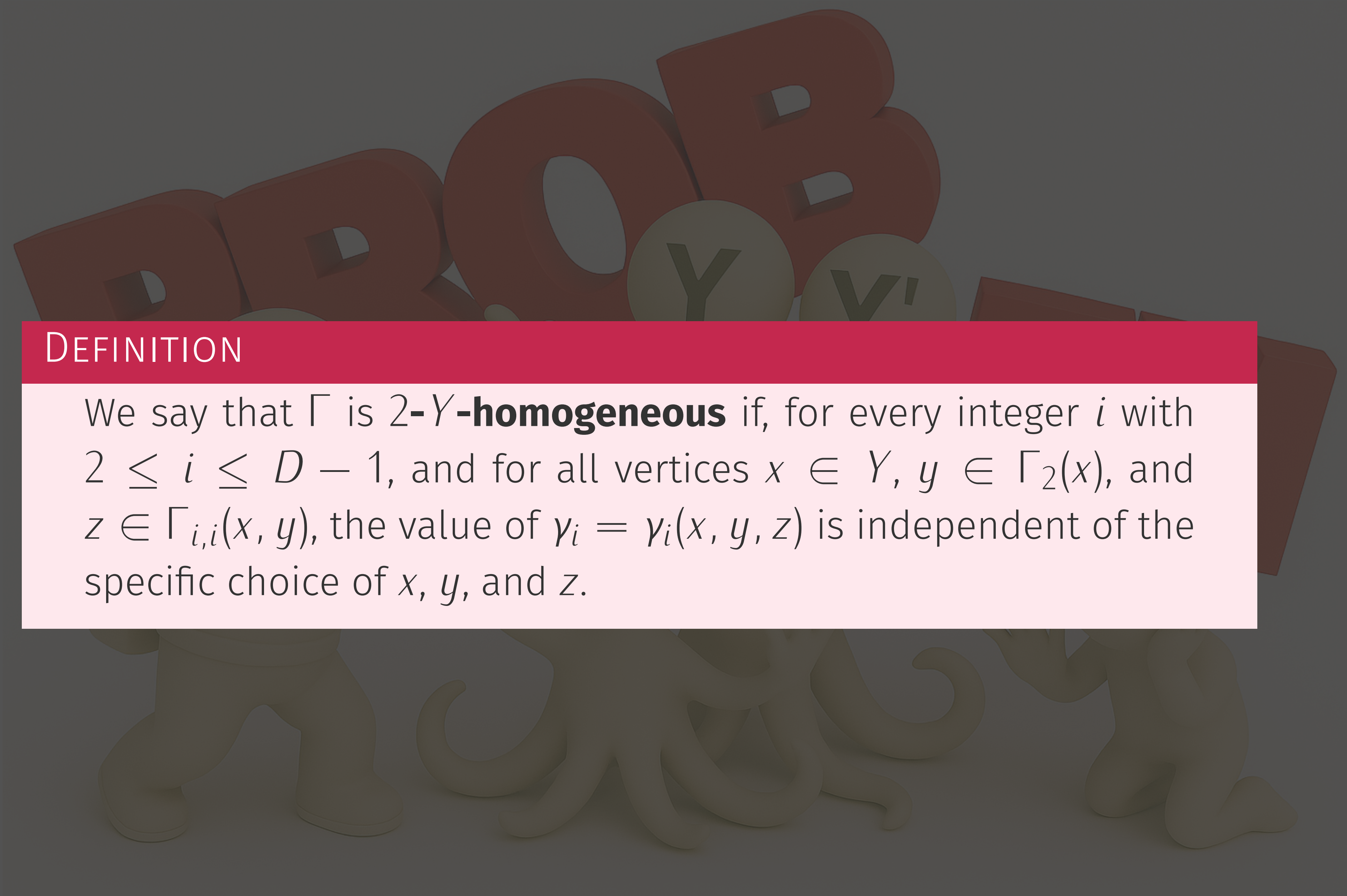
- Fix a pair of vertices (x, y) with $x \in Y$ and $y \in \Gamma_2(x) \subseteq Y$.
- Notice that $\Gamma_{1,1}(x, y) \neq \varnothing$.
- For a vertex $z \in \Gamma_{1,1}(x, y)$, the set $\Gamma_{1,1}(x, y) \cap \Gamma_0(z) = \{z\}$.


PROBLEM
Y
Y'
Y



DEFINITION

Let Γ be a (Y, Y')-DBRG with $D \geq 3$. Fix an integer i with $2 \leq i \leq D$. What conditions on Γ ensure that $\gamma_i(x, y, z)$ is constant across all triples (x, y, z) with $x \in Y$, $y \in \Gamma_2(x)$, and $z \in \Gamma_{i,i}(x, y)$?



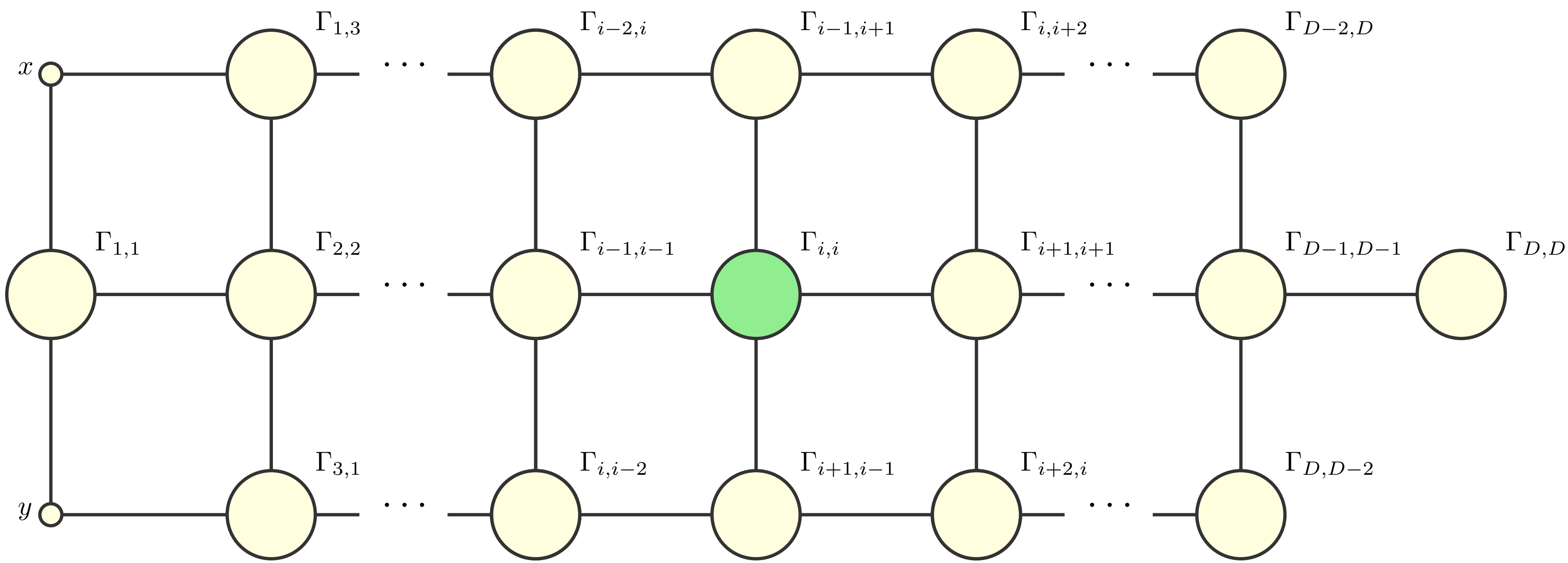
Definition

We say that Γ is **2-Y-homogeneous** if, for every integer i with $2 \leq i \leq D-1$, and for all vertices $x \in Y$, $y \in \Gamma_2(x)$, and $z \in \Gamma_{i,i}(x,y)$, the value of $\gamma_i = \gamma_i(x,y,z)$ is independent of the specific choice of x, y, and z.

The intersection diagrams of rank 2

Let Γ be a (Y, Y')-bipartite graph.

- Suppose that every vertex in Y has eccentricity $D \geq 3$.
- Fix an integer i with $1 \leq i \leq D$.
- Choose a pair of vertices (x, y) such that $x \in Y$ and $y \in \Gamma_2(x) \subseteq Y$.
- For integers i and j, define the set $\Gamma_{i,j} := \Gamma_{i,j}(x, y) = \Gamma_i(x) \cap \Gamma_j(y)$.
- Consider all vertices $z \in \Gamma_{i,i}(x, y)$.

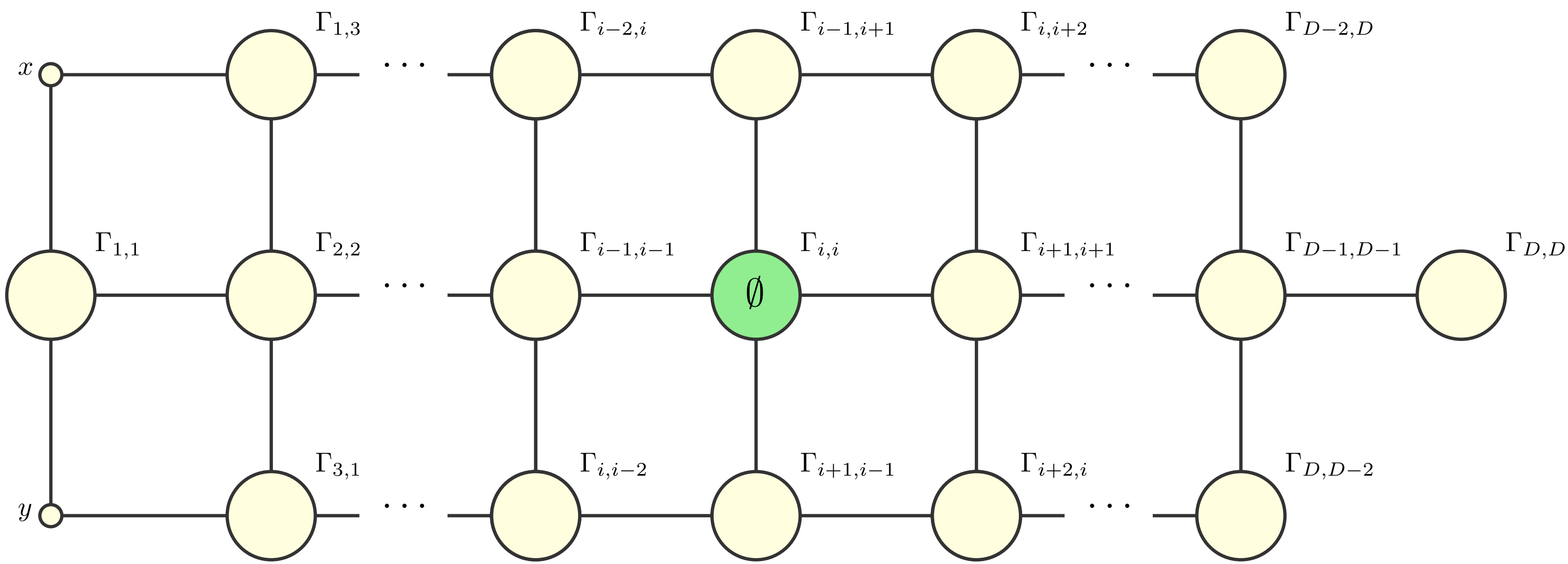


The scalar $\gamma_i(x, y, z)$ when $\Gamma_{i,i}(x, y) = \emptyset$

Definition

For i $(1 \leq i \leq D)$ and for $x \in Y$, $y \in \Gamma_2(x)$ and $z \in \Gamma_{i,i}(x, y)$, define

$$\gamma_i(x, y, z) = 0.$$

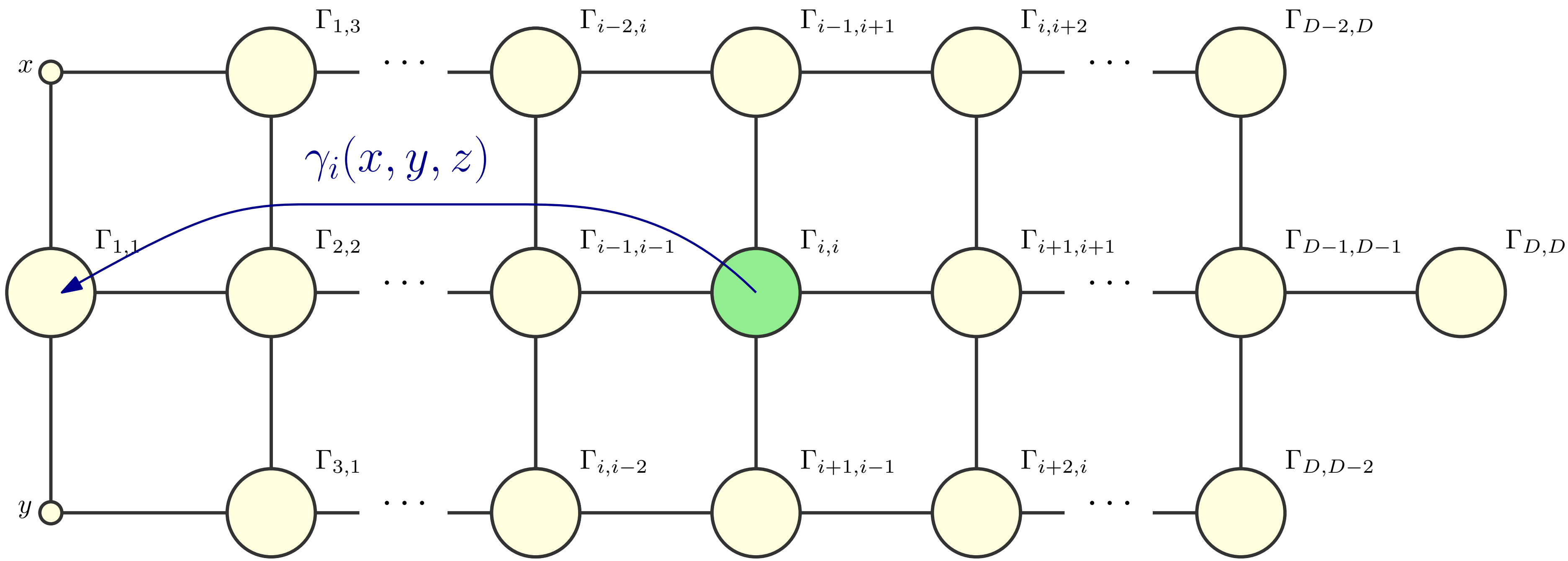


The scalar $\gamma_i(x, y, z)$ when $\Gamma_{i,i}(x, y) \neq \emptyset$

Definition

For i $(1 \leq i \leq D)$ and for $x \in Y$, $y \in \Gamma_2(x)$ and $z \in \Gamma_{i,i}(x, y)$, define

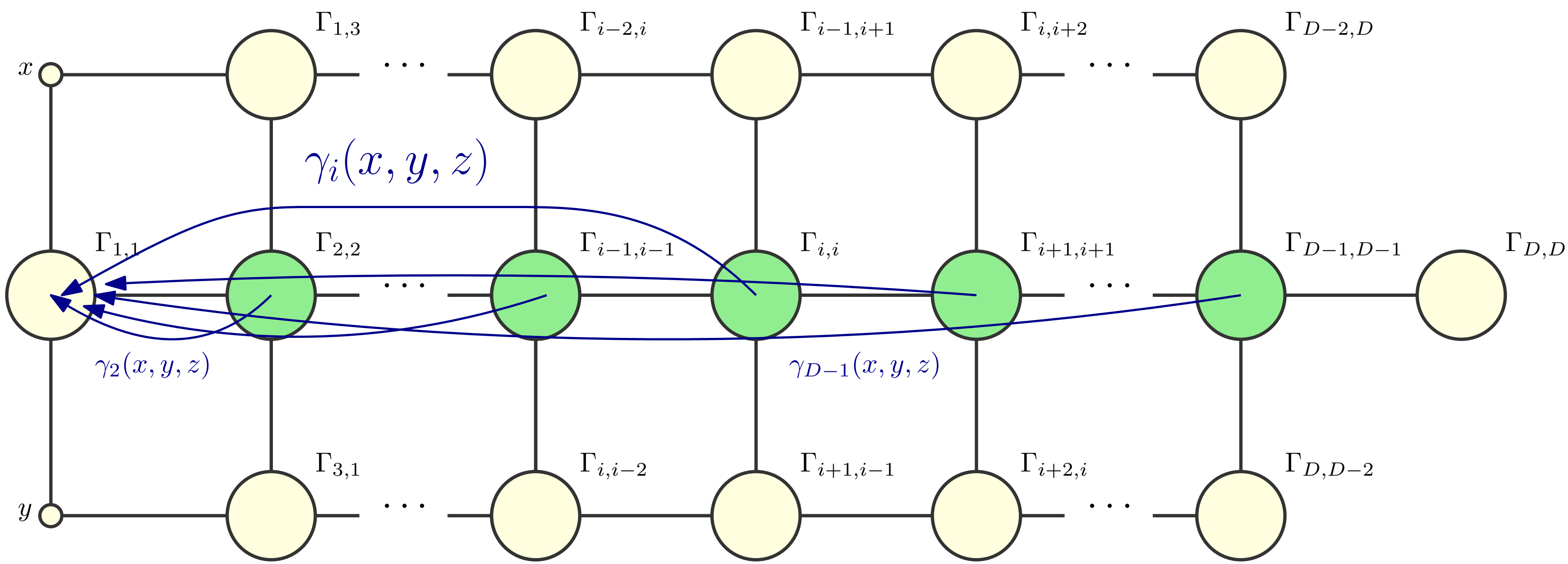
$$\gamma_i(x, y, z) = |\Gamma_{1,1}(x, y) \cap \Gamma_{i-1}(z)|.$$



The 2-Y-homogeneous condition

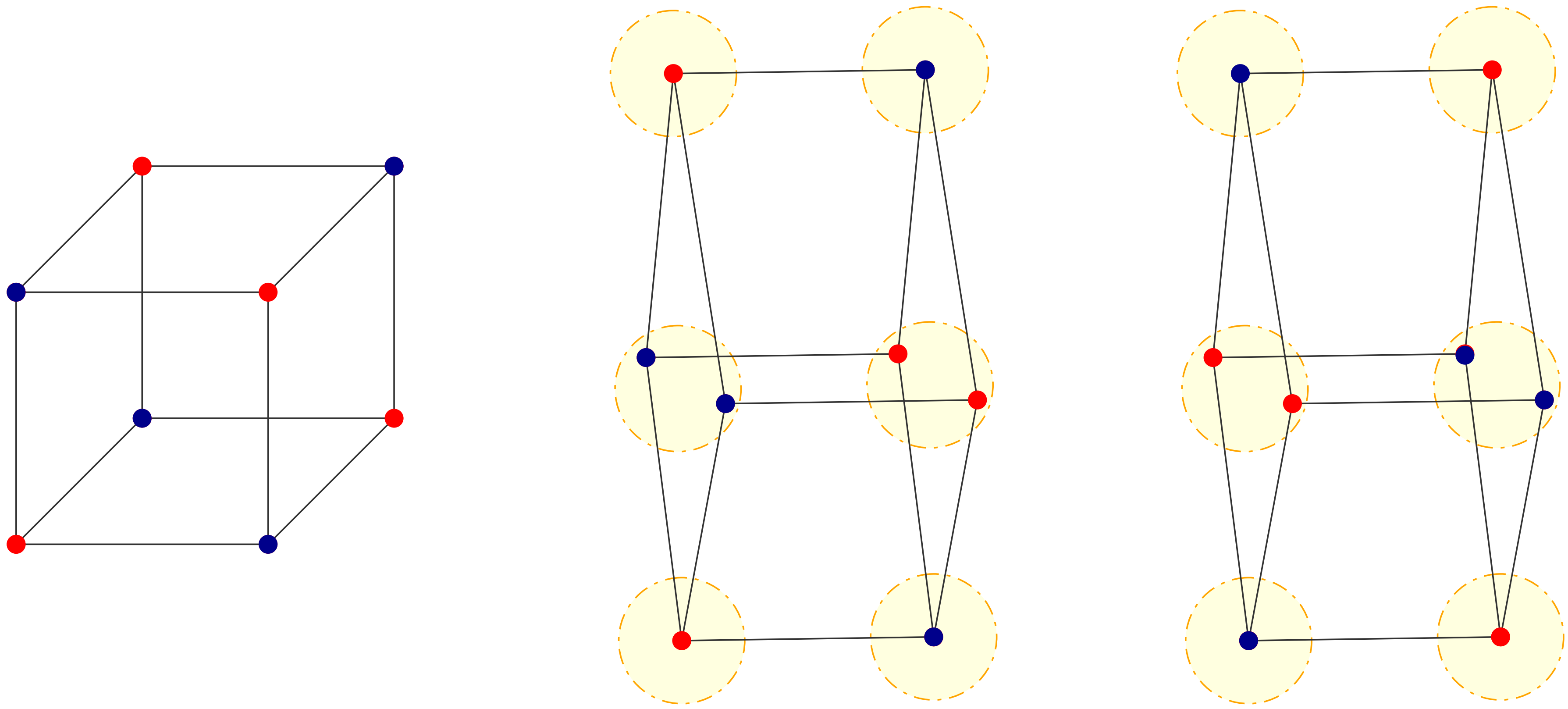
Definition

We say that Γ is **2-Y-homogeneous** if, for every integer i with $2 \leq i \leq D-1$, and for all vertices $x \in Y$, $y \in \Gamma_2(x)$, and $z \in \Gamma_{i,i}(x, y)$, the value of $\gamma_i = \gamma_i(x, y, z)$ is independent of the specific choice of x, y, and z.



