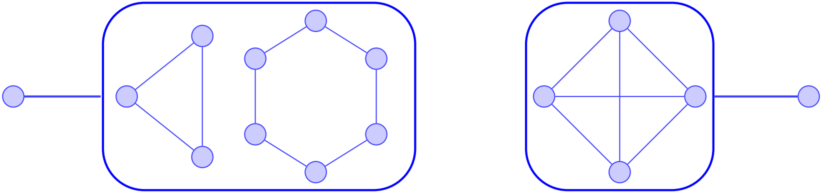
The coherent closure of a Neumaier graph

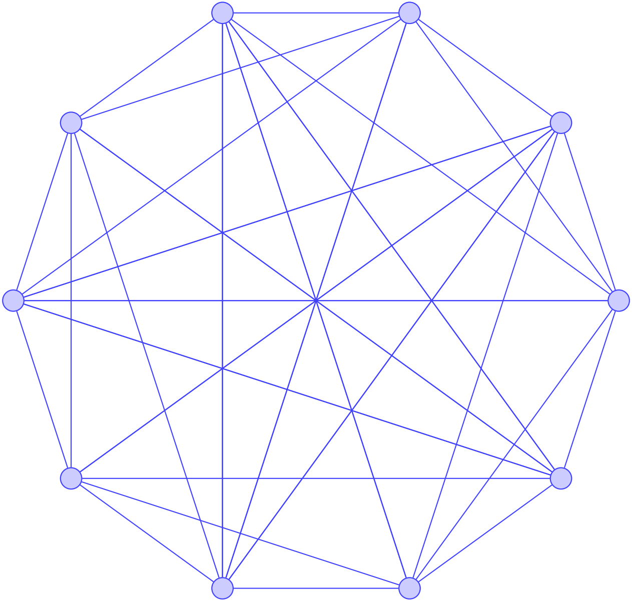
Gary Greaves

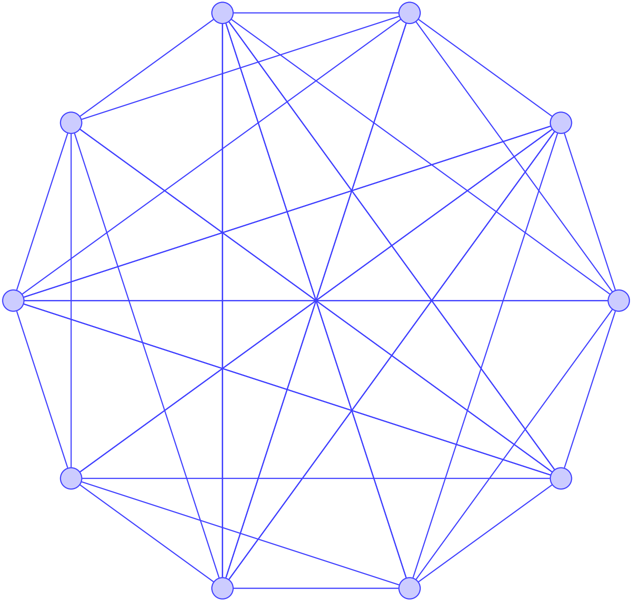
Nanyang Technological University

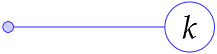
Singapore

23rd June 2025





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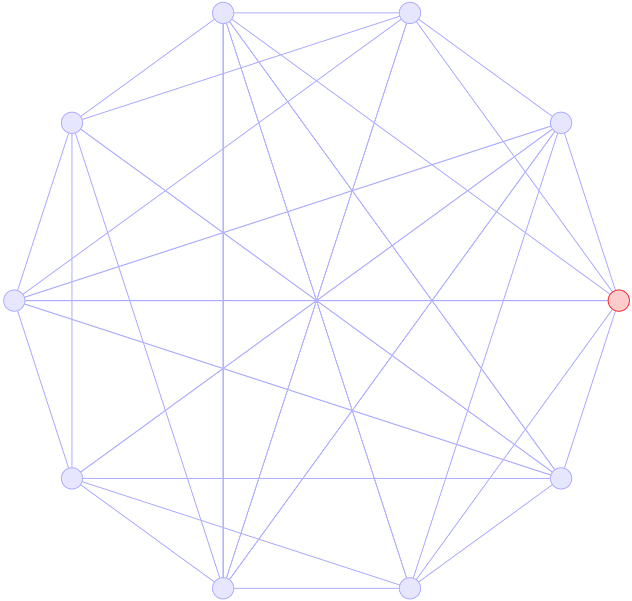

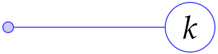

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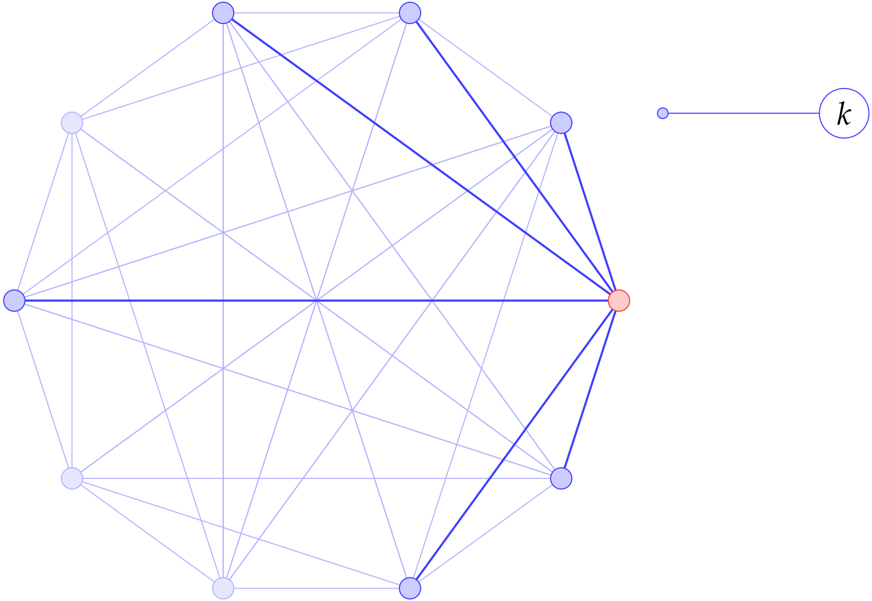
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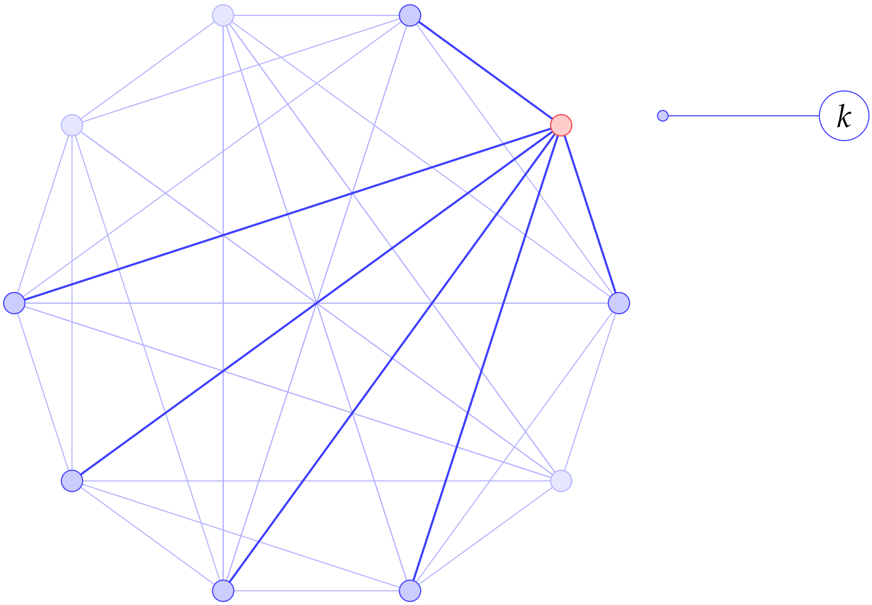
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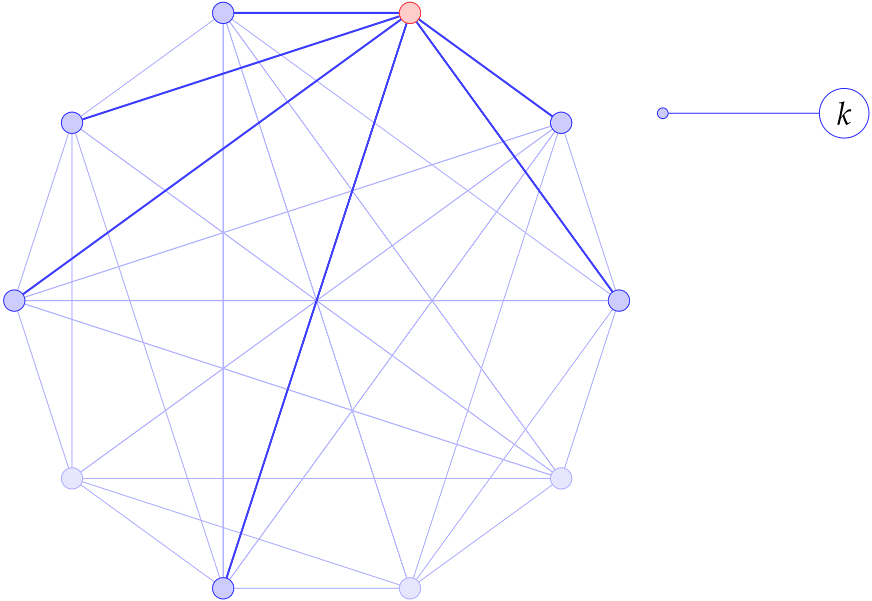
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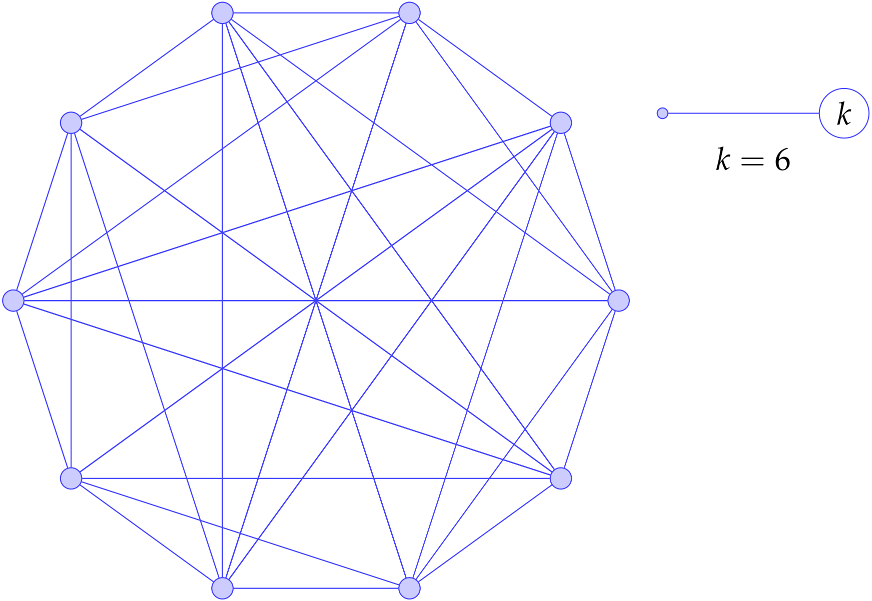
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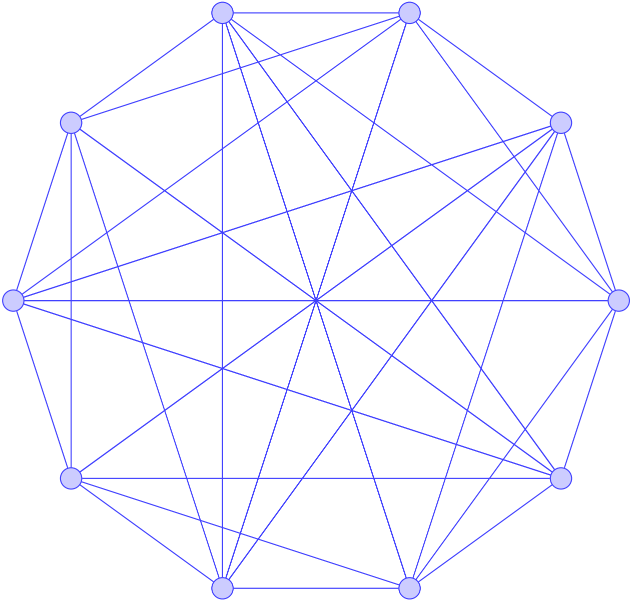