Example: the 3-cube $H(3,2)$

Let Γ be the 3-cube with parts $\mathcal{R}$ and $\mathcal{B}$.
Every vertex in Γ has eccentricity equal to 3.



Γ is 2-$\mathcal{R}$-homogeneous with $\gamma_2(\mathcal{R}) = 1$.
Γ is 2-$\mathcal{B}$-homogeneous with $\gamma_2(\mathcal{B}) = 1$.

Example: the 3-cube $H(3,2)$

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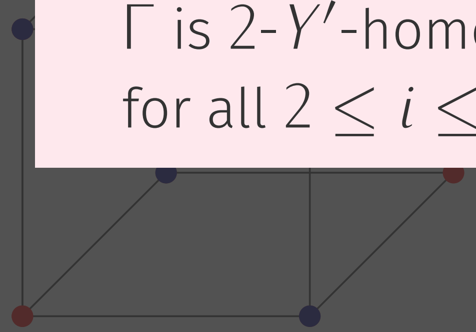
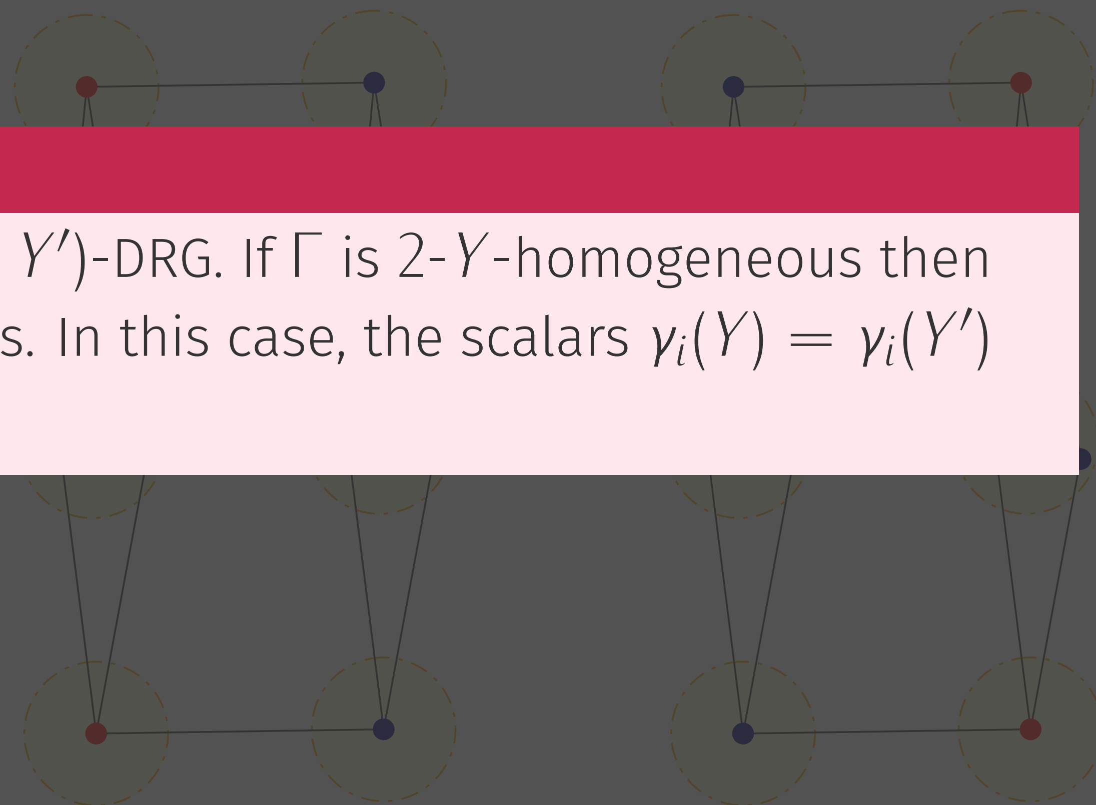


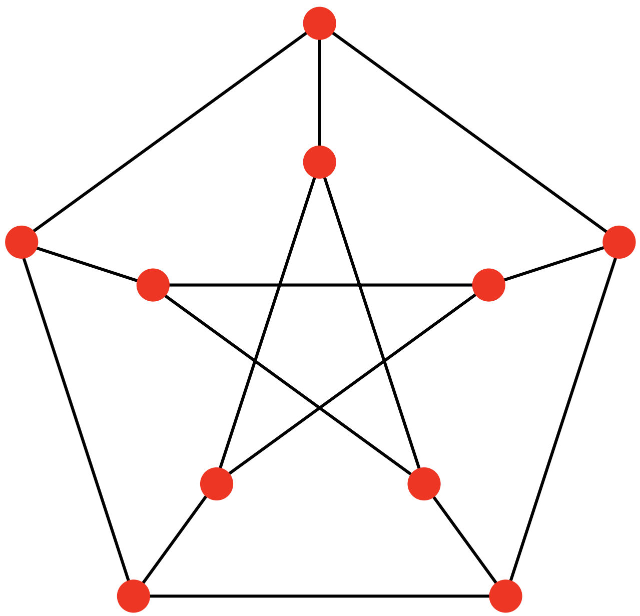


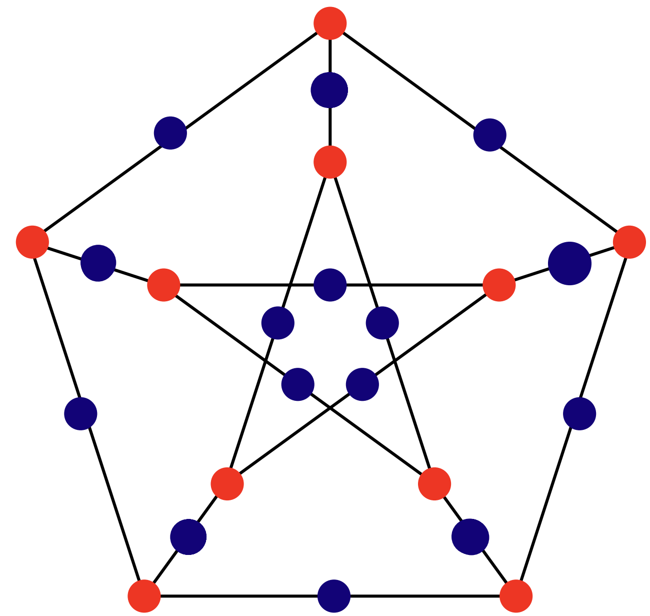
Remark

Let Γ be a bipartite (Y, Y')-DRG. If Γ is 2-Y-homogeneous then Γ is 2-Y'-homogeneous. In this case, the scalars $\gamma_i(Y) = \gamma_i(Y')$ for all $2 \leq i \leq D-1$.

Γ is 2-$\mathcal{R}$-homogeneous with $\gamma_2(\mathcal{R}) = 1$.
Γ is 2-$\mathcal{B}$-homogeneous with $\gamma_2(\mathcal{B}) = 1$.

Example: the subdivision of the Petersen graph



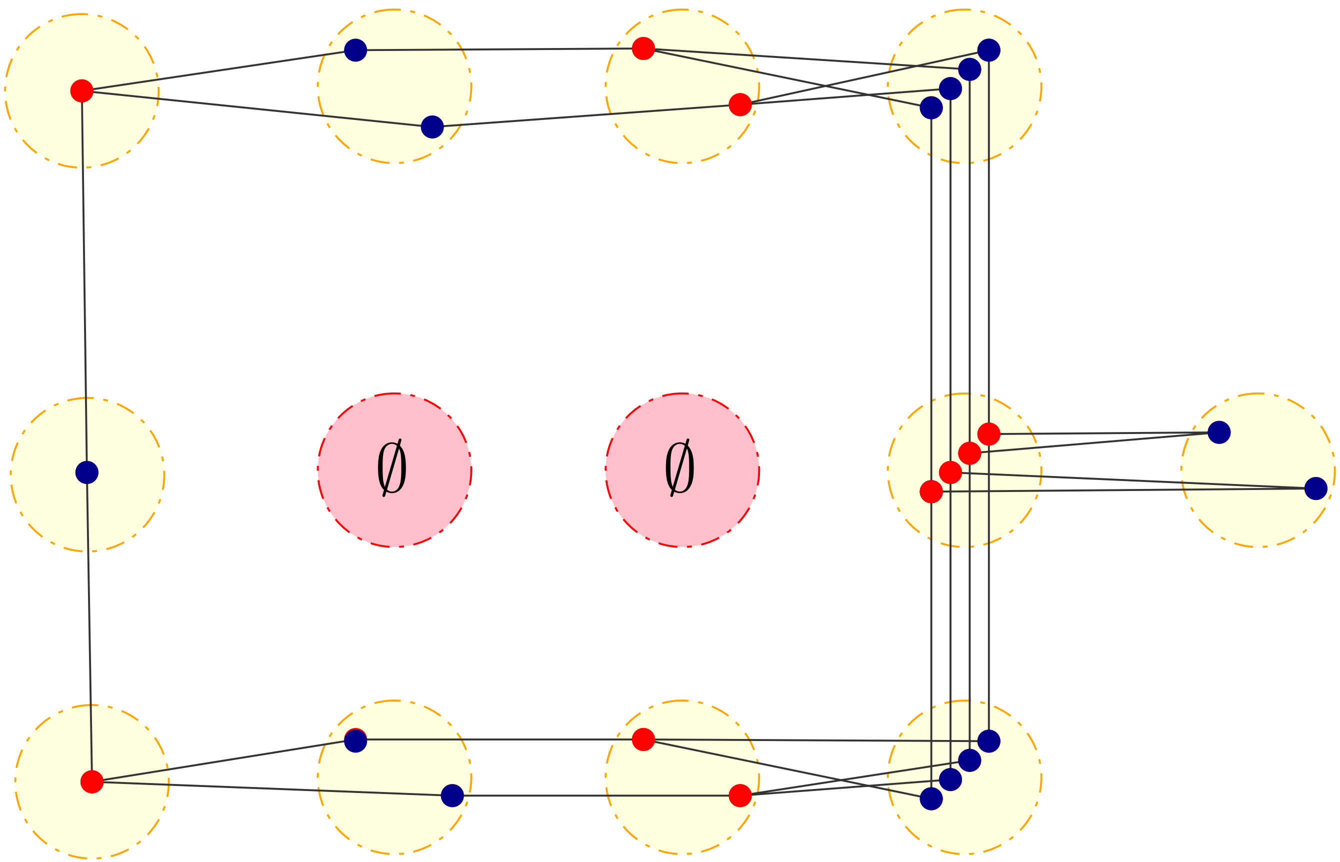


The subdivision graph Γ' of the Petersen graph Γ is the $(X, \mathcal{R})$-bipartite graph obtained from Γ by replacing each of its edges by a path of length 2.

The Petersen graph is a distance-regular graph and its subdivision graph is distance-biregular.

Example: the subdivision of the Petersen graph

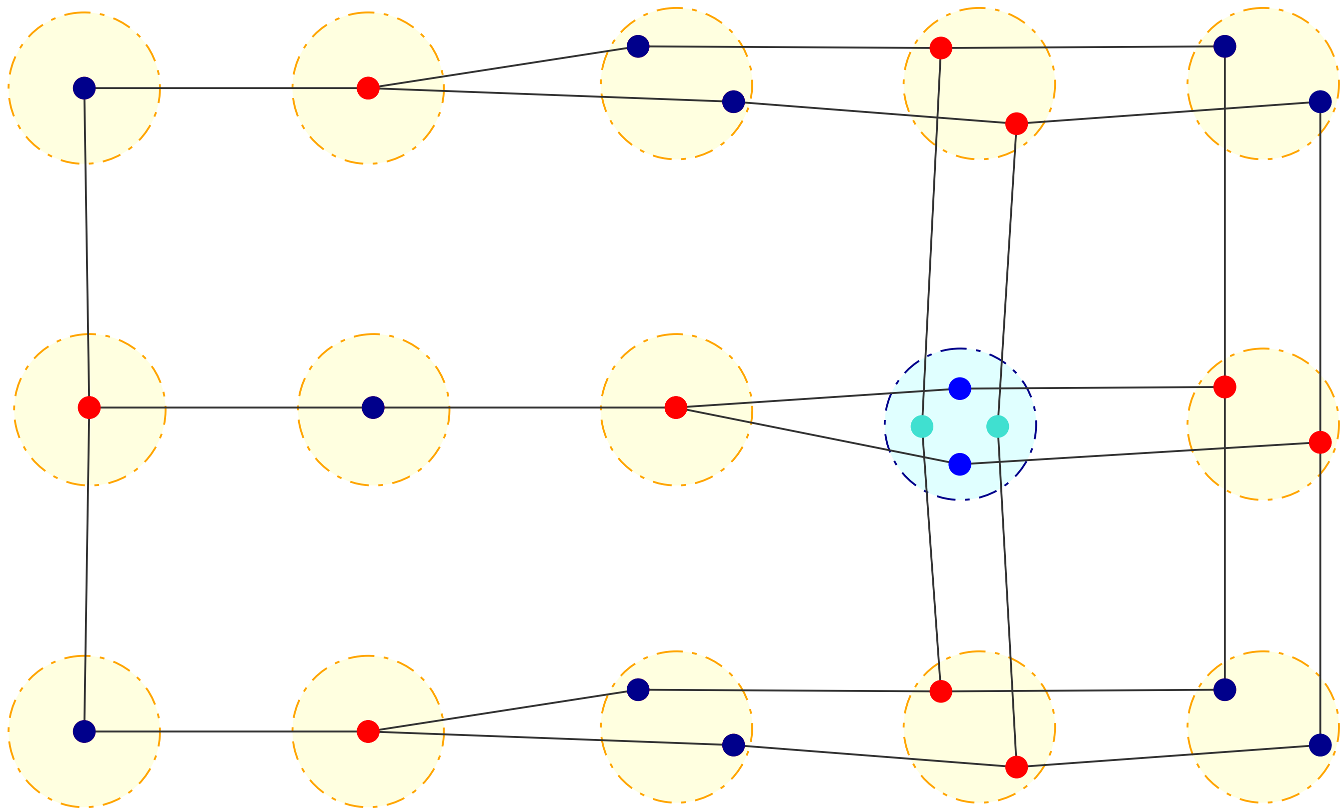
Let Γ be the subdivision of the Petersen graph with parts $\mathcal{R}$ and $\mathcal{B}$.
Every red vertex in Γ has eccentricity equal to 5.



Γ is 2-$\mathcal{R}$-homogeneous with $\gamma_i(\mathcal{R}) = 0$ $(2 \leq i \leq 4)$.

Example: the subdivision of the Petersen graph

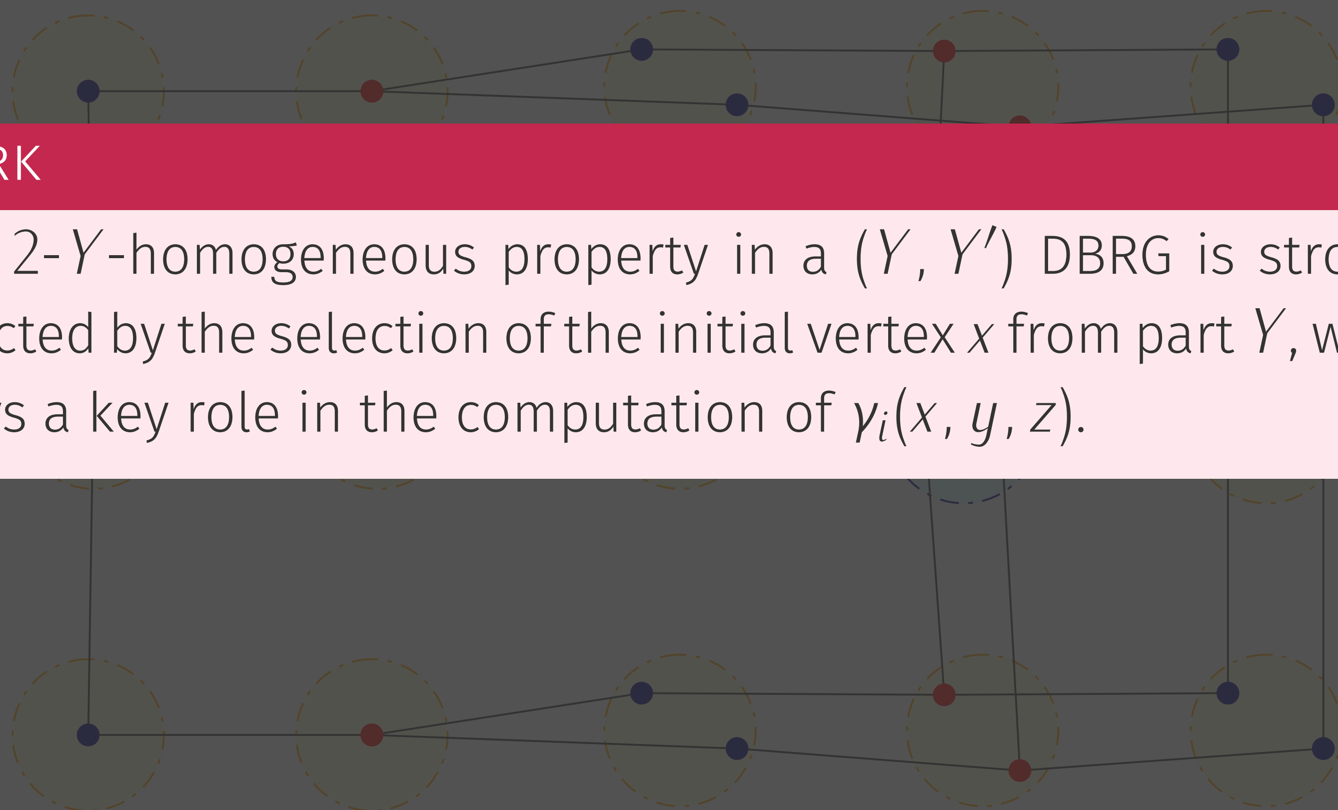
Let Γ be the subdivision of the Petersen graph with parts $\mathscr{R}$ and $\mathscr{B}$.
Every blue vertex in Γ has eccentricity equal to 6.