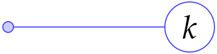

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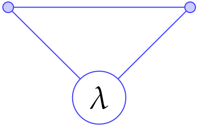

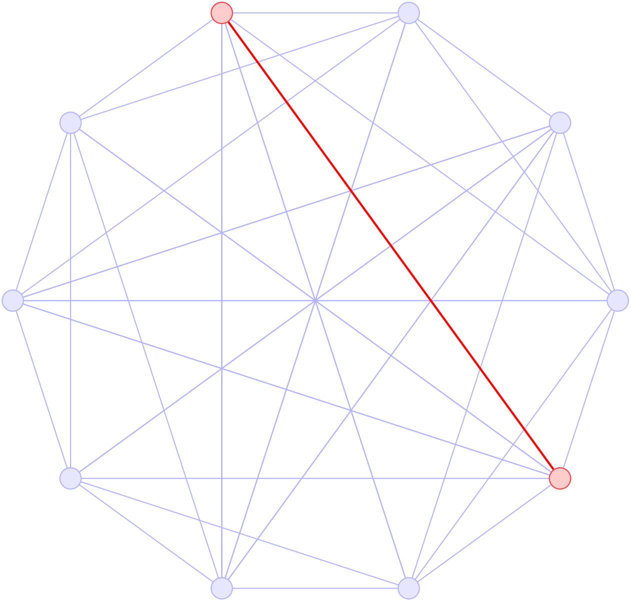

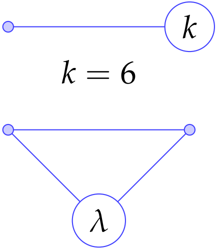
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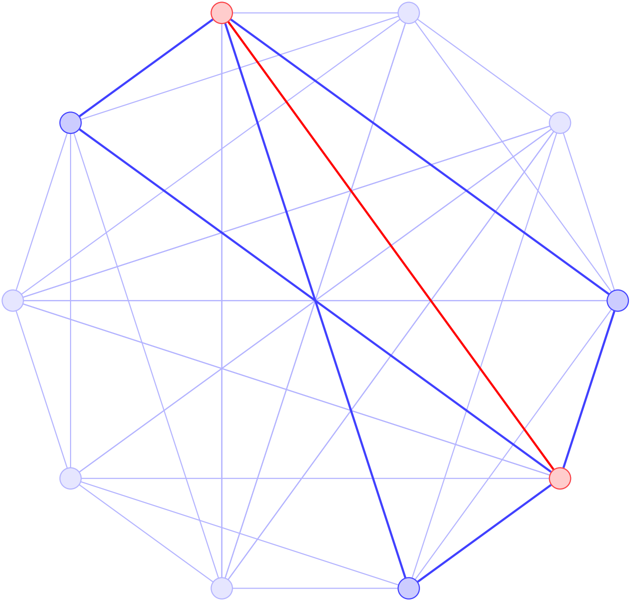
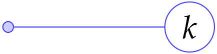
λ





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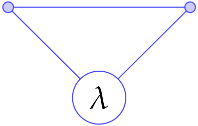

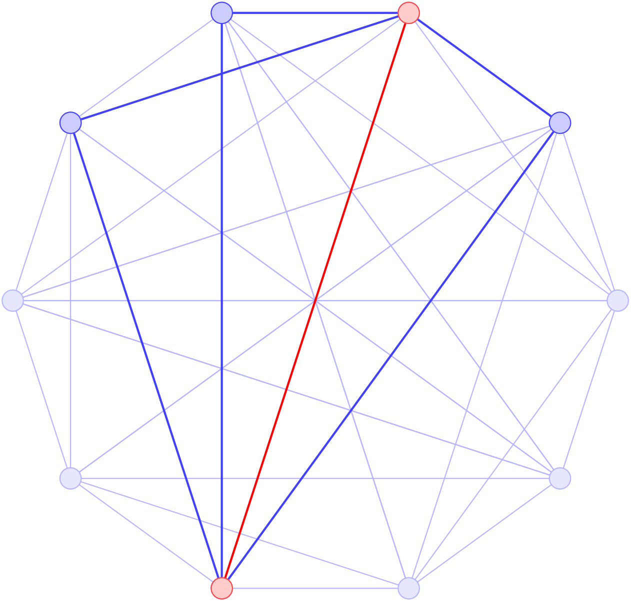

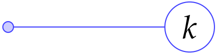

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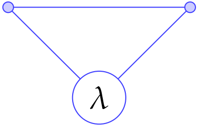