Γ is not 2-$\mathscr{B}$-homogeneous.

Example: the subdivision of the Petersen graph

Let Γ be the subdivision of the Petersen graph with parts $\mathcal{R}$ and $\mathcal{B}$.
Every blue vertex in Γ has eccentricity equal to 6.



Remark

The 2-Y-homogeneous property in a (Y, Y') DBRG is strongly affected by the selection of the initial vertex x from part Y, which plays a key role in the computation of $\gamma_i(x, y, z)$.

Γ is not 2-$\mathcal{B}$-homogeneous.

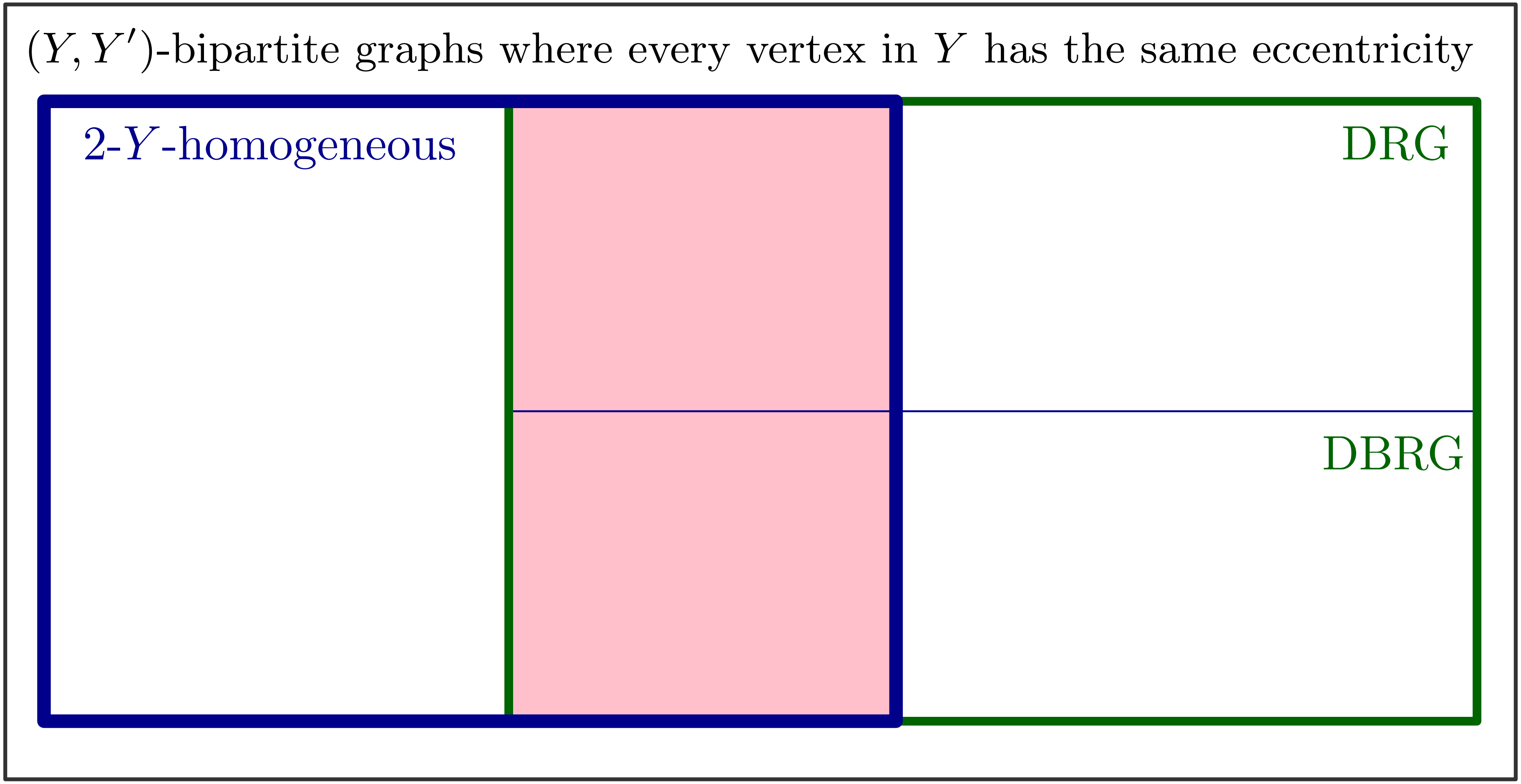
Main Motivation

- Let Γ be a (Y, Y')-DBRG.
- Suppose that every vertex in Y has eccentricity $D \geq 3$.

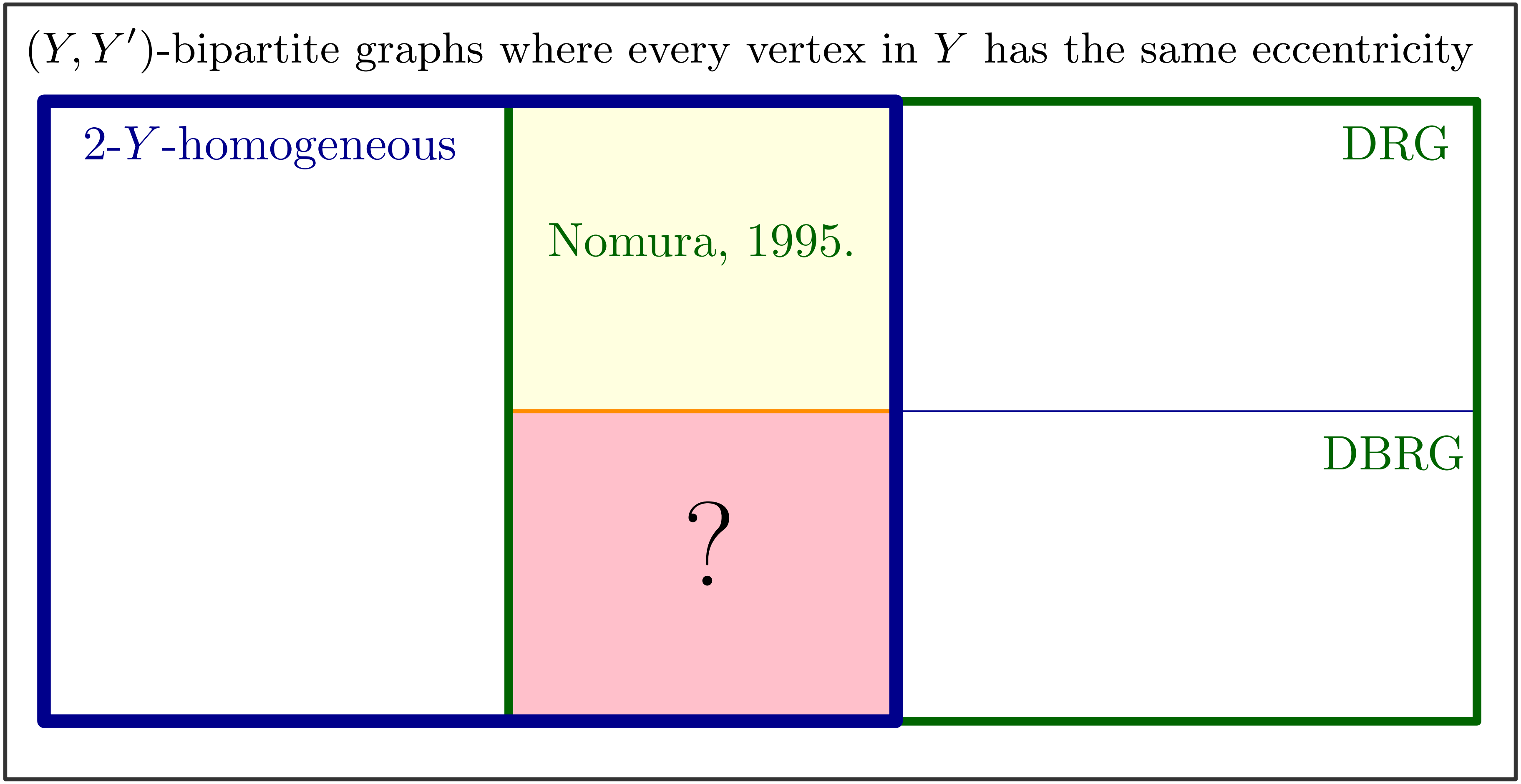
Problem

What are the necessary and sufficient conditions for Γ to be 2-Y-homogeneous?

Understanding the 2-Y-homogeneous condition

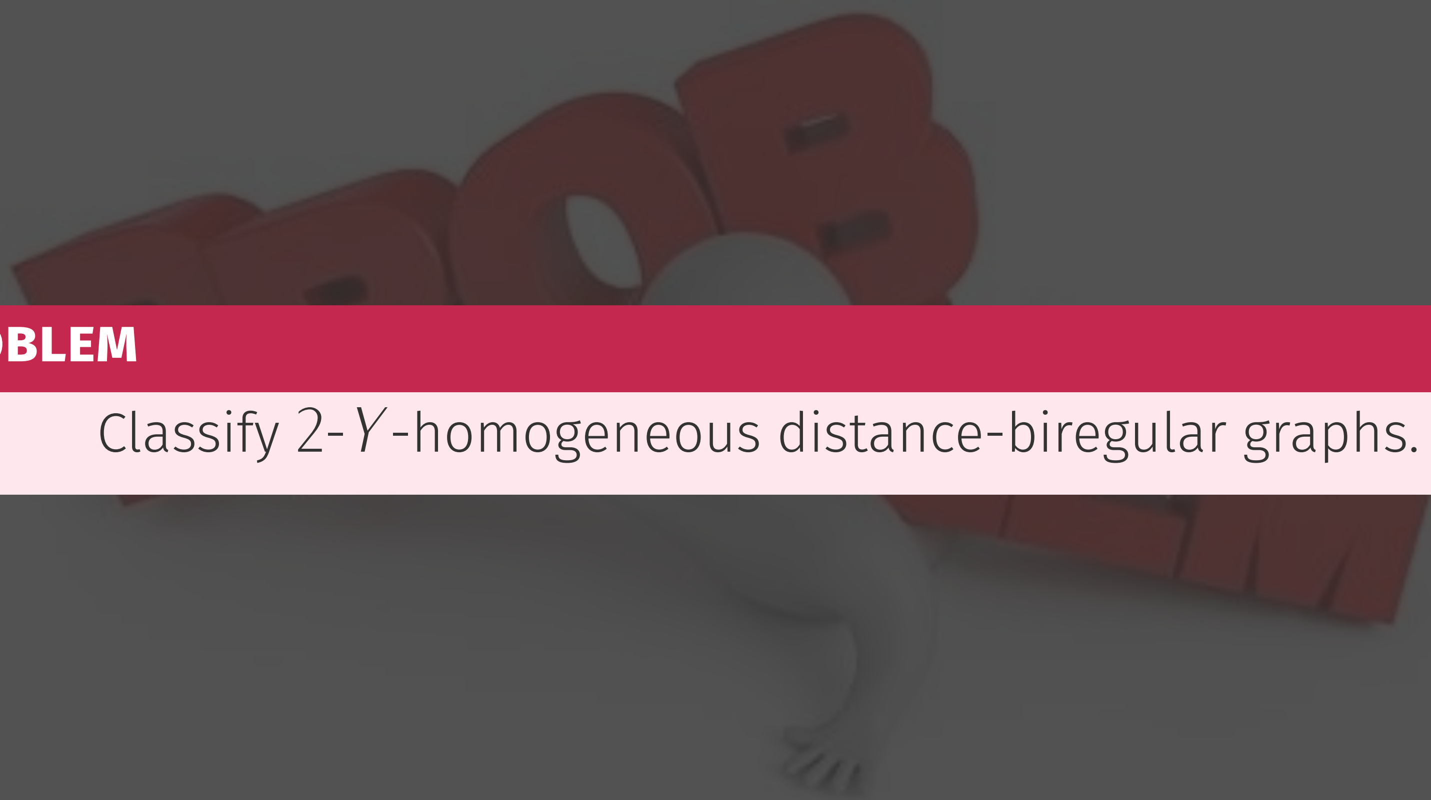


Understanding the 2-Y-homogeneous condition



Main Motivation

Main Motivation



Problem

Classify 2-Y-homogeneous distance-biregular graphs.

Case $b'_0 \geq 3$

The scalars $p^i_{2,i}(Y)$ and $\Delta_i(Y)$

Definition

For every integer i $(2 \leq i \leq D-1)$, let the scalar $p^i_{2,i} := p^i_{2,i}(Y)$ be defined as follows:

$$p^i_{2,i}(Y) = \begin{cases} \dfrac{b_i(c_{i+1}-1)+c_i(b_{i-1}-1)}{c_2} & \text{if } i \text{ is even,} \\ \dfrac{b'_i(c'_{i+1}-1)+c'_i(b'_{i-1}-1)}{c_2} & \text{if } i \text{ is odd.} \end{cases}$$

Definition

Define for every integer i $(1 \leq i \leq \min\{D-1, D'-1\})$ the scalar $\Delta_i(Y)$ as follows:

$$\Delta_i(Y) = \begin{cases} (b_{i-1}-1)(c_{i+1}-1) - p^i_{2,i}(Y)(c'_2-1) & \text{if } i \text{ is even,} \\ (b'_{i-1}-1)(c'_{i+1}-1) - p^i_{2,i}(Y)(c'_2-1) & \text{if } i \text{ is odd.} \end{cases}$$

A local condition

Theorem (Fernández, Penjić, 2023)

Let Γ be a (Y, Y')-DBRG with $D \geq 3$ and $b'_0 \geq 3$. Pick a vertex $x \in Y$. Fix an integer i $(2 \leq i \leq \min\{D-1, D'-1\})$. The following (1)–(2) are equivalent:

1. The scalar $\Delta_i(Y) = 0$.
2. For all $x, y \in Y$ with $\partial(x, y) = 2$ and $z \in \Gamma_{i,i}(x, y)$, the scalar $\gamma_i(x, y, z)$ does not depend on the choice of the triple (x, y, z).

Suppose (1)–(2) hold. Then,

$$\gamma_i(x, y, z) = \begin{cases} \dfrac{c_2 c_i (b_{i-1} - 1)}{b_i (c_{i+1} - 1) + c_i (b_{i-1} - 1)} & \text{if } i \text{ is even,} \\ \dfrac{c_2 c'_i (b'_{i-1} - 1)}{b'_i (c'_{i+1} - 1) + c'_i (b'_{i-1} - 1)} & \text{if } i \text{ is odd.} \end{cases}$$

A local condition

Theorem (Fernández, Penjić, 2023)

Let Γ be a (Y, Y')-DBRG with $D \geq 3$ and $b_0' \geq 3$. Pick a vertex $x \in Y$. Fix an integer i $(2 \leq i \leq \min\{D-1, D'-1\})$. The following (1)–(2) are equivalent:

Corollary (Fernández, Penjić, 2023)

Let Γ be a (Y, Y')-DBRG with $D \geq 3$ and $b_0' \geq 3$. Pick a vertex $x \in Y$. The following (1)–(2) are equivalent:

1. The number $\Delta_i(Y) = 0$ $(2 \leq i \leq \min\{D-1, D'-1\})$.
2. Γ is 2-Y-homogeneous.

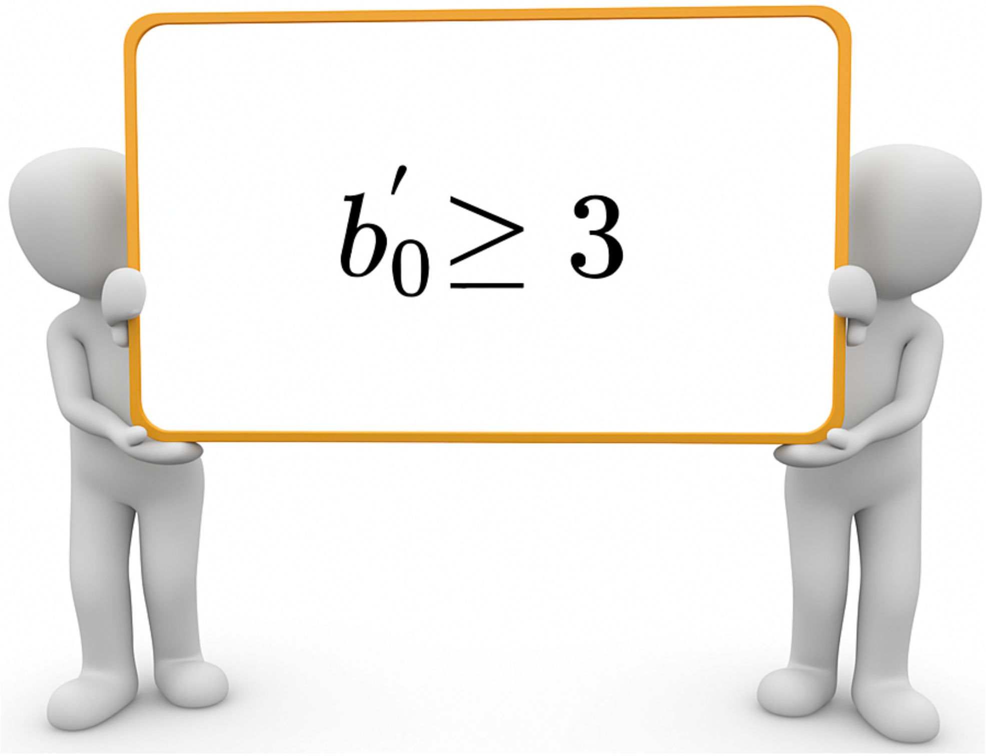
$$\gamma_i(x,y,z) = \begin{cases} \dfrac{c_2 c_i(b_{i-1}-1)}{b_i(c_{i+1}-1)+c_i(b_{i-1}-1)} & \text{if } i \text{ is even,} \\ \dfrac{c_2 c_i'(b_{i-1}'-1)}{b_i'(c_{i+1}'-1)+c_i'(b_{i-1}'-1)} & \text{if } i \text{ is odd.} \end{cases}$$

2-Y-homogeneous DBRGs with $c_2' = 1$

2-Y-homogeneous DBRG with $c_2' = 1$

Let Γ be a 2-Y-homogeneous DBRG with $D \geq 3$ and $c_2' = 1$.