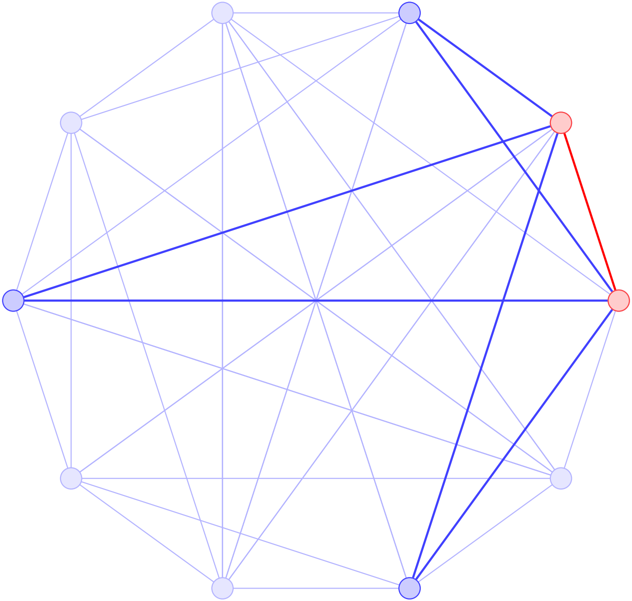

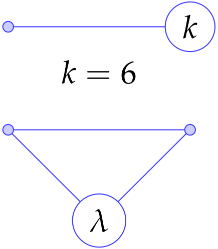

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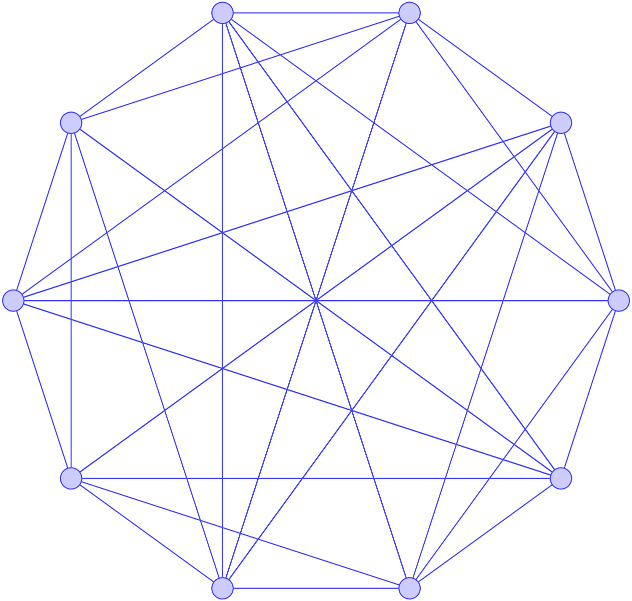

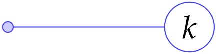

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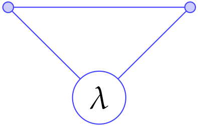

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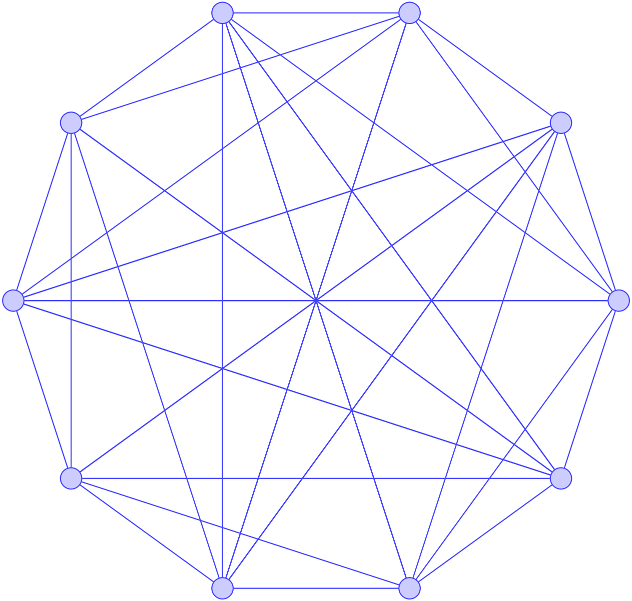

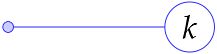

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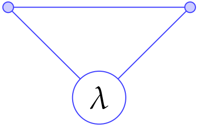

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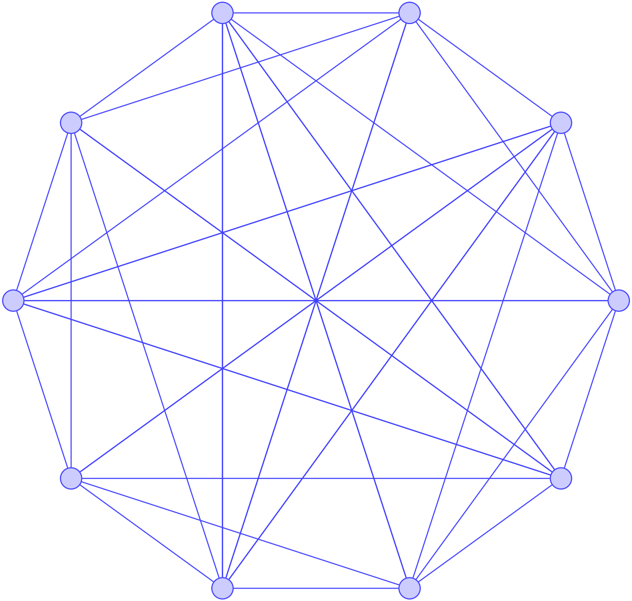
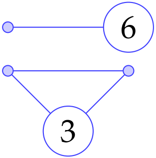
edge-regular

$\mathrm{erg}(10, 6, 3)$





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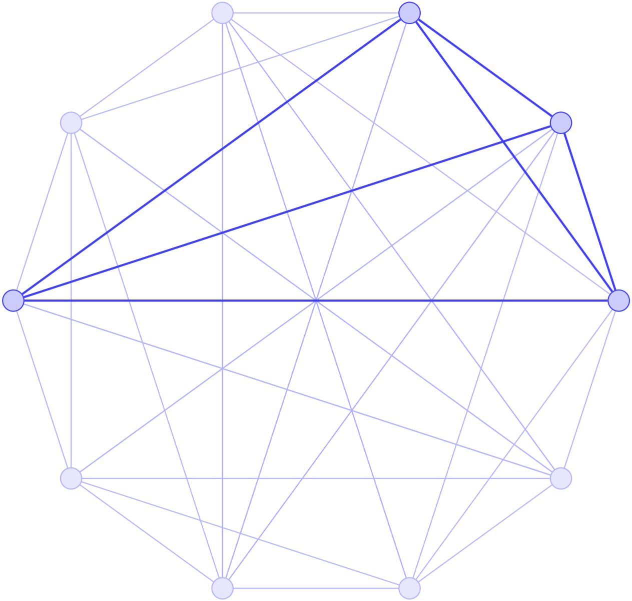

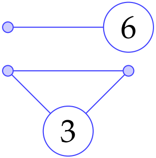

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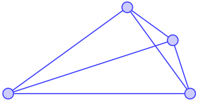
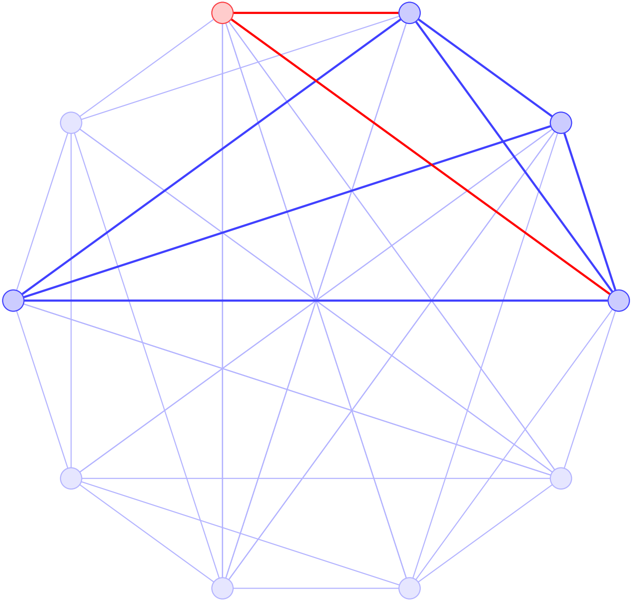
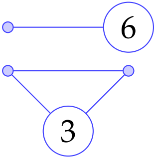


clique

of order 4





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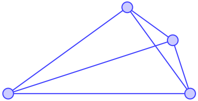