- Suppose that $b_0' \geq 3$.
- Then, the scalars $\Delta_i(Y) = 0$ $(2 \leq i \leq \min\{D-1, D'-1\})$.
- Given that $c_2' = 1$, the expressions for $\Delta_i(Y)$ can be significantly simplified for each i in this range.



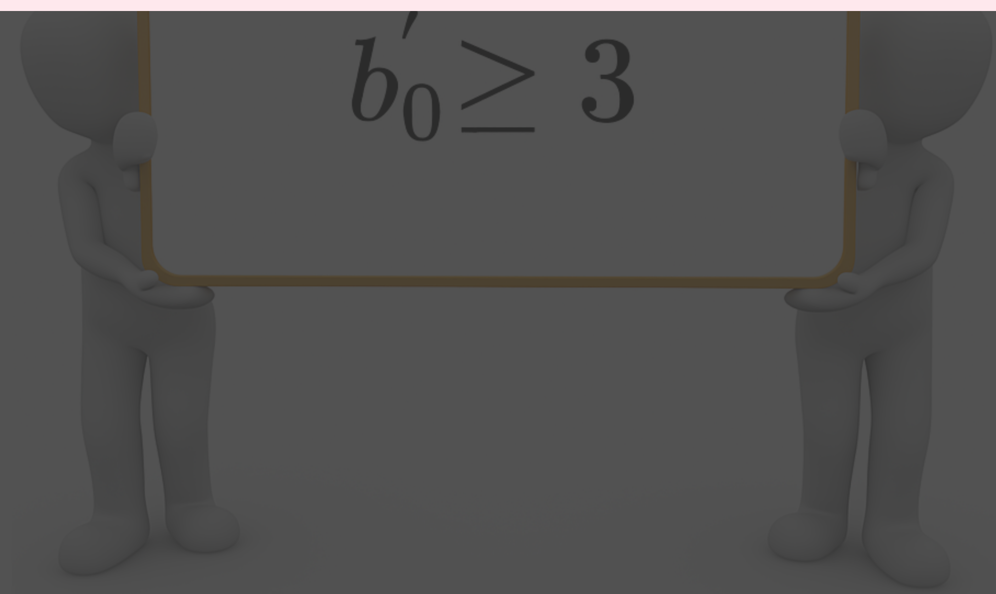
2-Y-homogeneous DBRG with $c'_2 = 1$

Let Γ be a 2-Y-homogeneous DBRG with $D \geq 3$ and $c'_2 = 1$.

- Suppose that $b'_0 \geq 3$.
- Then, the scalars $\Delta_i(Y) = 0$ $(2 \leq i \leq \min\{D-1, D'-1\})$.
- Given that $c'_2 = 1$, the expressions for $\Delta_i(Y)$ can be significantly

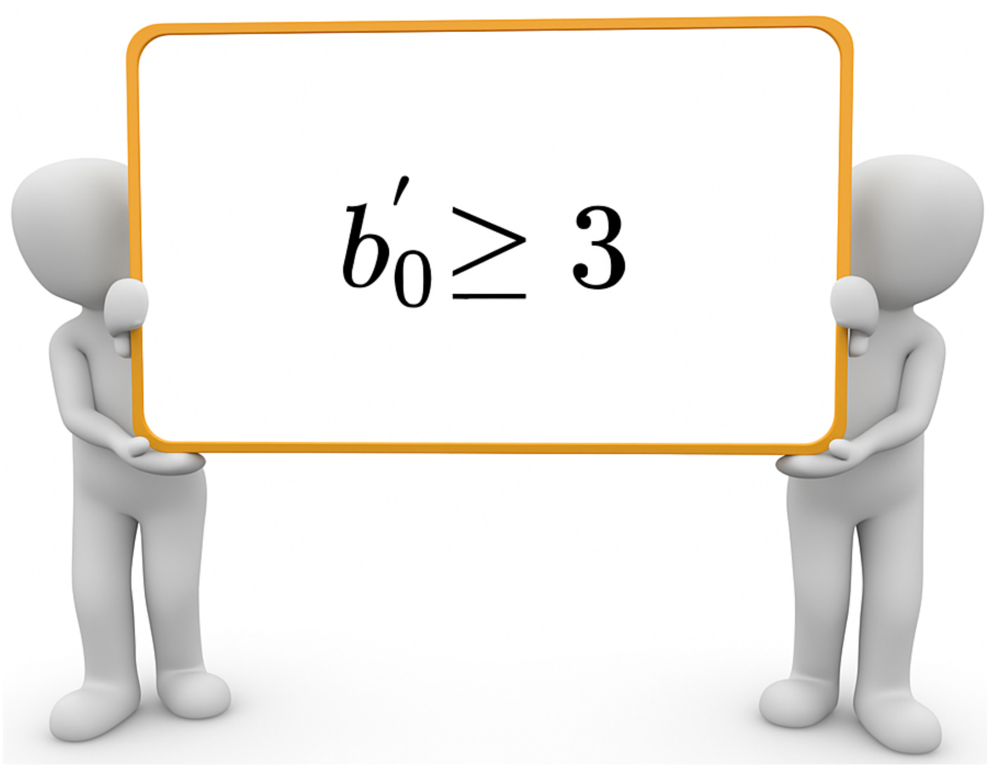
The scalars $\Delta_i(Y)$ $(2 \leq i \leq \min\{D-1, D'-1\})$

$$\Delta_i(Y) = \begin{cases} (b_{i-1} - 1)(c_{i+1} - 1) & \text{if } i \text{ is even,} \\ (b'_{i-1} - 1)(c'_{i+1} - 1) & \text{if } i \text{ is odd.} \end{cases}$$



Let Γ be a 2-Y-homogeneous DBRG with $D \geq 3$ and $c_2' = 1$.

- The case $D' < D$ is not possible!
- Therefore, $D' \geq D$ and $\min\{D-1, D'-1\} = D-1$.
- It follows that the scalar $\Delta_{D-1}(Y) = 0$.
- However, evaluating explicitly, we obtain $\Delta_{D-1}(Y) = (b_0' - 2)(b_0' - 1) = 0$, which yields a contradiction.



Let Γ be a 2-Y-homogeneous DBRG with $D \geq 3$ and $c_2' = 1$.

- The case $D' < D$ is not possible!
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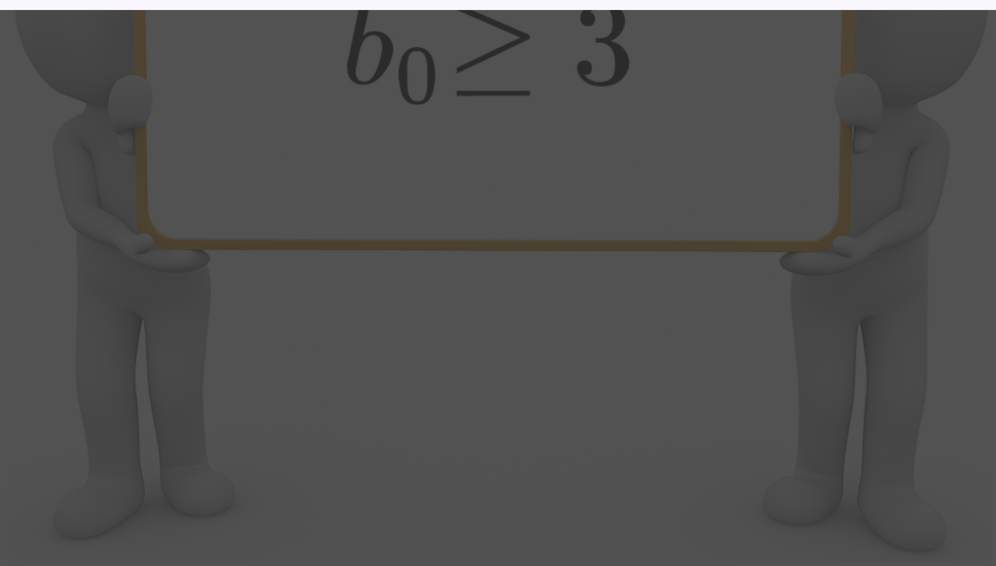
- It follows that the scalar $\Delta_{D-1}(Y) = 0$.
- However, evaluating explicitly, we obtain $\Delta_{D-1}(Y) = (b_0' - 2)(b_0' - 1) = 0$, which yields a contradiction.

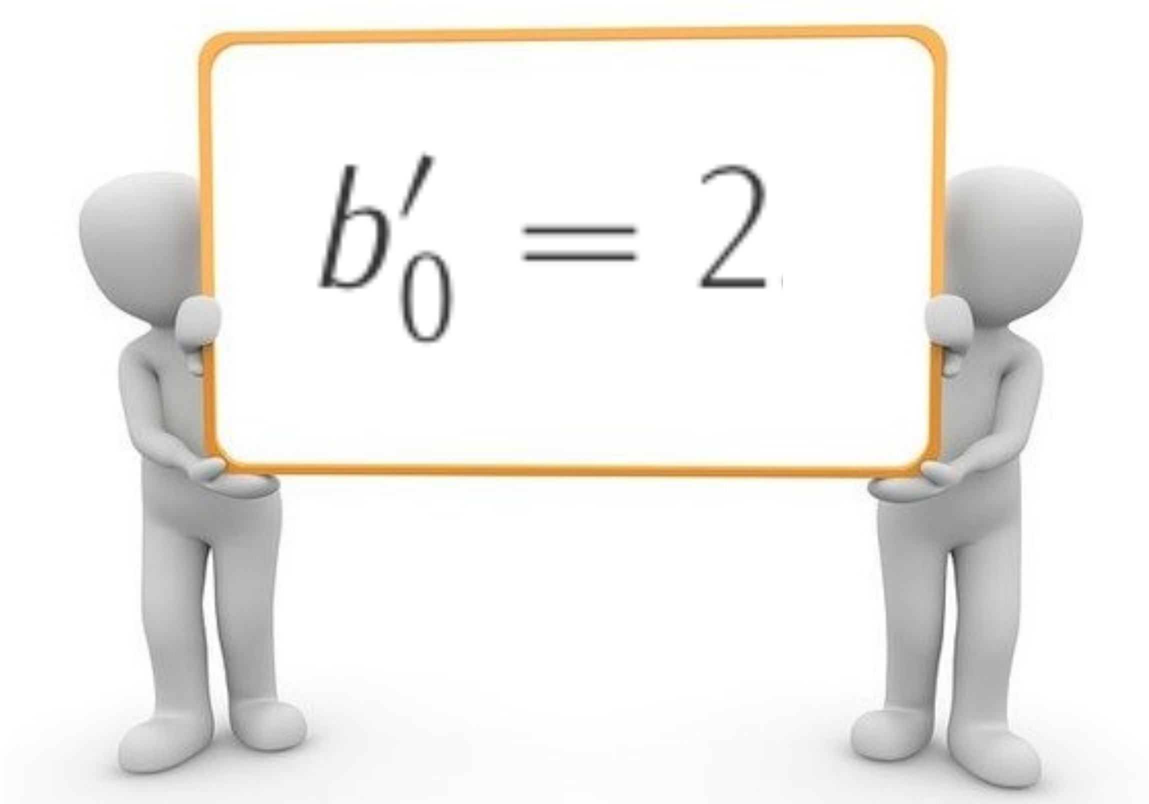
Theorem (Work in progress)

There does not exist a 2-Y-homogeneous DBRG with $D \geq 3$, $b_0' \geq 3$ and $c_2' = 1$.



Let Γ be a 2-Y-homogeneous DBRG with $D \geq 3$ and $c_2' = 1$.

- Hence, we conclude that $b_0' \leq 2$.
- Since $D \geq 3$, it follows that $b_1 = b_0' - 1 > 0$.
- Therefore, the only possibility is $b_0' = 2$.

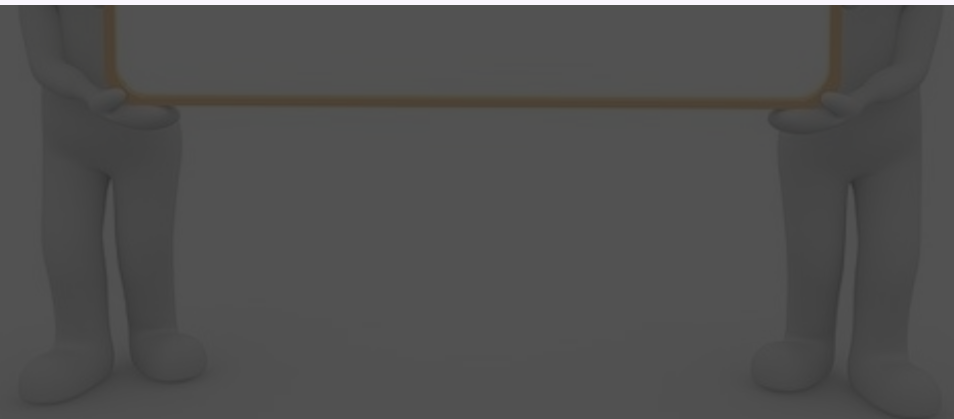


Let Γ be a 2-Y-homogeneous DBRG with $D \geq 3$ and $c_2' = 1$.

- Hence, we conclude that $b_0' \leq 2$.
- Since $D \geq 3$, it follows that $b_1 = b_0' - 1 > 0$.
- Therefore, the only possibility is $b_0' = 2$.

Theorem (Mohar, Shawe-Taylor, 1985)

A graph Γ with vertices of valency 2 is DBRG if and only if Γ is a complete bipartite graph $K_{2,n}$ $(n \geq 1)$ or Γ is the subdivision graph of a minimal (κ, g)-cage graph, where $\kappa, g \geq 3$.



Let Γ be a 2-Y-homogeneous DBRG with $D \geq 3$ and $c_2' = 1$.

- Hence, we conclude that $b_0' \leq 2$.
- Since $D \geq 3$, it follows that $b_1 = b_0' - 1 > 0$.
- Therefore, the only possibility is $b_0' = 2$.

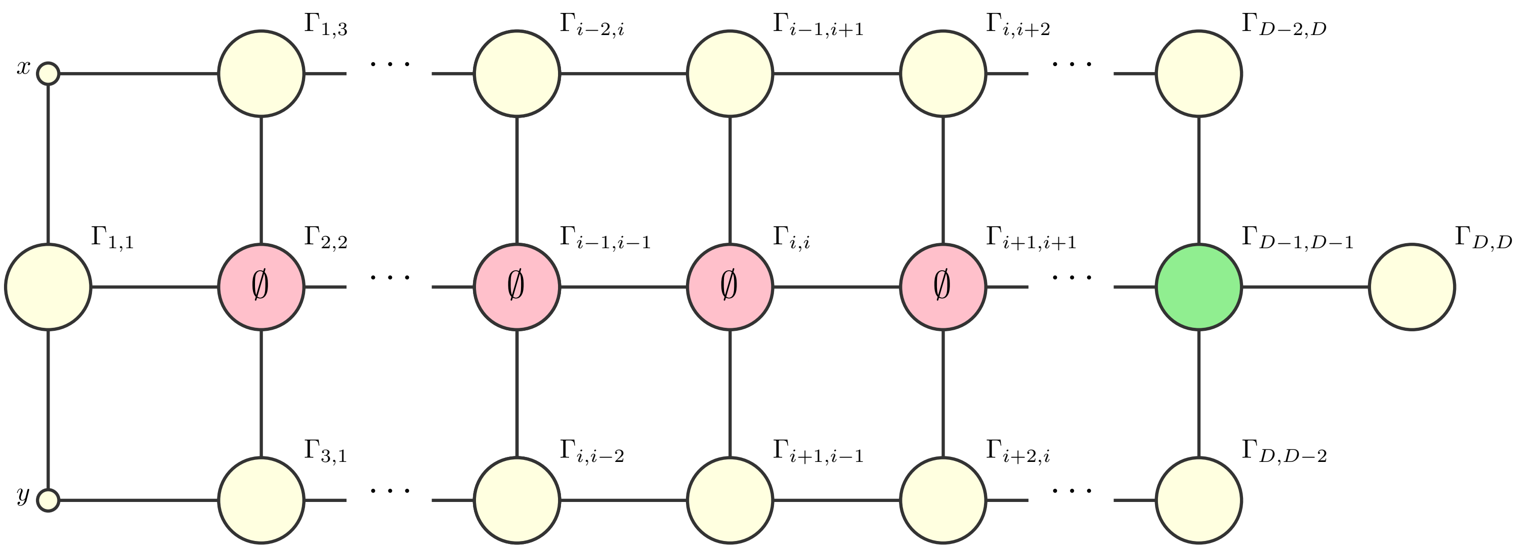
Theorem (Work in Progress)

A 2-Y-homogeneous DBRG Γ with $D \geq 3$ and $c_2' = 1$ is the subdivision graph of a minimal (κ, g)-cage graph $(\kappa, g \geq 3)$ with vertex set X, edge set $\mathcal{R}$, and parts $Y = X$ and $Y' = \mathcal{R}$.

The converse also holds!

Theorem (F.,Penjić, 2023.)

Let Γ' be a minimal (κ, g)-cage graph ($\kappa, g \geq 3$) with vertex set X, edge set $\mathcal{R}$. Then, the subdivision graph Γ of Γ' is a DBRG with parts $Y = X$ and $Y' = \mathcal{R}$, which is 2-Y-homogeneous.



Γ is 2-Y-homogeneous with $\gamma_i = 0$ $(2 \leq i \leq D-1)$.

The converse also holds!