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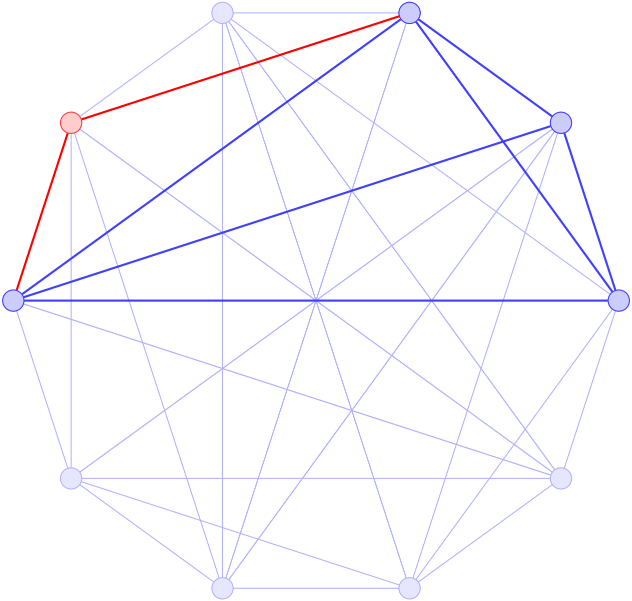

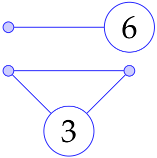

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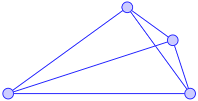

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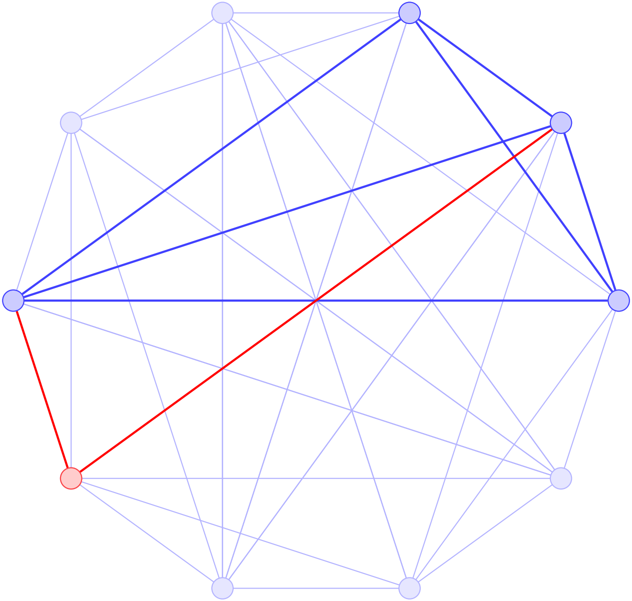

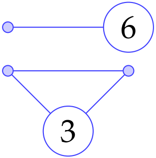

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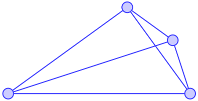

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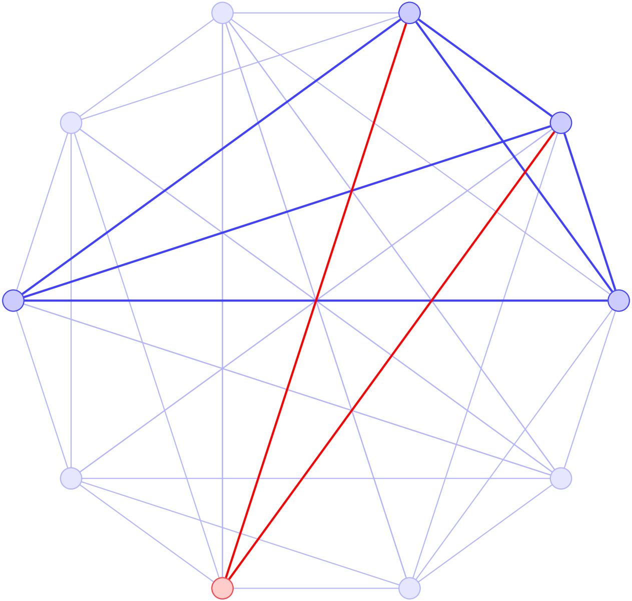

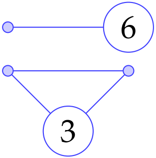

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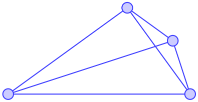

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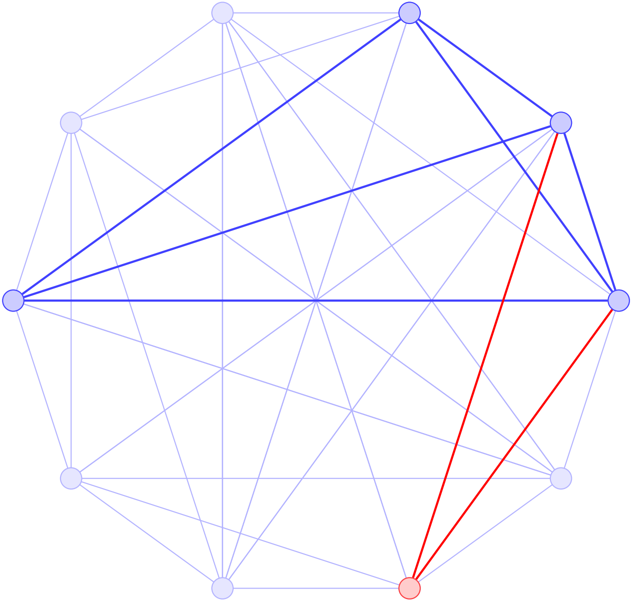

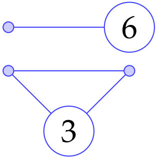

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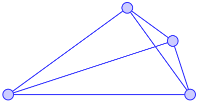

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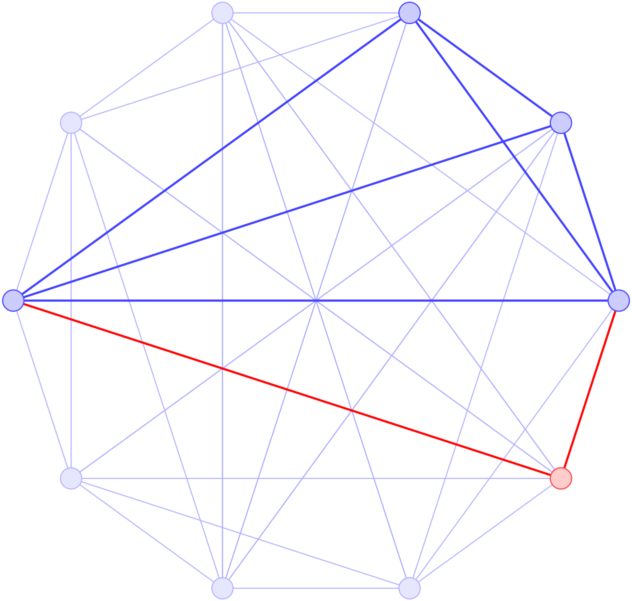

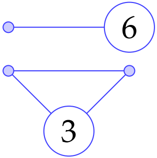

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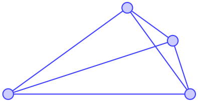

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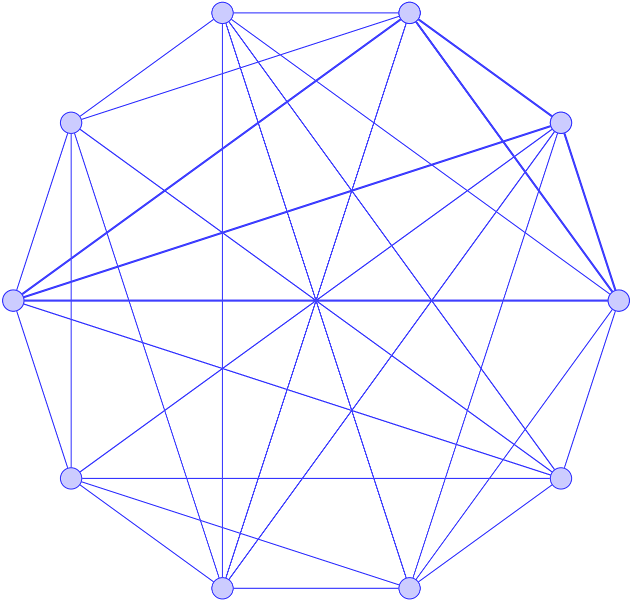

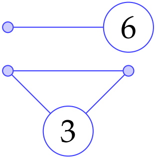

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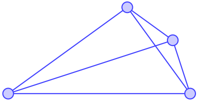
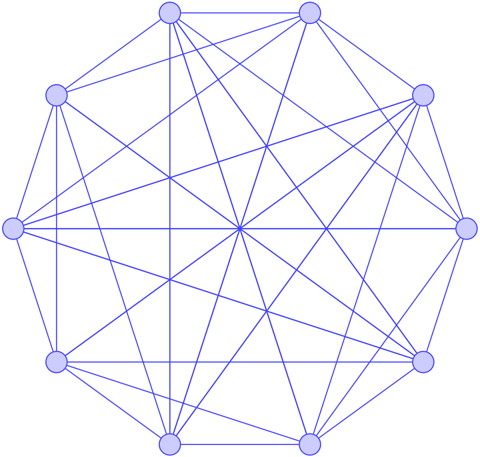
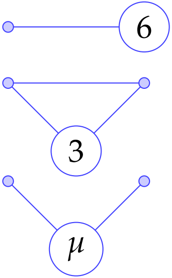


2-**regular** clique of order 4

Question (Neumaier 1981):
Must an edge-regular graph with a regular clique be strongly regular?