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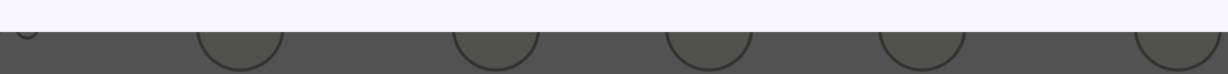
Let Γ' be a minimal (κ, g)-cage graph ($\kappa, g \geq 3$) with vertex set X, edge set $\mathcal{R}$. Then the subdivision graph Γ of Γ' is a DBRG with

Theorem (Work in progress)

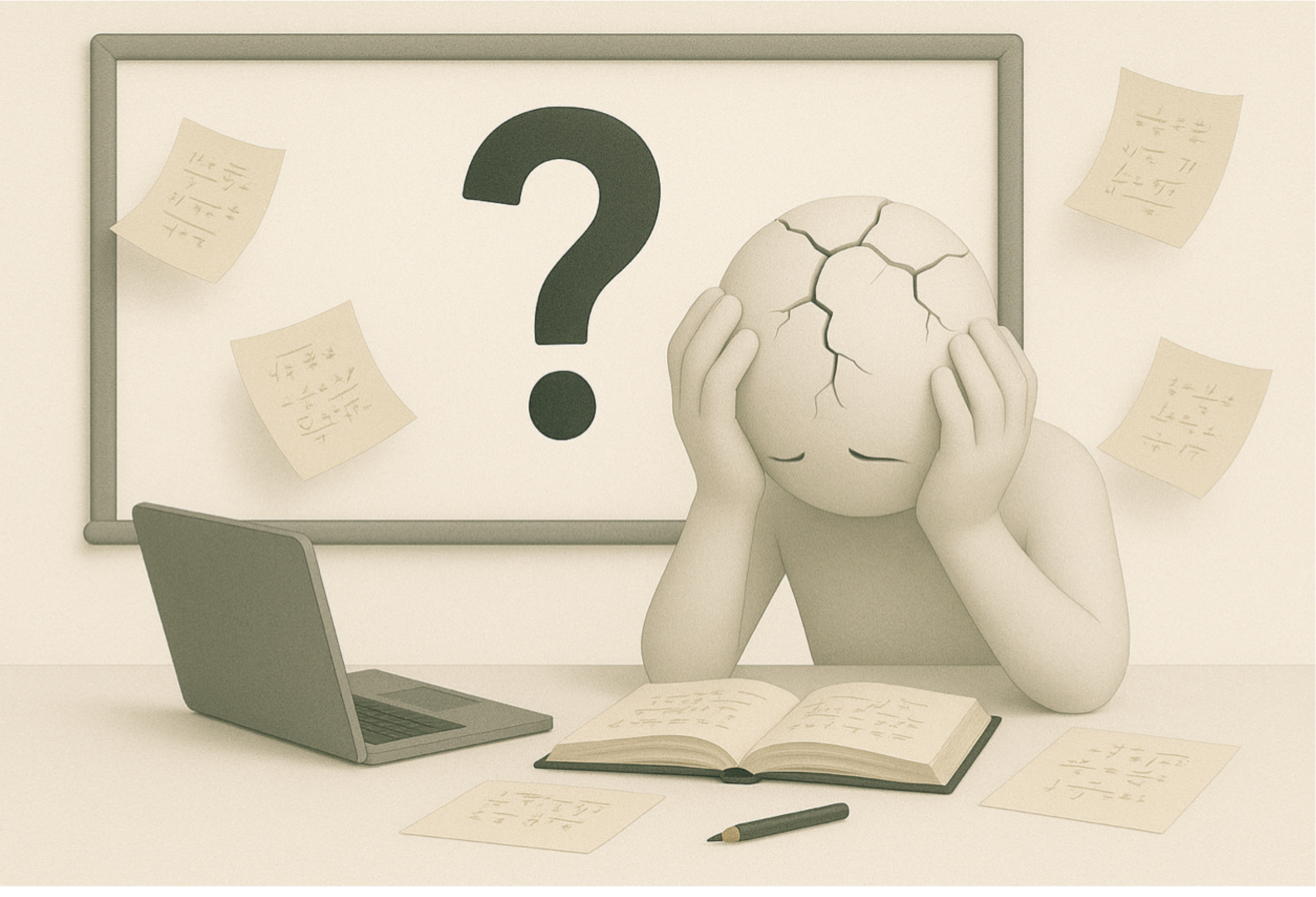
Let Γ be a (Y, Y')-distance-biregular graph with $D \geq 3$. Then, the following statements are equivalent:

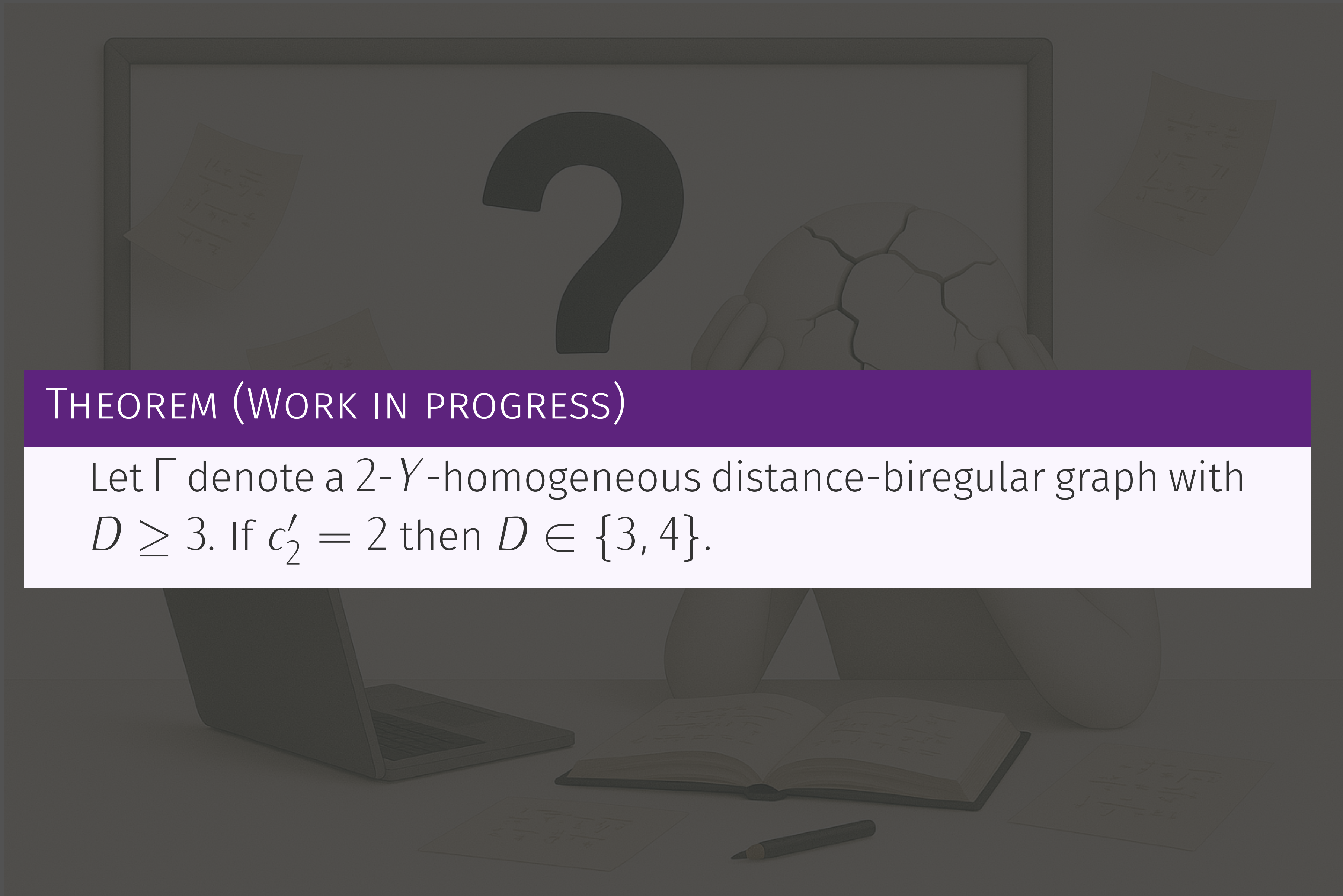
1. Γ is 2-Y-homogeneous with $c_2' = 1$.
2. Γ is the subdivision graph of a minimal (κ, g)-cage graph ($\kappa, g \geq 3$) with vertex set X, edge set $\mathcal{R}$, and parts $Y = X$, $Y' = \mathcal{R}$.
3. Γ is 2-Y-homogeneous with $b_0' = 2$.



Γ is 2-Y-homogeneous with $\gamma_i = 0$ $(2 \leq i \leq D-1)$.

2-Y-homogeneous DBRGs with $c_2' = 2$





Theorem (Work in progress)

Let Γ denote a 2-Y-homogeneous distance-biregular graph with $D \geq 3$. If $c_2' = 2$ then $D \in \{3, 4\}$.

2-Y-homogeneous DBRGs with $c_2' \geq 3$

Bounding the Diameter via Inequalities

Let Γ be a 2-Y-homogeneous (Y, Y')-distance-biregular graph with $c_2' \geq 3$ and $D \geq 3$.

- There exist constants $c \geq 3$, $s \geq 3$, and $g \geq 2$ such that:

$$c_3 - b_3 = \frac{g(s-2) + c[2 + s(g-2)]}{g(g-1)}.$$

- Under these assumptions, the inequality $c_3 - b_3 > 0$ holds, implying $b_3 < c_3$.

Lemma (F., Penjić, 2023)

Let Γ be a (Y, Y')-distance-biregular graph. If $i + j \leq D$ and $i + j$ is even, then $c_i \leq b_j$.

- Applying the lemma with $i = j = 3$, we obtain $c_3 \leq b_3$ when $D \geq 6$, which contradicts the earlier conclusion that $b_3 < c_3$.

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Theorem (F., Penjić, 2023)

Let Γ denote a 2-Y-homogeneous distance-biregular graph with $D \geq 3$. If $c_2' \geq 3$ then $D \in \{3, 4, 5\}$.

Lemma (F., Penjić, 2023)

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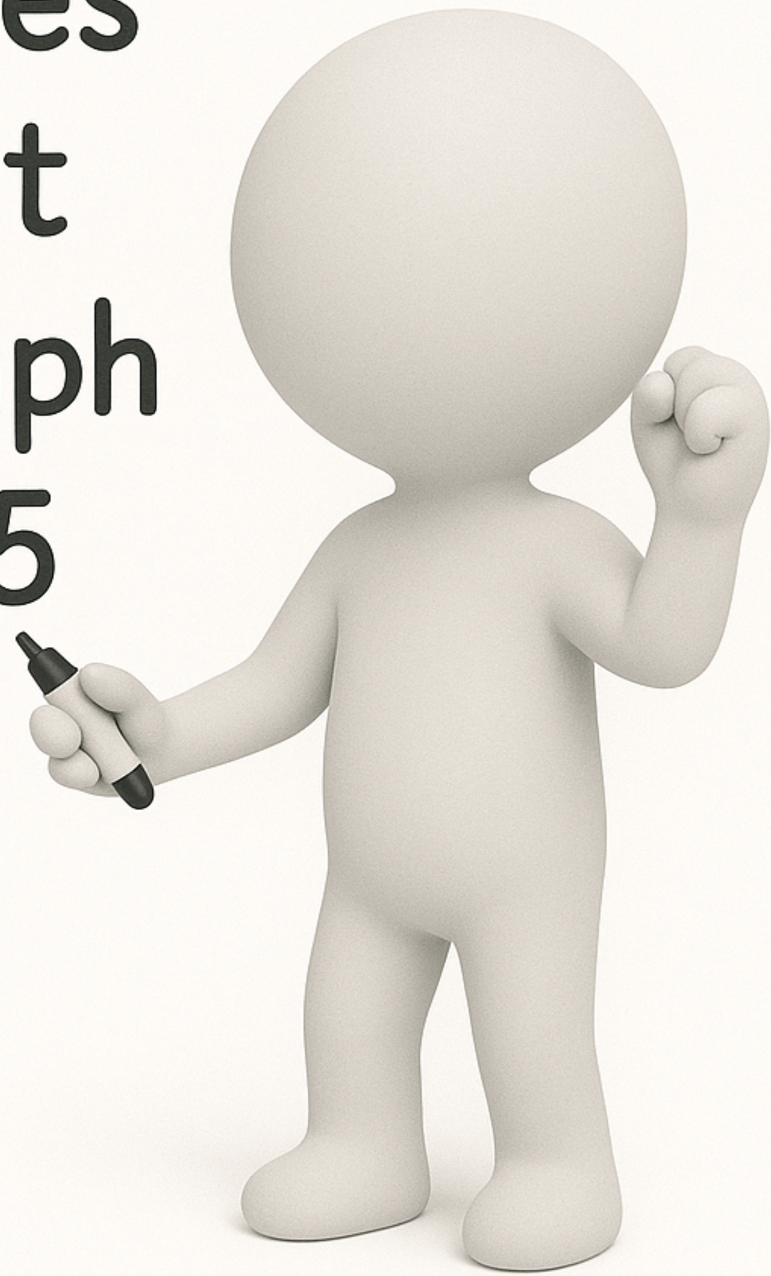
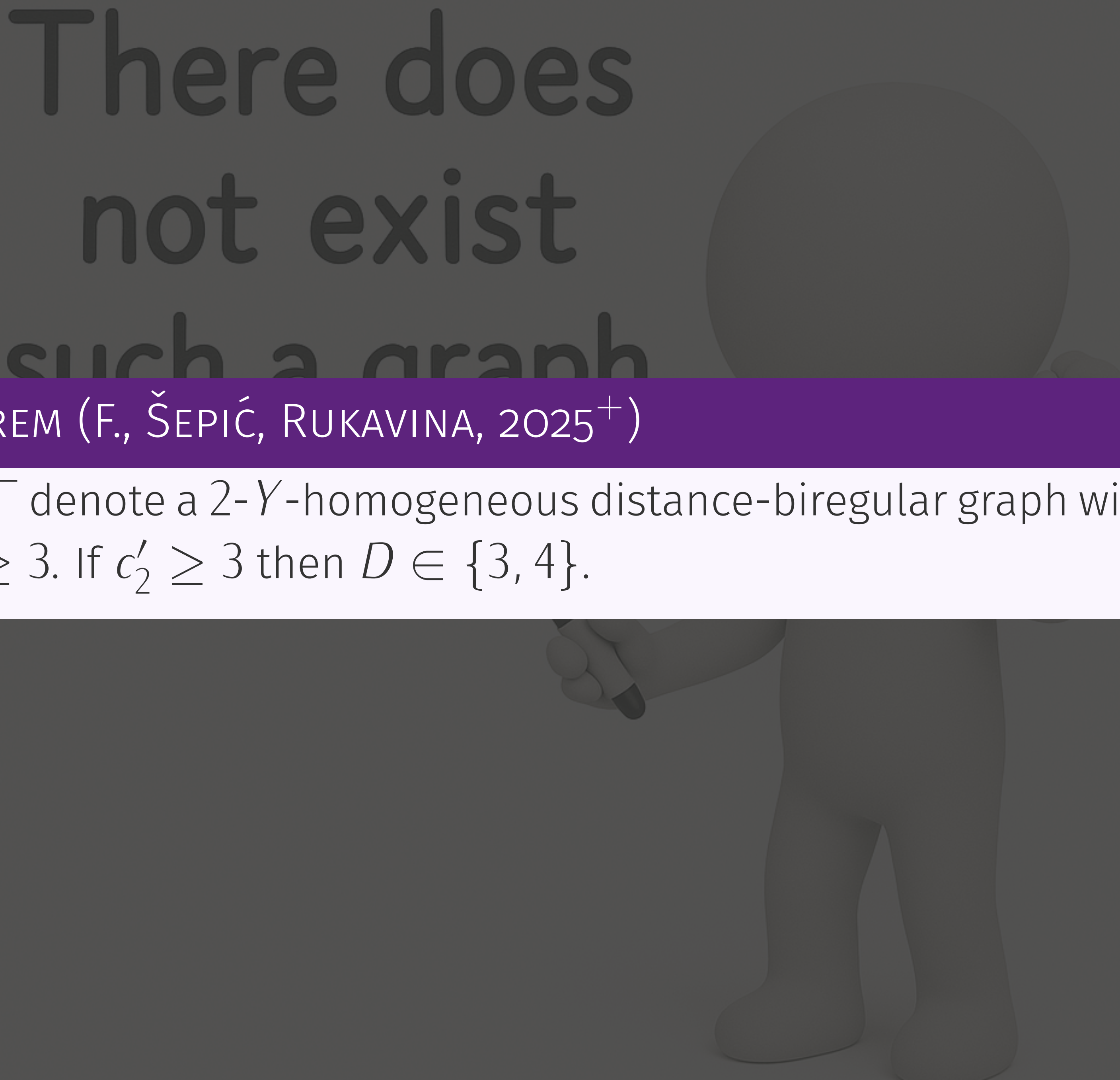
There does not exist such a graph with $D = 5$





Theorem (F., Šepić, Rukavina, 2025^+)

Let Γ denote a 2-Y-homogeneous distance-biregular graph with $D \geq 3$. If $c_2' \geq 3$ then $D \in \{3, 4\}$.