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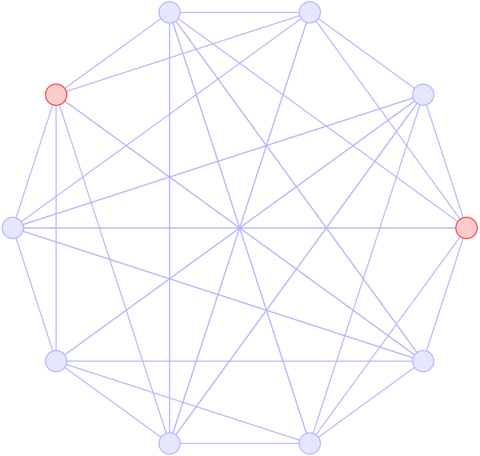

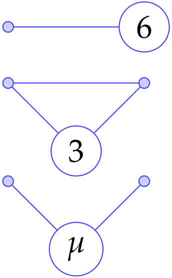

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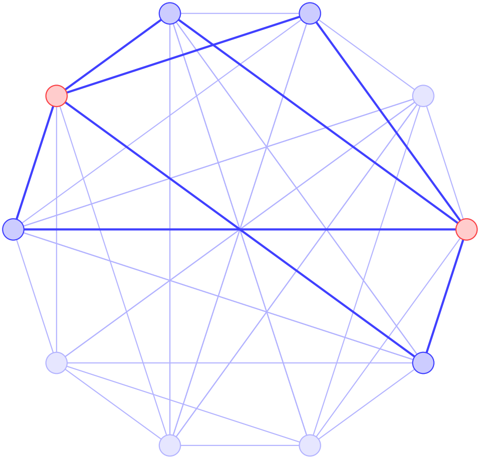

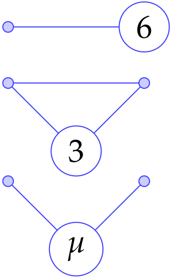

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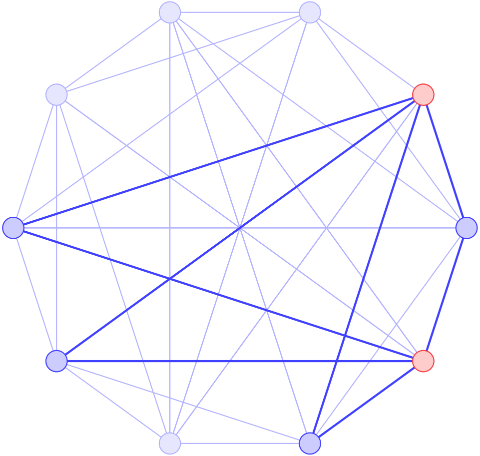

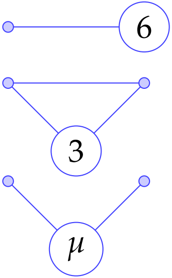

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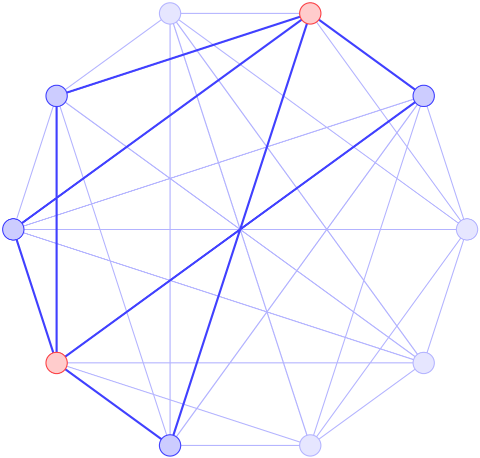

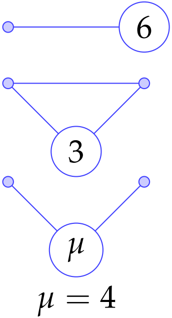

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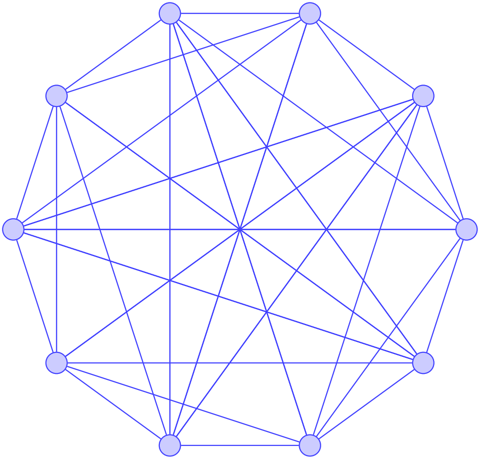

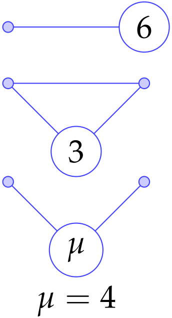

strongly regular
$\mathrm{srg}(10, 6, 3, 4)$

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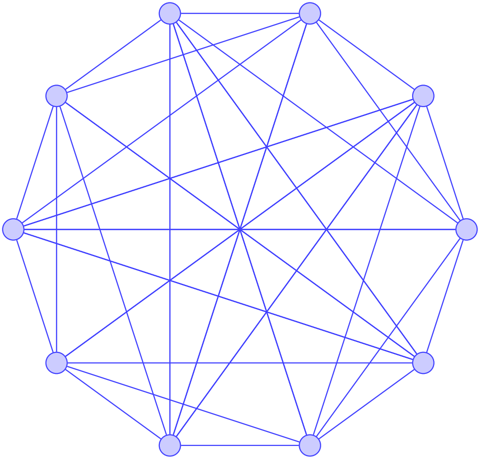

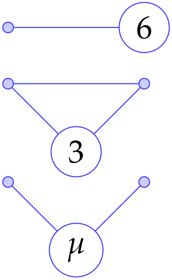

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Question (Neumaier 1981):
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Answer (GG and Koolen 2018):
No.

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Answer (GG and Koolen 2018):
No.

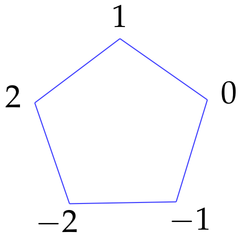
```
Gary — magma.exe ◂ magma — 94×34
Graph
Vertex  Neighbours

1       8 9 14 15 18 19 22 24 27 ;
2       8 9 10 16 19 20 23 25 28 ;
3       9 10 11 17 20 21 22 24 26 ;
4       10 11 12 15 18 21 23 25 27 ;
5       11 12 13 15 16 19 24 26 28 ;
6       12 13 14 16 17 20 22 25 27 ;
7       8 13 14 17 18 21 23 26 28 ;
8       1 2 7 15 17 20 22 25 26 ;
9       1 2 3 16 18 21 23 26 27 ;
10      2 3 4 15 17 19 24 27 28 ;
11      3 4 5 16 18 20 22 25 28 ;
12      4 5 6 17 19 21 22 23 26 ;
13      5 6 7 15 18 20 23 24 27 ;
14      1 6 7 16 19 21 24 25 28 ;
15      1 4 5 8 10 13 22 23 28 ;
16      2 5 6 9 11 14 22 23 24 ;
17      3 6 7 8 10 12 23 24 25 ;
18      1 4 7 9 11 13 24 25 26 ;
19      1 2 5 10 12 14 25 26 27 ;
20      2 3 6 8 11 13 26 27 28 ;
21      3 4 7 9 12 14 22 27 28 ;
22      1 3 6 8 11 12 15 16 21 ;
23      2 4 7 9 12 13 15 16 17 ;
24      1 3 5 10 13 14 16 17 18 ;
25      2 4 6 8 11 14 17 18 19 ;
26      3 5 7 8 9 12 18 19 20 ;
27      1 4 6 9 10 13 19 20 21 ;
28      2 5 7 10 11 14 15 20 21 ;
```