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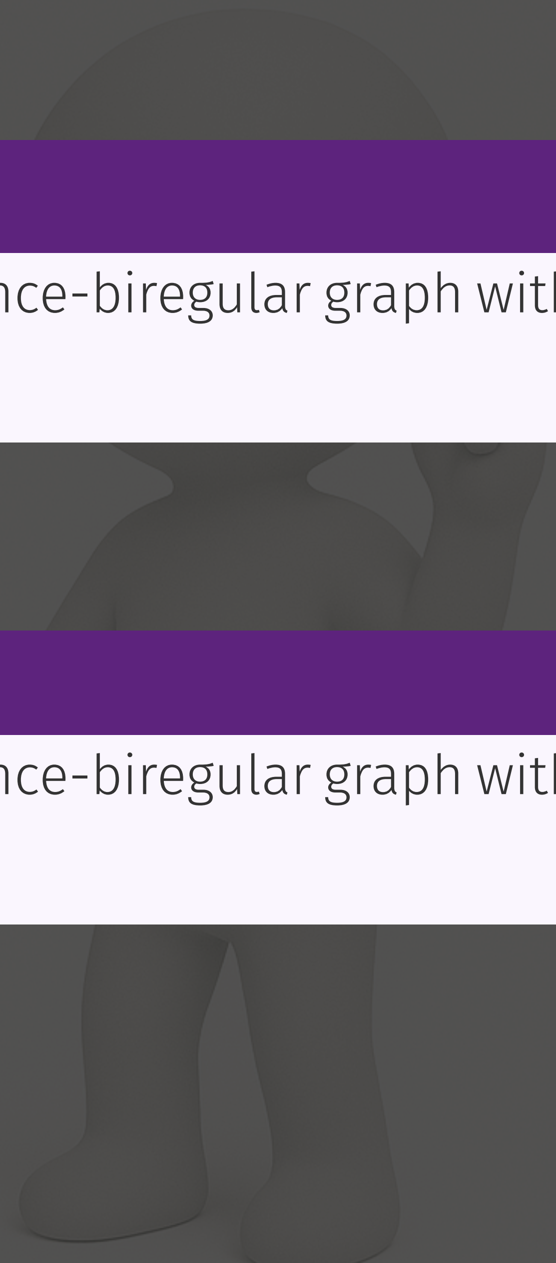
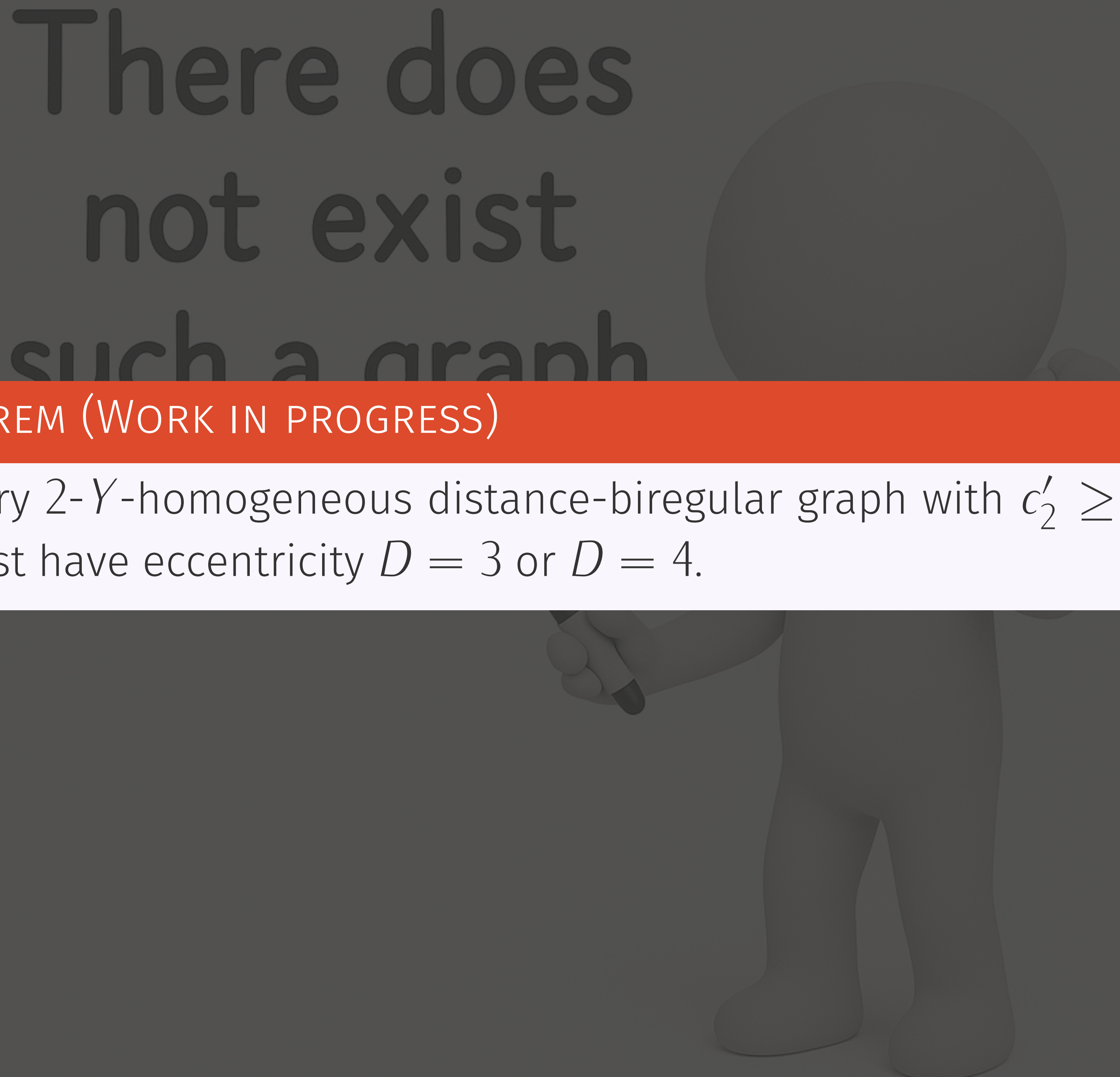


Theorem (F., Šepić, Rukavina, 2025^{+})

Let Γ denote a 2-Y-homogeneous distance-biregular graph with $D \geq 3$. If $c_2' \geq 3$ then $D \in \{3, 4\}$.

Theorem (Work in progress)

Let Γ denote a 2-Y-homogeneous distance-biregular graph with $D \geq 3$. If $c_2' = 2$ then $D \in \{3, 4\}$.



Theorem (Work in progress)

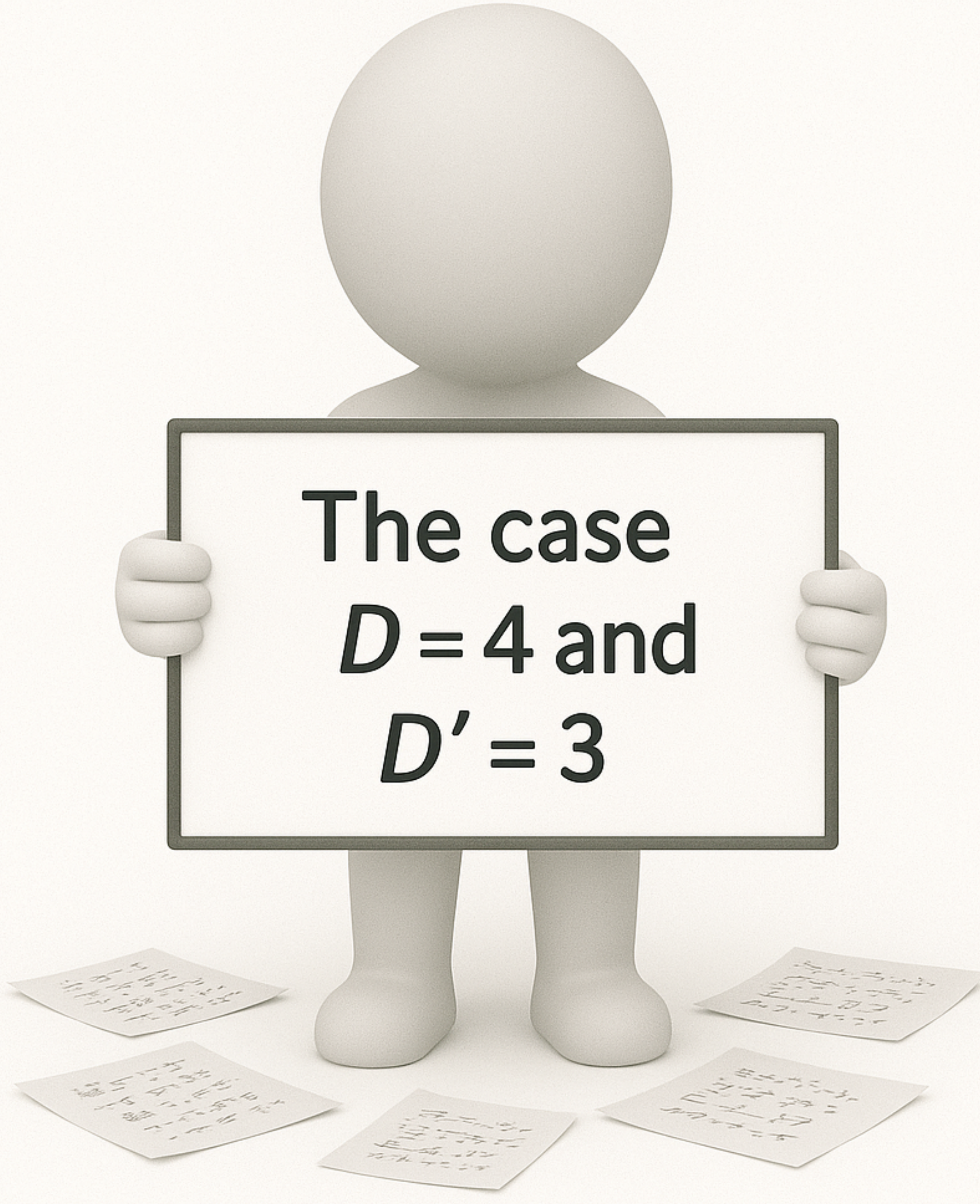
Every 2-Y-homogeneous distance-biregular graph with $c_2' \geq 2$ must have eccentricity $D = 3$ or $D = 4$.

2-Y-homogeneous DBRGs with $c_2' \geq 2$:

(1) The case $D = 4$ and $D' = 3$

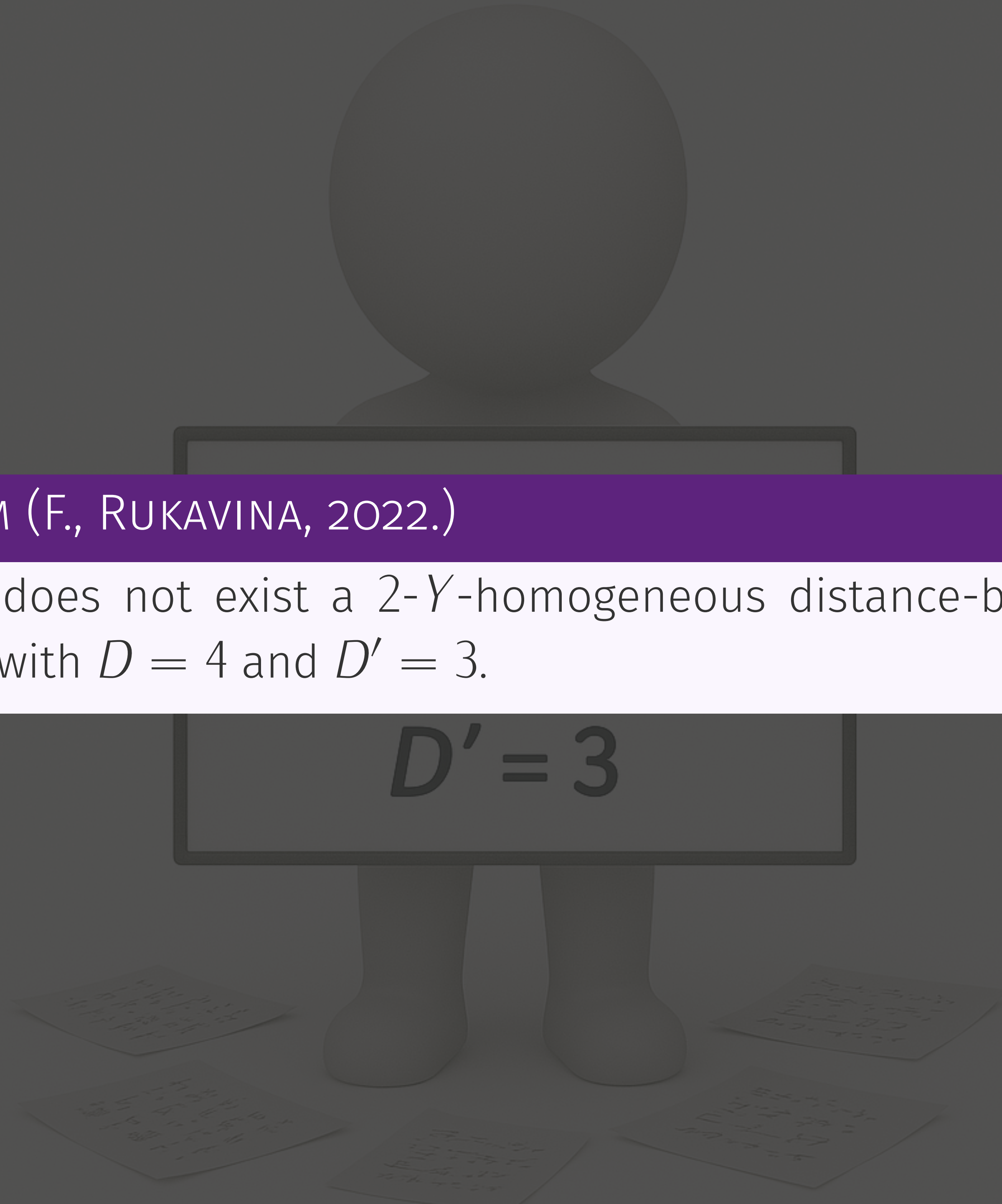
(2) The case $D = D' = 4$

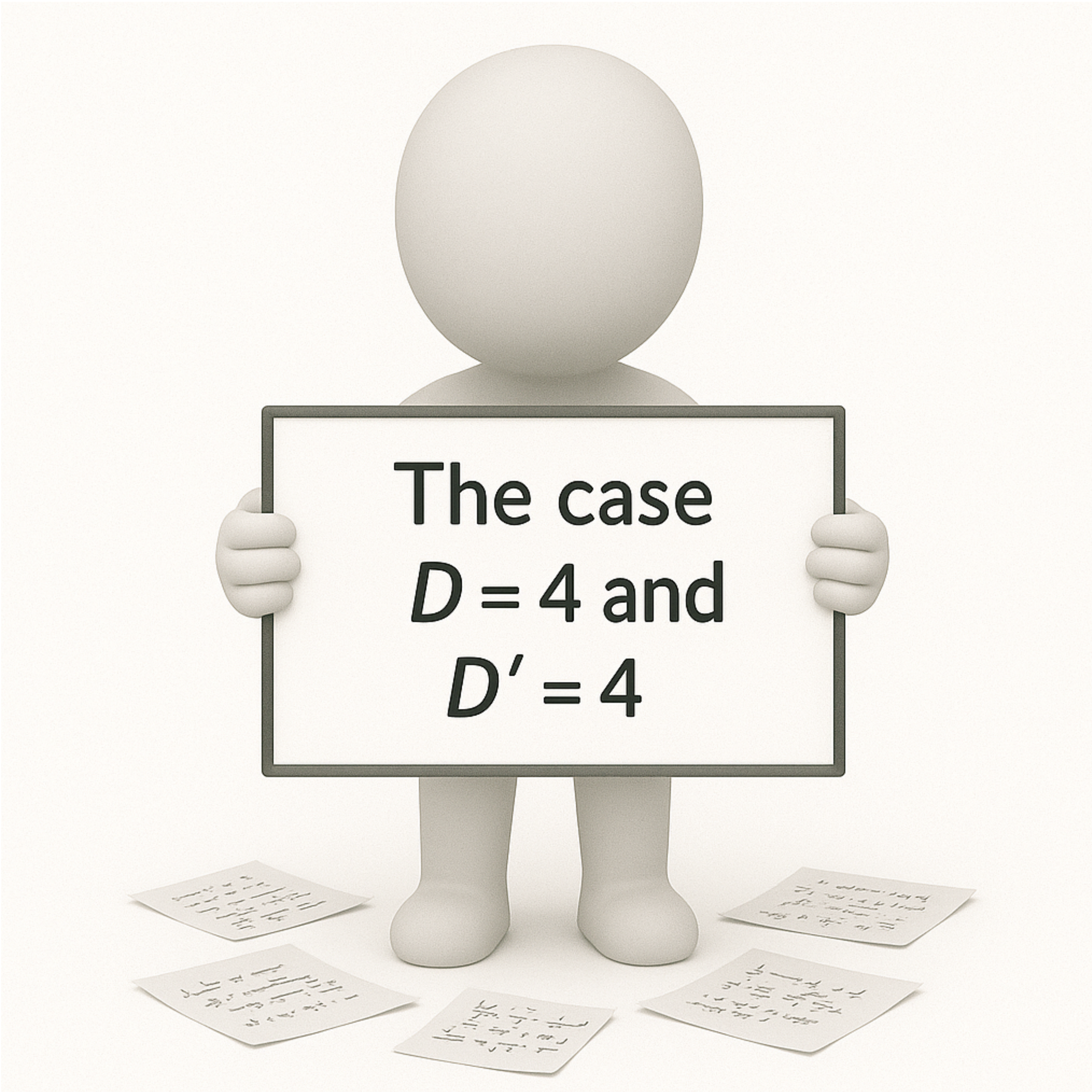
(3) The case $D = 3$


The case
$D = 4$ and
$D' = 3$

Theorem (F., Rukavina, 2022.)

There does not exist a 2-Y-homogeneous distance-biregular graph with $D = 4$ and $D' = 3$.