Cayley graphs

- Let G be an (additive) group and $S \subseteq G$ a generating subset, i.e., $G = \langle S \rangle$.
- The **Cayley graph** $\mathrm{Cay}(G, S)$ has vertex set G and arc set
$$\{(g, g+s) : g \in G \text{ and } s \in S\}.$$

Example

$\Gamma = \mathrm{Cay}(\mathbb{Z}_5, S)$ Generating set $S = \{-1, 1\}$



An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \times \mathbb{Z}_7, S)$

Generating set S

$(01,0)$	$(01,\pm 1)$
$(10,0)$	$(10,\pm 2)$
$(11,0)$	$(11,\pm 3)$

An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \times \mathbb{Z}_7, S)$
- Γ is **edge-regular** $(28, 9, 2)$:

Generating set S

$(01, 0)$	$(01, \pm 1)$
$(10, 0)$	$(10, \pm 2)$
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$(00, 0)$

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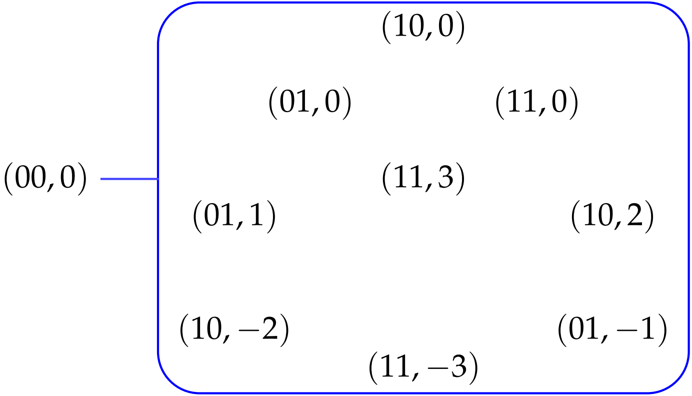

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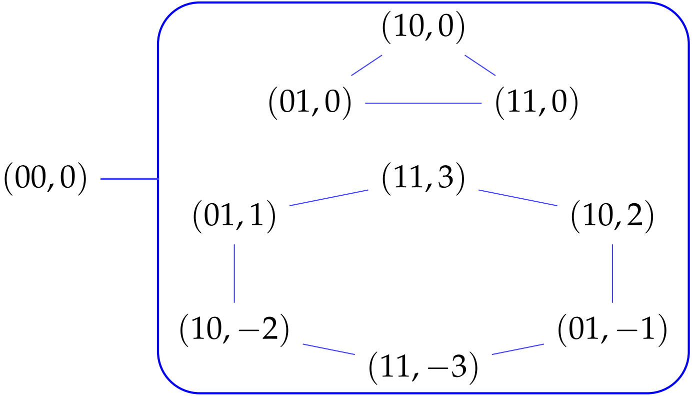

An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \times \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;

- Γ has a 1-**regular** 4-**clique**:

Generating set S

$(01, 0)$	$(01, \pm 1)$
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$(11, 0)$	$(11, \pm 3)$

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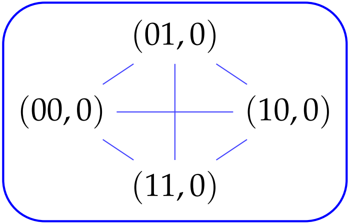

Generating set S

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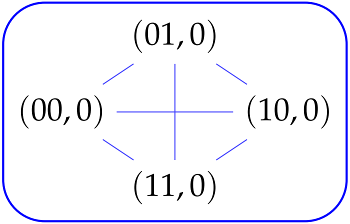

$(a, b) \quad b \neq 0$

An example

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Generating set S

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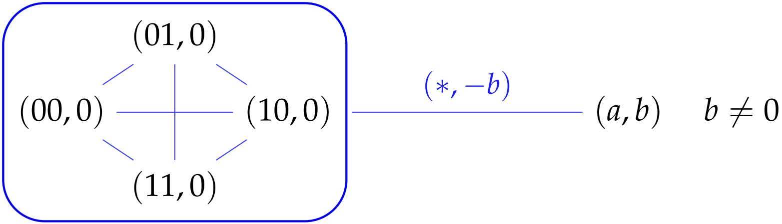

An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \times \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;
- Γ has a 1-regular 4-clique;

- Γ is **not strongly regular**:

Generating set S

$(01,0)$	$(01,\pm 1)$
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$(11,0)$	$(11,\pm 3)$

An example

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- Γ is edge-regular $(28, 9, 2)$;
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Generating set S

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$(10, 0)$	$(10, \pm 2)$
$(11, 0)$	$(11, \pm 3)$

- Γ is **not strongly regular**:

$(00, 0)$ —

	$(11, 0)$
$(01, 0)$	$(01, 1)$
$(11, 3)$	$(10, 0)$
$(10, -2)$	$(01, -1)$
$(11, -3)$	$(10, 2)$

An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \times \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;
- Γ has a 1-regular 4-clique;

Generating set S

$(01, 0)$	$(01, \pm 1)$
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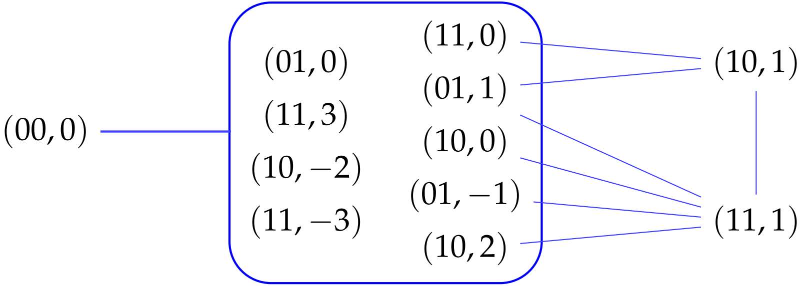

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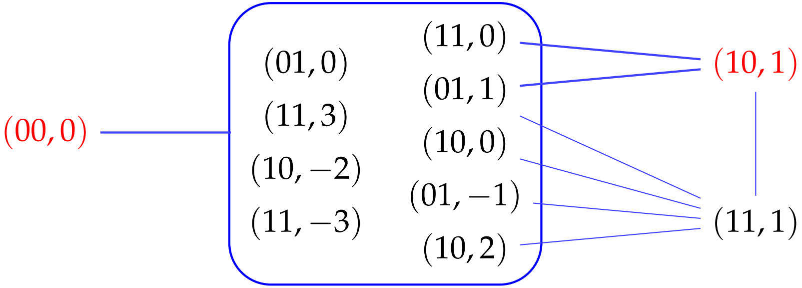

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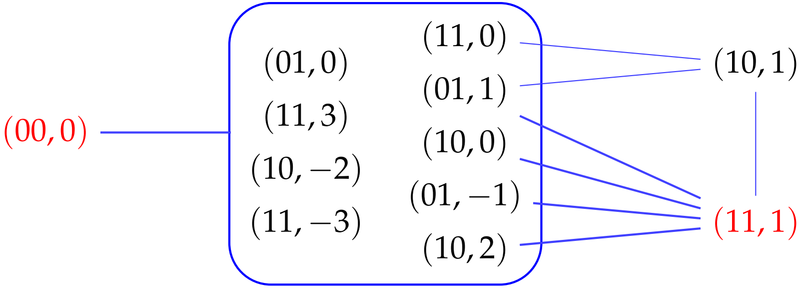


Neumaier graphs

- A non-complete edge-regular graph with a regular clique is called a **Neumaier graph**.

- A Neumaier graph Γ has parameters $(v, k, \lambda; e, s)$.
  - Γ is an $\mathrm{erg}(v, k, \lambda)$.
  - Γ has an e-regular clique K_s.

  } Every $K_s \subset \Gamma$ is e-regular; we call e the **nexus** of Γ.

- Many strongly regular graphs are Neumaier graphs.
  - block graph of orthogonal arrays
  - line graphs of complete graphs

- Evans, Goryainov, and Panasenko (2019): Smallest non-strongly-regular Neumaier graph has parameters $(16, 9, 4; 2, 4)$.

More constructions (not strongly regular)

- Evans-Goryainov-Panasenko (2019):
  infinite families; parameters:
  $(2^{2n}, (2^{n-1}+1)(2^n-1), 2(2^{n-2}+1)(2^{n-1}-1); 2^{n-1}, 2^n)$.
- Evans-Goryainov-Konstantinova-Mednykh (2021):
  infinite families, with parameters: $(*,*,*;1,*)$
- Abiad-Castryck-De Boeck-Koolen-Zeijlemaker (2023):
  infinite families, with parameters: $(*,*,*;1,*)$

Nexus

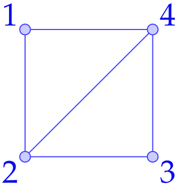
Questions:

- Evans-Goryainov-Panasenko (2019):
  Must the nexus be a power of 2?
- Which Neumaier graphs are *closest* to being SRGs?