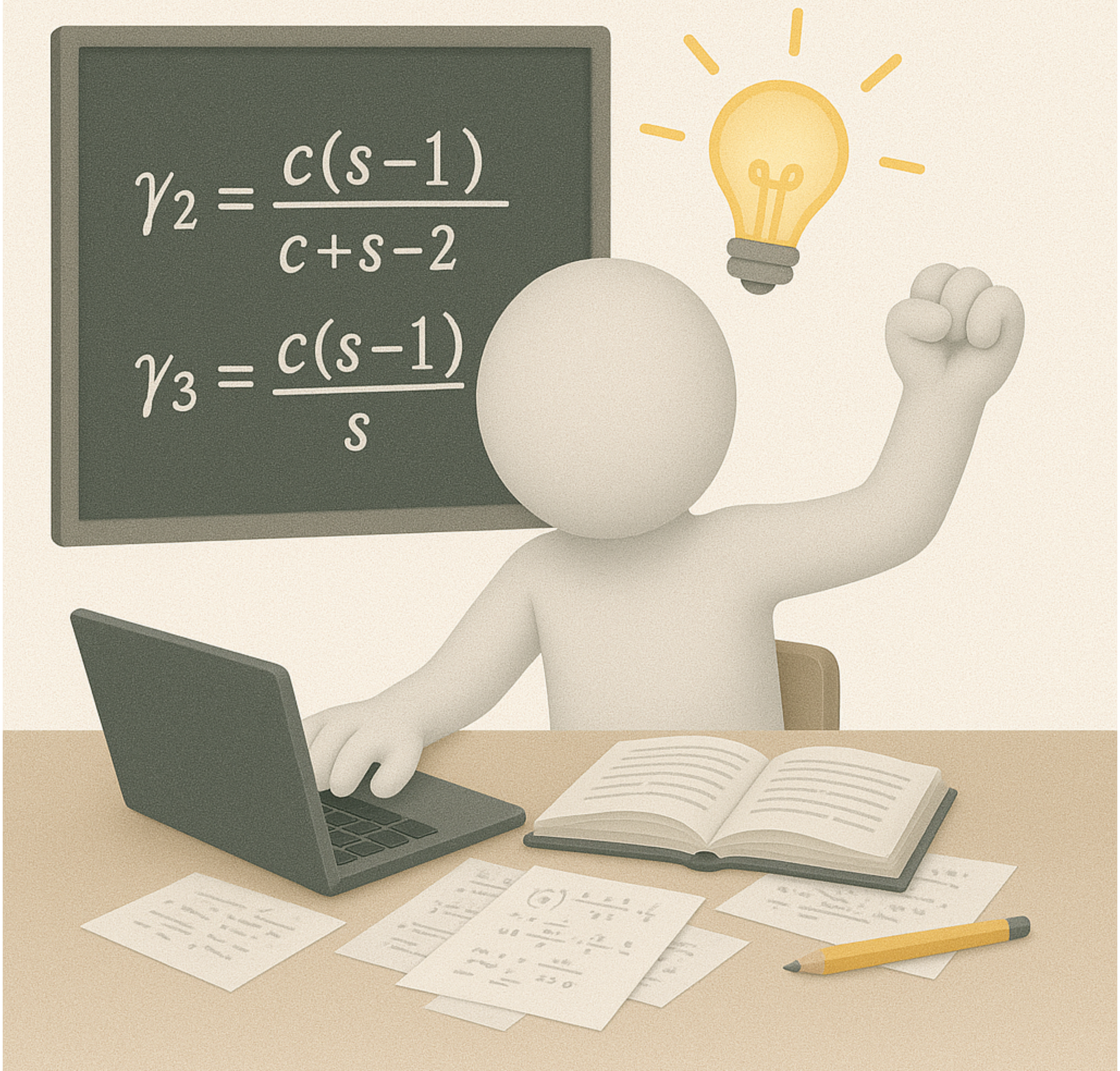


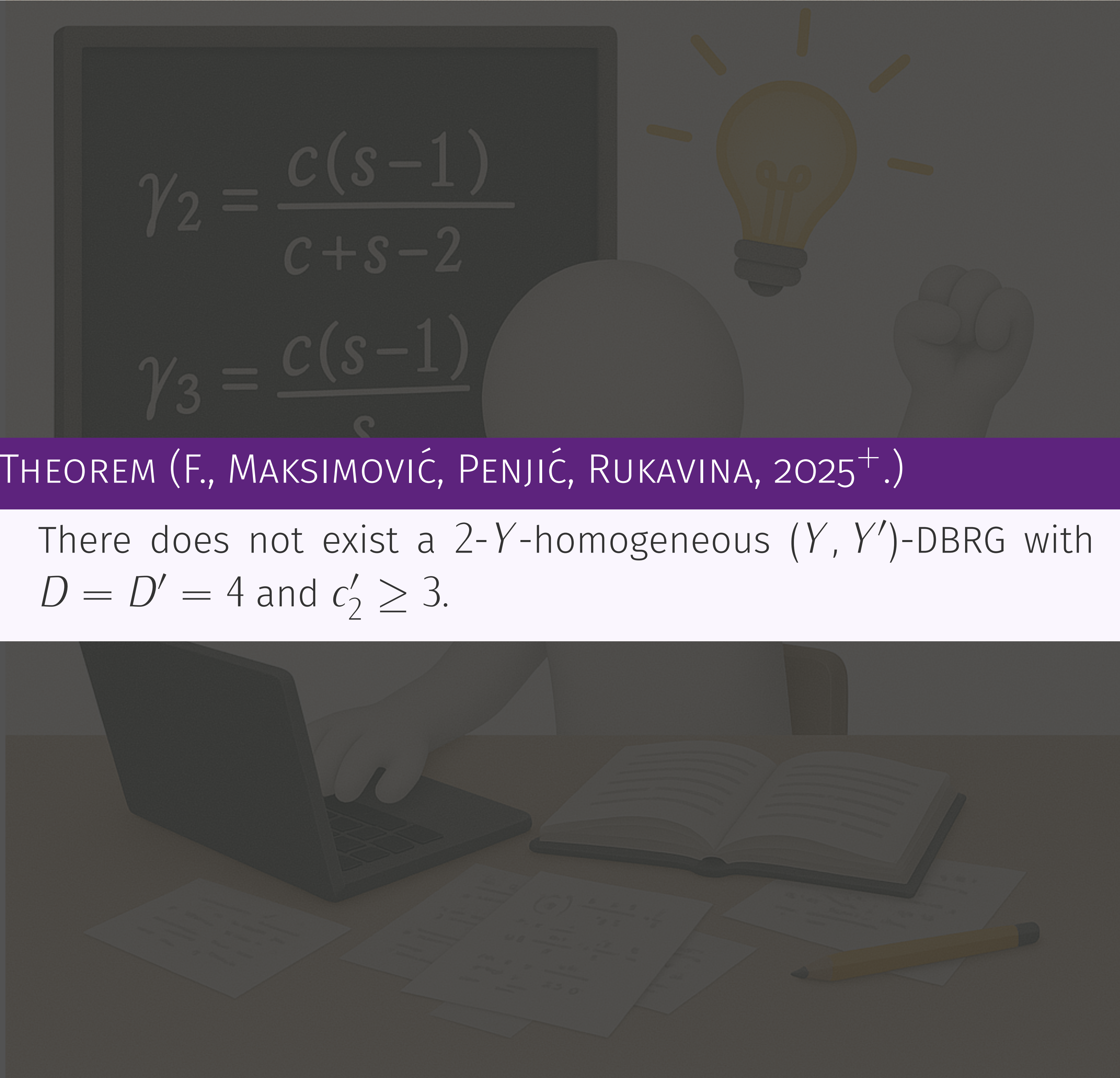

The case
$D = 4$ and
$D' = 4$

Theorem (F., Maksimović, Penjić, Rukavina, 2025^{+}.)

Let Γ denote a (Y, Y')-DBRG with $D = D' = 4$ and $c_2' \geq 2$. Then, Γ is 2-Y-homogeneous if and only if there exist positive integers $c > s \geq 2$ such that Γ has the following intersection array:

$$\left| \begin{array}{ccccc} \frac{c(c+s-2)}{s-1}; & 1, & c, & c+s-1, & \frac{c(c+s-2)}{s-1} \\ c+s; & 1, & s, & \frac{c(c+s-1)}{s}, & c+s \end{array} \right|.$$


$\gamma_2 = \frac{c(s-1)}{c+s-2}$
$\gamma_3 = \frac{c(s-1)}{s}$



Theorem (F., Maksimović, Penjić, Rukavina, 2025^{+}.)

There does not exist a 2-Y-homogeneous (Y, Y')-DBRG with $D = D' = 4$ and $c_2' \geq 3$.


$c_2' = 2$

Theorem (F., Maksimović, Penjić, Rukavina, 2025^{+}.)

Let Γ denote a (Y, Y')-DBRG with $D = D' = 4$ and $c_2' \geq 2$. Then, Γ is 2-Y-homogeneous if and only if there exist an even integer $c \geq 4$ such that Γ has the following intersection array:

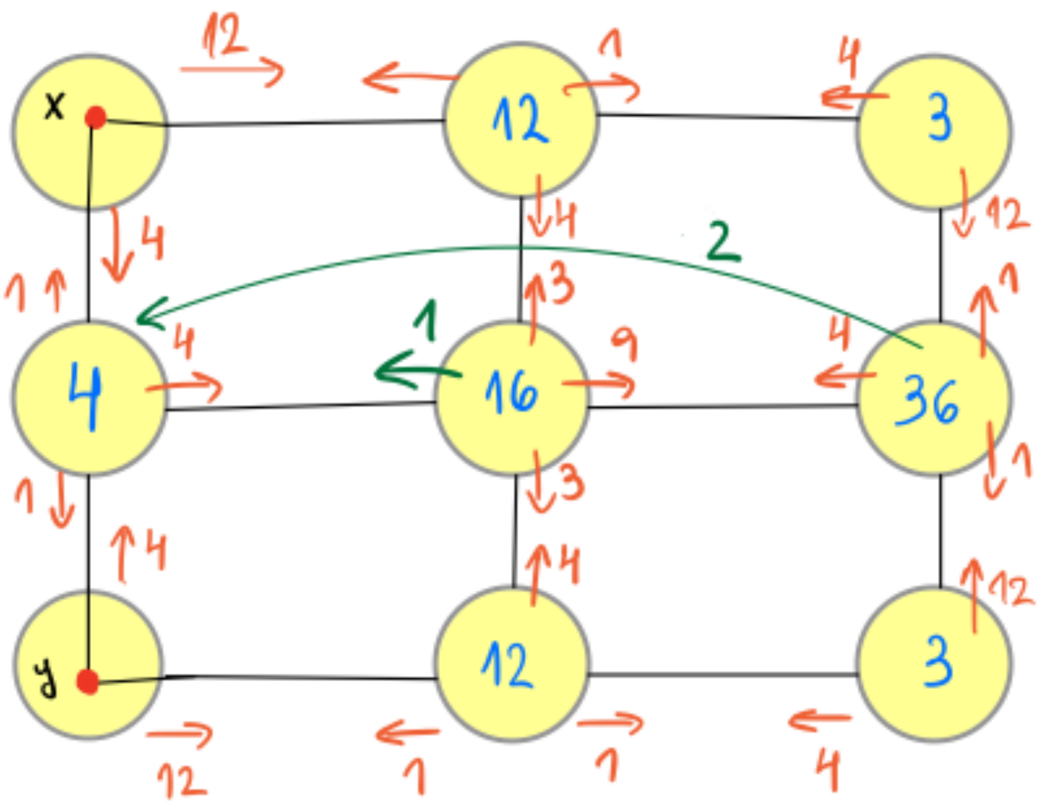
$$\left| \begin{array}{cccccc} c^2; & & 1, & c, & c+1, & c^2 \\ c+2; & & 1, & 2, & \frac{c(c+1)}{2}, & c+2 \end{array} \right| .$$

Europ. J. Combinatorics (1994) **15**, 223–238

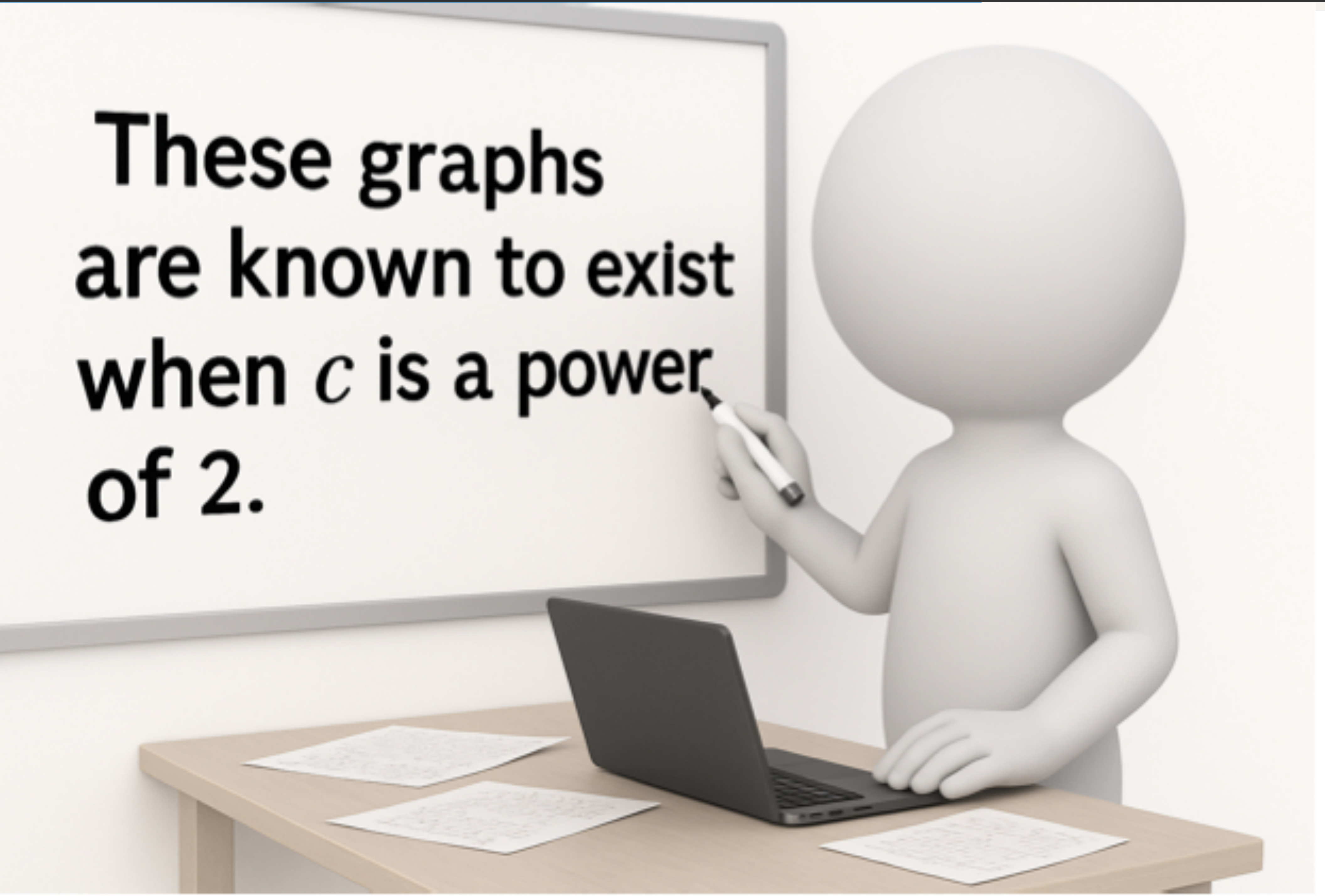
Distance Biregular Bipartite Graphs

C. Delorme

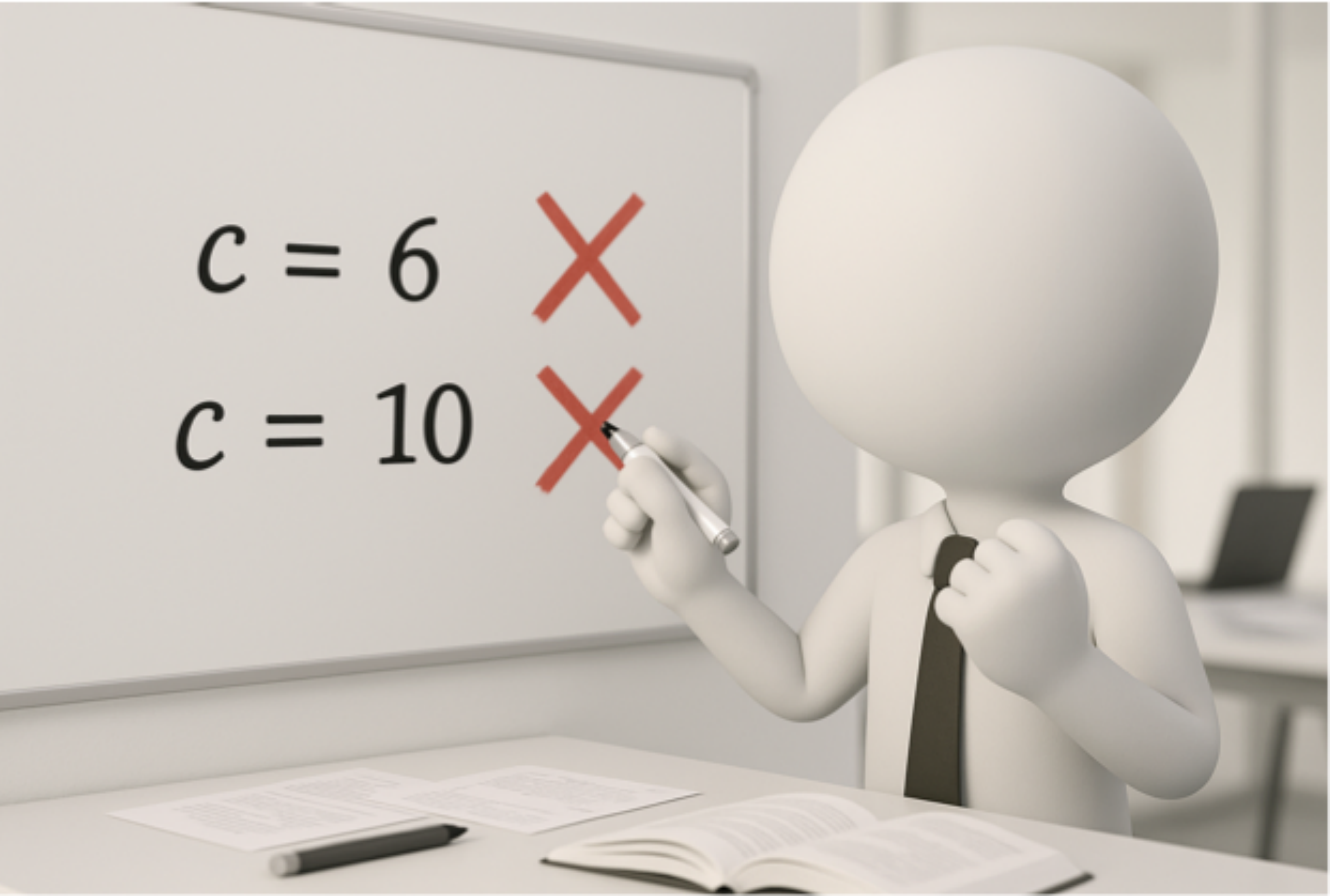
16,	1,	4,	5,	16
6,	1,	2,	10,	6



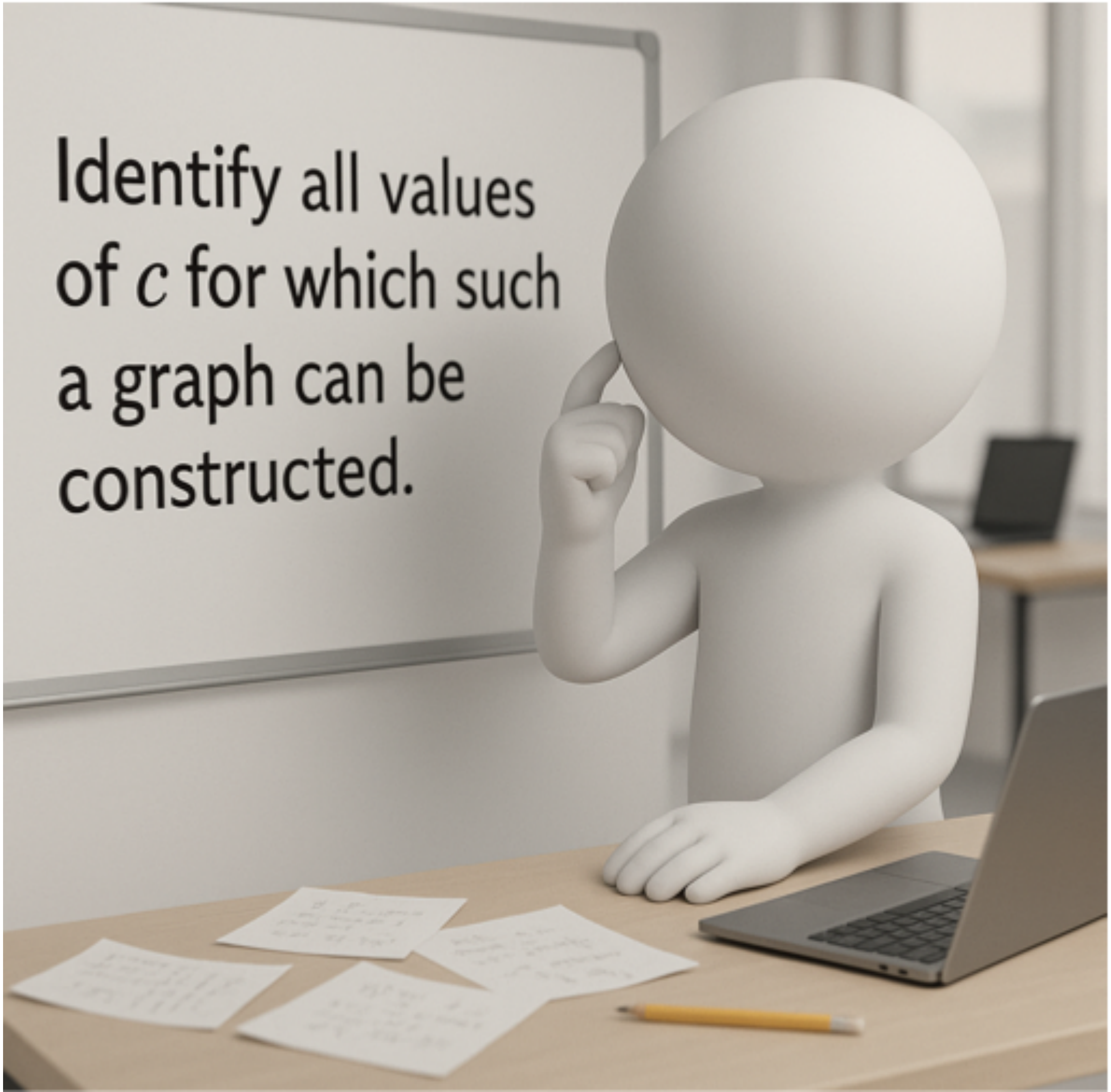
Denniston (1969) and van den Akker (1990)



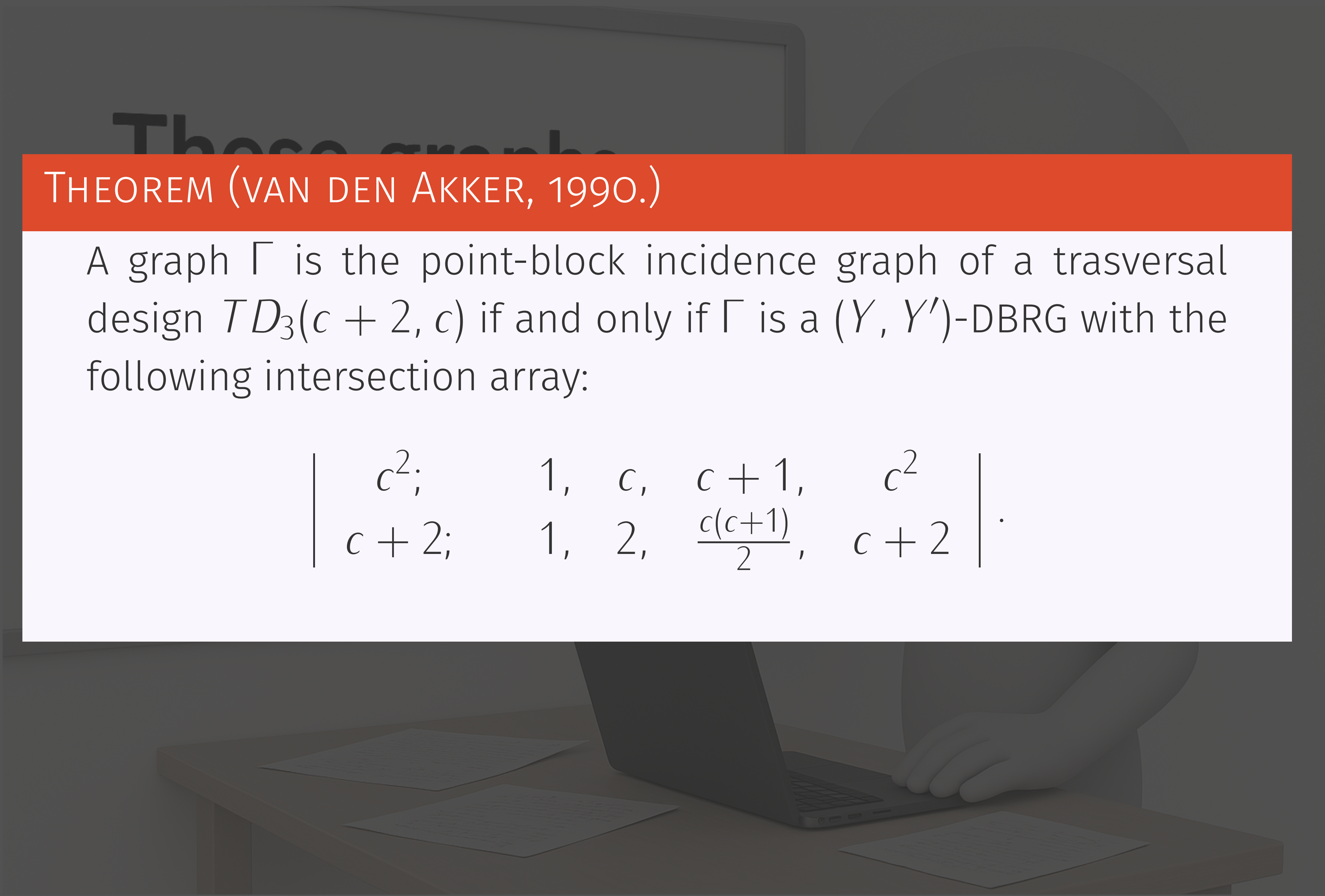
van den Akker (1990)



Open Problem



Denniston (1969) and van den Akker (1990)

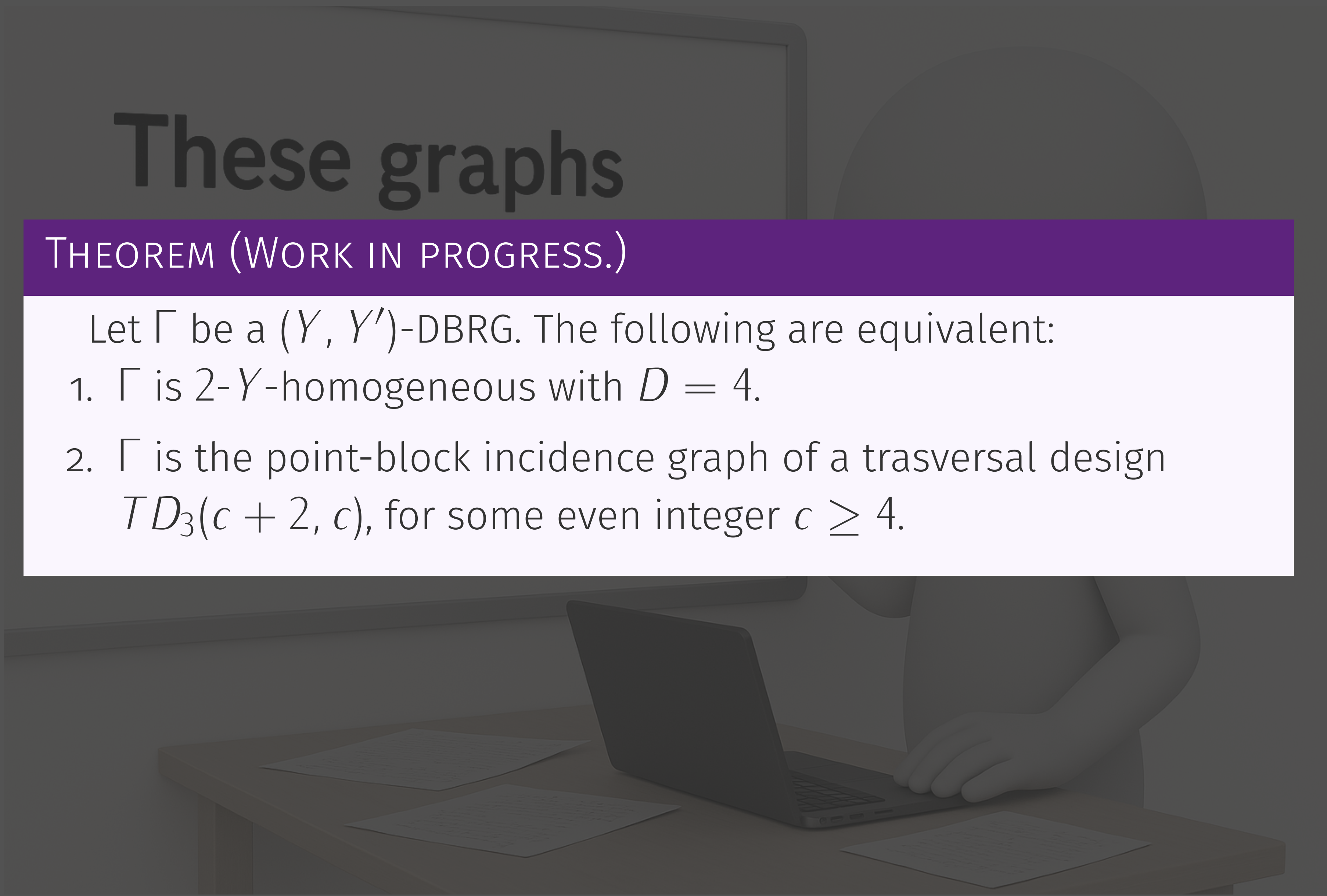


Theorem (van den Akker, 1990.)

A graph Γ is the point-block incidence graph of a trasversal design $TD_3(c+2, c)$ if and only if Γ is a (Y, Y')-DBRG with the following intersection array:

$$\left|\begin{array}{ccccc} c^2; & 1, & c, & c+1, & c^2 \\ c+2; & 1, & 2, & \frac{c(c+1)}{2}, & c+2 \end{array}\right|.$$

Denniston (1969) and van den Akker (1990)



Theorem (Work in progress.)

Let Γ be a (Y, Y')-DBRG. The following are equivalent:

1. Γ is 2-Y-homogeneous with $D = 4$.
2. Γ is the point-block incidence graph of a trasversal design $TD_3(c+2, c)$, for some even integer $c \geq 4$.

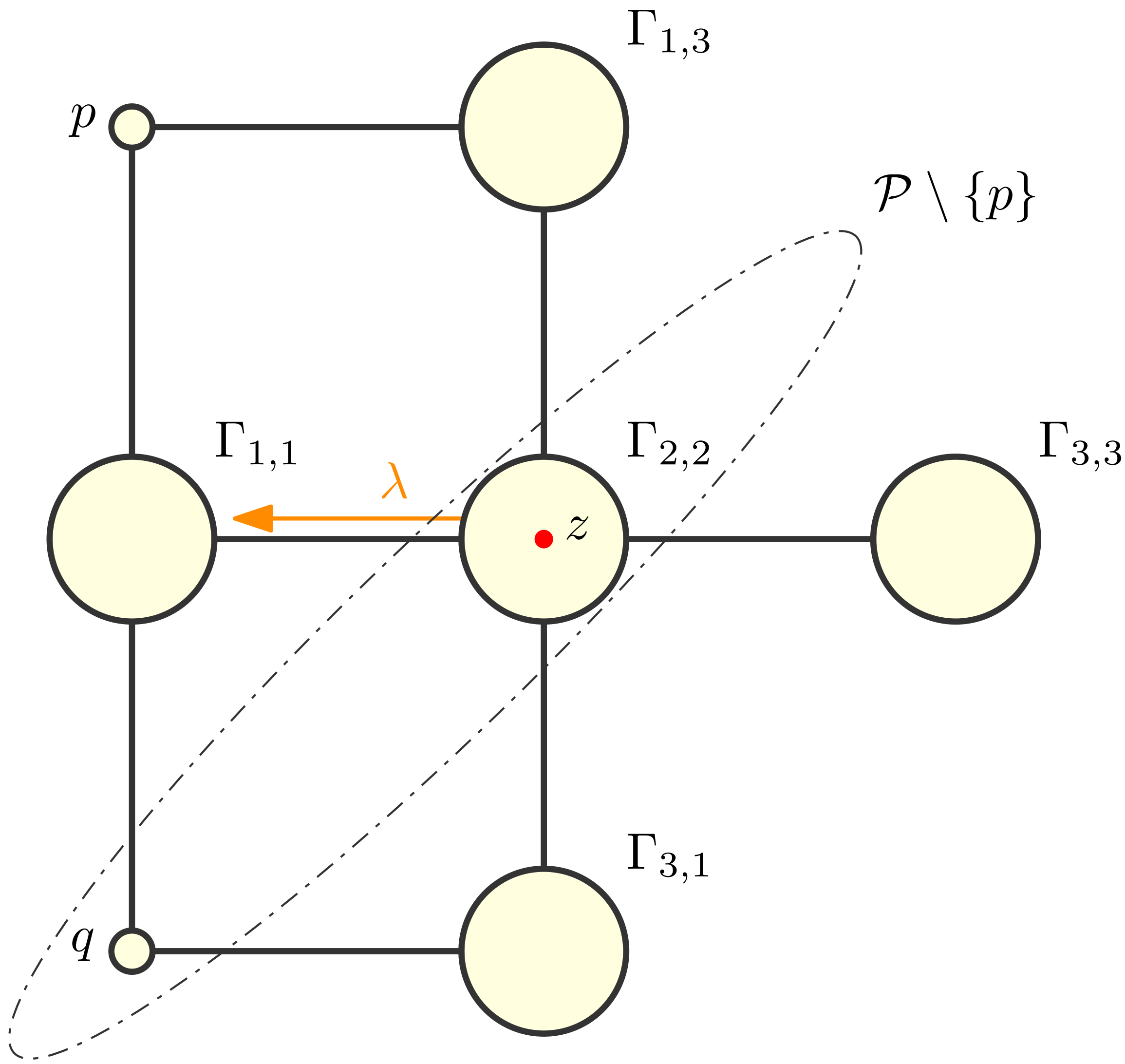

The case
$D = 3$

Theorem (F., Penjić, 2023.)

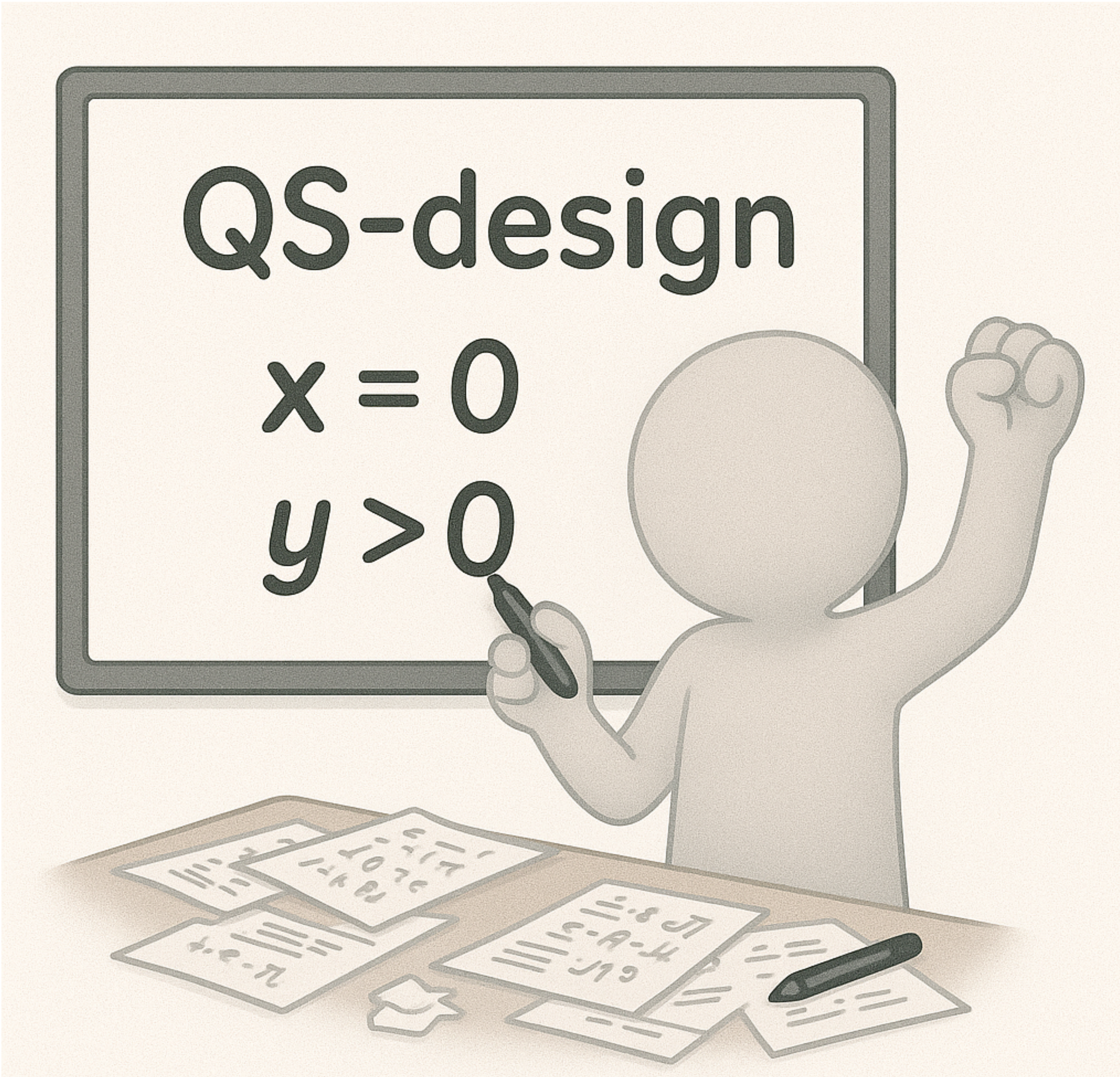
Let Γ denote a (Y, Y')-DBRG with $D = 3$. The following are equivalent:

1. Γ is 2-Y-homogeneous.
2. $b_1 = c_2$.
3. $|\Gamma_2(x)| = \deg(x)$ for all $x \in Y$.

Combinatorial structure of Γ when $D = 3$.



The point-block $(\mathscr{P}, \mathscr{B})$-incidence graph of a 3-$(v, k, \lambda)$ design.


QS-design
x = 0
y > 0

Theorem (F., Rukavina, 2022)

Let Γ denote a (Y, Y')-DBRG. The following are equivalent:

1. Γ is 2-Y-homogeneous with $D = 3$.
2. Γ is the incidence graph of a quasi-symmetric 3-(v, k, λ) design with $x = 0$, $y > 0$, where the following cases may occur:
   - $\mathcal{D}$ is a Hadamard 3-design with $v = 4(\lambda + 1)$ and $k = 2(\lambda + 1)$.
   - $v = (\lambda + 1)(\lambda^2 + 5\lambda + 5)$ and $k = (\lambda + 1)(\lambda + 2)$.
   - $v = 496$, $k = 40$ and $\lambda = 3$.


WE DID IT!

Theorem (Work in progress)

Let Γ denote a (Y, Y')-DBRG. Then, Γ is 2-Y-homogeneous if and only if one of the following holds:

1. Γ is the subdivision graph of a minimal (κ, g)-cage graph $(\kappa, g \geq 3)$ with vertex set X, edge set $\mathcal{R}$, and parts $Y = X$, $Y' = \mathcal{R}$.
2. Γ is the point-block $(\mathcal{P}, \mathcal{B})$-incidence graph of a trasversal design $TD_3(c+2, c)$, for some even integer $c \geq 4$, and parts $Y = \mathcal{P}$, $Y' = \mathcal{B}$.
3. Γ is the $(\mathcal{P}, \mathcal{B})$-incidence graph of a quasi-symmetric 3-design with $x = 0$, $y > 0$, and parts $Y = \mathcal{P}$, $Y' = \mathcal{B}$.


WHY DOES
THIS
MATTER?

Terwilliger algebras of DBRGs


T(x)


1
0
2
0
T(x)

Example (Work in progress)

Let Γ be a (Y, Y')-DBRG, and fix a vertex $x \in Y$. Define $T := T(x)$. Suppose that Γ is the point-block $(\mathscr{P}, \mathscr{B})$-incidence graph of a transversal design $TD_3(c+2, c)$, for some even integer $c \geq 4$, where the parts are given by $Y = \mathscr{P}$ and $Y' = \mathscr{B}$. Then Γ has, up to isomorphism, exactly one irreducible T-module with endpoint i for each $0 \leq i \leq 2$, and each such module is thin.

Happy Birthday, Paul!


Thank you!