Weisfeiler-Leman Stabilization

[Klin and Gyürki 2015]

$$M_1 = \begin{bmatrix} a & b & c & b \\ b & a & b & b \\ c & b & a & b \\ b & b & b & a \end{bmatrix}$$



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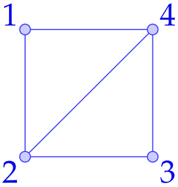

$$M_1^2 = \begin{bmatrix} a^2+2b^2+c^2 & ab+ba+b^2+cb & ac+2b^2+ca & ab+ba+b^2+cb \\ ab+ba+b^2+bc & a^2+3b^2 & ab+ba+b^2+bc & ab+ba+2b^2 \\ ac+2b^2+ca & ab+ba+b^2+cb & a^2+2b^2+c^2 & ab+ba+b^2+cb \\ ab+ba+b^2+bc & ab+ba+2b^2 & ab+ba+b^2+bc & a^2+3b^2 \end{bmatrix}$$

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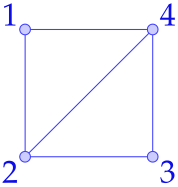

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$$M_2 = \begin{bmatrix} a & & & \\ & & & \\ & & a & \\ & & & \end{bmatrix}$$

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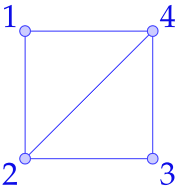
[Klin and Gyürki 2015]

$$M_1 = \begin{bmatrix} a & b & c & b \\ b & a & b & b \\ c & b & a & b \\ b & b & b & a \end{bmatrix}$$



$$M_1^2 = \begin{bmatrix} a^2+2b^2+c^2 & {\color{red} ab+ba+b^2+cb} & ac+2b^2+ca & {\color{red} ab+ba+b^2+cb} \\ ab+ba+b^2+bc & a^2+3b^2 & ab+ba+b^2+bc & ab+ba+2b^2 \\ ac+2b^2+ca & {\color{red} ab+ba+b^2+cb} & a^2+2b^2+c^2 & {\color{red} ab+ba+b^2+cb} \\ ab+ba+b^2+bc & ab+ba+2b^2 & ab+ba+b^2+bc & a^2+3b^2 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} a & b & & b \\ & & & \\ & b & a & b \\ & & & \end{bmatrix}$$

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[Klin and Gyürki 2015]

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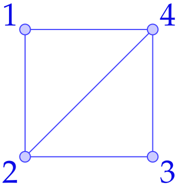

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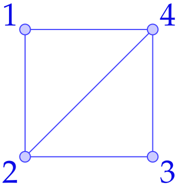

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$$M_2 = \begin{bmatrix} \begin{smallmatrix} a \\ d \end{smallmatrix} & b & \begin{smallmatrix} c \\ d \end{smallmatrix} & b \\ \begin{smallmatrix} c \\ d \end{smallmatrix} & b & \begin{smallmatrix} a \\ d \end{smallmatrix} & b \end{bmatrix}$$

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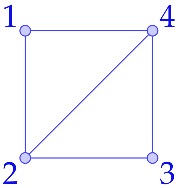

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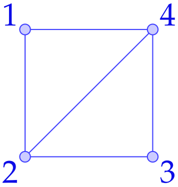

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Weisfeiler-Leman Stabilization

[Klin and Gyürki 2015]

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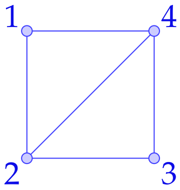

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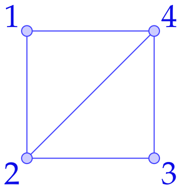

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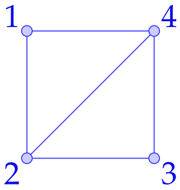

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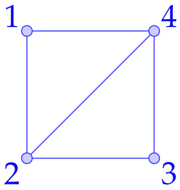

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[Klin and Gyürki 2015]

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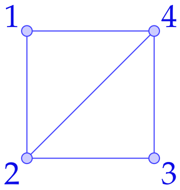

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Weisfeiler-Leman Stabilization

[Klin and Gyürki 2015]

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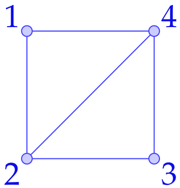


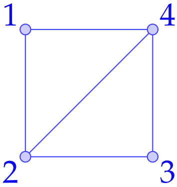
$$M_2^2 = \begin{bmatrix} a^2+2bd+c^2 & ab+be+bf+cb & ac+2bd+ca & ab+be+bf+cb \\ da+dc+ed+fd & 2db+e^2+f^2 & da+dc+ed+fd & 2db+ef+fe \\ ac+2bd+ca & ab+be+bf+cb & a^2+2bd+c^2 & ab+be+bf+cb \\ da+dc+ed+fd & 2db+ef+fe & da+dc+ed+fd & 2db+e^2+f^2 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} a & b & c & b \\ d & & d & \\ c & b & a & b \\ d & & d & \end{bmatrix}$$

Weisfeiler-Leman Stabilization

[Klin and Gyürki 2015]

$$M_2 = \begin{bmatrix} a & b & c & b \\ d & e & d & f \\ c & b & a & b \\ d & f & d & e \end{bmatrix}$$



$$M_2^2 = \begin{bmatrix} a^2+2bd+c^2 & ab+be+bf+cb & ac+2bd+ca & ab+be+bf+cb \\ da+dc+ed+fd & {\color{red}2db+e^2+f^2} & da+dc+ed+fd & 2db+ef+fe \\ ac+2bd+ca & ab+be+bf+cb & a^2+2bd+c^2 & ab+be+bf+cb \\ da+dc+ed+fd & 2db+ef+fe & da+dc+ed+fd & {\color{red}2db+e^2+f^2} \end{bmatrix}$$

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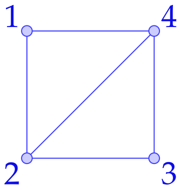

$$M_2^2 = \begin{bmatrix} a^2+2bd+c^2 & ab+be+bf+cb & ac+2bd+ca & ab+be+bf+cb \\ da+dc+ed+fd & 2db+e^2+f^2 & da+dc+ed+fd & 2db+ef+fe \\ ac+2bd+ca & ab+be+bf+cb & a^2+2bd+c^2 & ab+be+bf+cb \\ da+dc+ed+fd & 2db+ef+fe & da+dc+ed+fd & 2db+e^2+f^2 \end{bmatrix}$$

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Weisfeiler-Leman Stabilization

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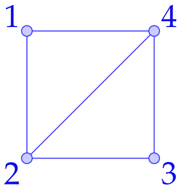
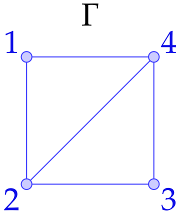


$$M_2^2 = \begin{bmatrix} a^2+2bd+c^2 & ab+be+bf+cb & ac+2bd+ca & ab+be+bf+cb \\ da+dc+ed+fd & 2db+e^2+f^2 & da+dc+ed+fd & 2db+ef+fe \\ ac+2bd+ca & ab+be+bf+cb & a^2+2bd+c^2 & ab+be+bf+cb \\ da+dc+ed+fd & 2db+ef+fe & da+dc+ed+fd & 2db+e^2+f^2 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} a & b & c & b \\ d & e & d & f \\ c & b & a & b \\ d & f & d & e \end{bmatrix} = M_2$$

Coherent closure

$$\mathcal{W}(\Gamma) = \begin{bmatrix} a & b & c & b \\ d & e & d & f \\ c & b & a & b \\ d & f & d & e \end{bmatrix}$$



$$\overset{a}{\begin{bmatrix} 1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix}}, \overset{b}{\begin{bmatrix} 0&1&0&1\\0&0&0&0\\0&1&0&1\\0&0&0&0 \end{bmatrix}}, \overset{c}{\begin{bmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{bmatrix}}, \overset{d}{\begin{bmatrix} 0&0&0&0\\1&0&1&0\\0&0&0&0\\1&0&1&0 \end{bmatrix}}, \overset{e}{\begin{bmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1 \end{bmatrix}}, \overset{f}{\begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0 \end{bmatrix}}$$

$$\mathcal{W}(\Gamma) = \left\langle \begin{bmatrix} 1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0&1\\0&0&0&0\\0&1&0&1\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0\\1&0&1&0\\0&0&0&0\\1&0&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1 \end{bmatrix}, \begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0 \end{bmatrix} \right\rangle$$

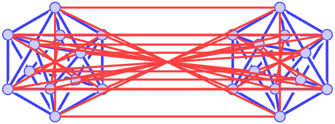
We call $\mathcal{W}(\Gamma)$ the **coherent closure** of Γ.
We say "Γ has **coherent rank** 6" (aka WL-rank).

Coherent rank of a Neumaier graph

Let Γ be a Neumaier graph.

- $\operatorname{rk}\mathcal{W}(\Gamma) = 3$ iff Γ is strongly regular;
- $\operatorname{rk}\mathcal{W}(\Gamma) \neq 4$;
- $\operatorname{rk}\mathcal{W}(\Gamma) \neq 5$?; ← **Open Problem!**
- $\geqslant 6$ otherwise.

Neumaier graphs with coherent rank 6:

- $\mathrm{Cay}(\mathbb{Z}_2^2 \times \mathbb{Z}_7, \{(01,0),(10,0),(11,0),(01,\pm 1),(10,\pm 2),(11,\pm 3)\})$
- 

Neumaier graphs with coherent rank 6

Let $\Gamma = \mathrm{Cay}\left(\mathbb{Z}_2^2 \times \mathbb{Z}_7, \left\{ \begin{array}{c} (01,0),(10,0),(11,0), \\ (01,\pm 1),(10,\pm 2),(11,\pm 3) \end{array} \right\}\right)$

Then $\mathcal{W}(\Gamma) = \langle I, A(7K_4), A(4K_7), A(\Delta_1), A(\Delta_2), A(\Delta_3)\rangle$, where $\Delta_1 \cong \Delta_2 \cong \Delta_3 \cong \mathrm{Cay}\left(\mathbb{Z}_2^2 \times \mathbb{Z}_7, \{(01,\pm 1),(10,\pm 2),(11,\pm 3)\}\right)$.

Neumaier graphs with coherent rank 6

Let $\Gamma = \operatorname{Cay}\left(\mathbb{Z}_2^2 \times \mathbb{Z}_7, \left\{ \begin{array}{c} (01,0),(10,0),(11,0), \\ (01,\pm 1),(10,\pm 2),(11,\pm 3) \end{array} \right\}\right)$

Then $\mathcal{W}(\Gamma) = \langle I, A(7K_4), A(4K_7), A(\Delta_1), A(\Delta_2), A(\Delta_3)\rangle$, where $\Delta_1 \cong \Delta_2 \cong \Delta_3 \cong \operatorname{Cay}\left(\mathbb{Z}_2^2 \times \mathbb{Z}_7, \{(01,\pm 1),(10,\pm 2),(11,\pm 3)\}\right)$.

Problem:
Do there exist *more* Neumaier graphs Γ s.t.
$\mathcal{W}(\Gamma) = \langle I, \mathsf{A}_1, \mathsf{A}_2, \mathsf{A}_3, \mathsf{A}_4, \mathsf{A}_5\rangle$ and $A(\Gamma) = \mathsf{A}_1 + \mathsf{A}_2$?

Neumaier graphs with coherent rank 6

Let $\Gamma = \operatorname{Cay}\left(\mathbb{Z}_2^2 \times \mathbb{Z}_7, \left\{ \begin{array}{c} (01,0),(10,0),(11,0), \\ (01,\pm 1),(10,\pm 2),(11,\pm 3) \end{array} \right\}\right)$

Then $\mathcal{W}(\Gamma) = \langle I, A(7K_4), A(4K_7), A(\Delta_1), A(\Delta_2), A(\Delta_3) \rangle$, where $\Delta_1 \cong \Delta_2 \cong \Delta_3 \cong \operatorname{Cay}\left(\mathbb{Z}_2^2 \times \mathbb{Z}_7, \{(01,\pm 1),(10,\pm 2),(11,\pm 3)\}\right)$.

Problem:

Do there exist *more* Neumaier graphs Γ s.t. $\mathcal{W}(\Gamma) = \langle I, \mathsf{A}_1, \mathsf{A}_2, \mathsf{A}_3, \mathsf{A}_4, \mathsf{A}_5 \rangle$ and $A(\Gamma) = \mathsf{A}_1 + \mathsf{A}_2$?

Theorem. (cf. Abiad et al. 2021):

Suppose Γ is a Neumaier graph with *commutative* $\mathcal{W}(\Gamma) = \langle I, \mathsf{A}_1, \dots, \mathsf{A}_r \rangle$ and $A(\Gamma) \in \{\mathsf{A}_1, \dots, \mathsf{A}_r\}$. Then Γ is strongly regular.

NEW! construction (coherent rank 6)

q_1 and q_2: prime powers congruent to 1 modulo 3.
α_1 and α_2: primitive elements of $\mathrm{GF}(q_1)$ and $\mathrm{GF}(q_2)$.

$$S := \left\{(\alpha_1^i, 0) \;:\; 0 \leqslant i \leqslant q_1 - 2\right\} \cup \left\{(\alpha_1^{i_1}, \alpha_2^{i_2}) \;:\; i_1 \equiv i_2 \mod 3\right\}$$

Theorem. (GG and Tan 2025+):
$\mathrm{Cay}\left(\mathrm{GF}(q_1) \times \mathrm{GF}(q_2), S\right)$ is Neumaier iff

$$4(2q_1 - q_2) = a(q_1)a(q_2) \pm 27b(q_1)b(q_2)$$

For prime power $q \equiv 1 \mod 3$, $a(q)$ and $b(q)$ are defined as

$$4q = a(q)^2 + 27b(q)^2 \quad \text{and } a(q) \equiv 1 \mod 3.$$

$\mathcal{W}(\Gamma) = \langle I, A(q_2K_{q_1}), A(q_1K_{q_2}), A(\Delta_1), A(\Delta_2), A(\Delta_3)\rangle$, where
$\Delta_i \cong \mathrm{Cay}\left(\mathrm{GF}(q_1) \times \mathrm{GF}(q_2), \left\{(\alpha_1^{i_1}, \alpha_2^{i_2}) \;:\; i_1 \equiv i_2 \mod 3\right\}\right)$.

NEW! (infinite?) construction

$$4q = a(q)^2 + 27b(q)^2 \text{ and } a(q) \equiv 1 \mod 3.$$

Theorem. (GG and Tan 2025+):
$\mathrm{Cay}\left(\mathrm{GF}(q_1) \times \mathrm{GF}(q_2), S\right)$ is Neumaier iff

$$4(2q_1 - q_2) = a(q_1)a(q_2) \pm 27b(q_1)b(q_2)$$

Parameters: $\left(q_1q_2, \frac{(q_1-1)(q_2+2)}{3}, q_1 - 2 + \frac{(q_1-4)(q_2-1)}{9}; \frac{q_1-1}{3}, q_1\right)$

Nexus

q_1	q_2	q_1	q_2	q_1	q_2	q_1	q_2
2^2	7	13	2^4	61	97	163	211
2^2	13	19	67	97	241	13^2	313
7	2^4	31	43	109	433	193	769
7	19	37	73	139	331	199	787
13	7^2	7^2	193	151	163	223	811

Neumaier graphs with coherent rank 6

Let Γ be an a-antipodal distance-regular graph of diameter 3 with distance matrices A_1, A_2, and A_3.

- Set $t = (a_1 + 2)/a$.
- $\Gamma^\dagger :=$ graph whose adjacency matrix is
$$I_t \otimes (\mathsf{A}_1 + \mathsf{A}_3) + (J_t - I_t) \otimes (I + \mathsf{A}_3).$$

Theorem. (GG and Koolen 2018):
$\Gamma^\dagger$ is a Neumaier graph.

Theorem. (GG and Tan 2025+):

$$\mathcal{W}(\Gamma^\dagger) = \left\langle \begin{matrix} I,\ I_t \otimes \mathsf{A}_1,\ I_t \otimes \mathsf{A}_2,\ I_t \otimes \mathsf{A}_3, \\ (J_t - I_t) \otimes (I + \mathsf{A}_3), (J_t - I_t) \otimes (\mathsf{A}_1 + \mathsf{A}_2) \end{matrix} \right\rangle$$

Thank you for your attention

Happy Birthday, Paul

Further reading:

G.R.W. Greaves and Z.K. Tan, *Neumaier graphs from cyclotomy with small coherent rank*, arXiv:2504.12026.

G.R.W. Greaves and J.H. Koolen, *Another construction of edge-regular graphs with regular cliques*, Discrete Math. **88** (2019), pp. 2818–2820.

G.R.W. Greaves and J.H. Koolen, *Edge-regular graphs with regular cliques*, European J. Combin. **71** (2018), pp. 194–201.