Bipartite Coherent Configurations

Sabrina Lato

Umeå Universitet

sabrina.lato@umu.se

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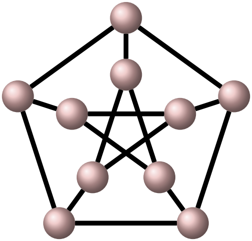
Distance-Biregular Graphs

Distance-Regular Graphs



Definition

Let G be a graph with diameter d. Then G is **distance-regular** *if for any vertex u and any v at distance i from u for some $0 \leq i \leq d$, the numbers of vertices*

$$c_i := \left|\{w \sim v : d(u,w) = i-1\}\right|,$$

$$a_i := \left|\{w \sim v : d(u,w) = i\}\right|,$$

and

$$b_i := \left|\{w \sim v : d(u,w) = i+1\}\right|$$

are independent of the choice of u and v.

Locally Distance-Regular Vertices

Definition

Let G be a graph with vertex u of eccentricity e. Then u is **locally distance-regular** *if for any vertex v at distance i from u for some $0 \leq i \leq e$, the numbers of vertices*

$$c_i := \left|\{w \sim v : d(u,w) = i-1\}\right|,$$

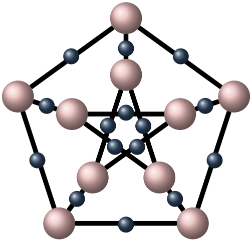
$$a_i := \left|\{w \sim v : d(u,w) = i\}\right|,$$

and

$$b_i := \left|\{w \sim v : d(u,w) = i+1\}\right|$$

are independent of the choice of v.

Distance-Biregular Graphs



Definition

Let G be a bipartite graph. Then G is **distance-biregular** *if for any vertex u and any v at distance i from u for some $0 \leq i \leq e$, the numbers of vertices*

$$c_i := |\{w \sim v : d(u, w) = i - 1\}|$$

and

$$b_i := |\{w \sim v : d(u, w) = i + 1\}|$$

depend only on the cell of the bipartition that u lies in.

Theorem (Godsil and Shawe-Taylor)

Let G be a graph where every vertex is locally distance-regular. Then G is either distance-regular or distance-biregular.

Examples

- Complete bipartite graphs
- Bipartite distance-regular graphs

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- (Tits 1959) Generalized polygons

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- (Bose 1942) Affine resolvable designs
- (Goethals and Seidel 1970) Sporadic quasi-symmetric design related to Golay code
- (Witt 1944) Sporadic quasi-symmetric design related to Witt design

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- (Witt 1944) Sporadic quasi-symmetric design related to Witt design
- (Bose 1963) Partial geometries

- (Delorme 1983) Infinite family related to a cone in affine space
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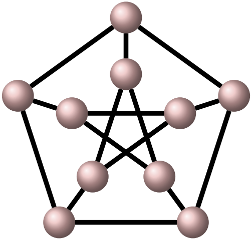
- (Fernández, Ihringer, Lato, and Munemasa 2025+) Sporadic example generalizing Delorme's maximal arc family
- (Fernández, Ihringer, Lato, and Munemasa 2025+) Infinite family coming from subgraphs of Delorme's maximal arc family or an independent construction using hyperovals.

Alternative Definition



$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Definition

The i-distance adjacency matrix A_i of a graph G is the matrix indexed by the vertices of G with u, v-entry equal to one if $d(u, v) = i$ and zero otherwise.

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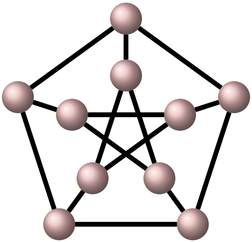
The i-distance adjacency matrix A_i of a graph G is the matrix indexed by the vertices of G with u, v-entry equal to one if $d(u, v) = i$ and zero otherwise.

- $A_i^T = A_i$
- $A_0 = I$
- If G is connected with diameter d, then

$$\sum_{i=0}^{d} A_i = J$$



$$A^2 = \begin{pmatrix} 3 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 3 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 3 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 3 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 3 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 3 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 3 \end{pmatrix}$$

Distance-Regular Graphs

Definition

Let G be a graph with diameter d. It is **distance-regular** *if G has $d+1$ distinct eigenvalues and there exists a sequence of polynomials $F_0, \ldots, F_d$ such that, for all $0 \leq i \leq d$, the polynomial F_i has degree i and*

$$F_i(A) = A_i.$$

$$A = \begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix}$$

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$$A^{2i} = \begin{pmatrix} \left(NN^T\right)^i & 0 \\ 0 & \left(N^T N\right)^i \end{pmatrix}$$

$$A^{2i+1} = \begin{pmatrix} 0 & \left(NN^T\right)^i N \\ \left(N^T N\right)^i N^T & 0 \end{pmatrix}$$

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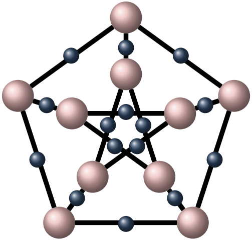
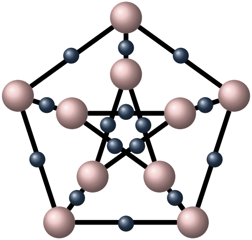
$$A^{2i+1} = \begin{pmatrix} 0 & \left(NN^T\right)^i N \\ \left(N^TN\right)^i N^T & 0 \end{pmatrix}$$

$$A_{2i} = \begin{pmatrix} X_i & 0 \\ 0 & Y_i \end{pmatrix}$$

$$A_{2i+1} = \begin{pmatrix} 0 & N_{2i+1} \\ N_{2i+1}^T & 0 \end{pmatrix}.$$



$$N = \begin{pmatrix} 1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&1&1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&1&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&1&1&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&1&0&0&0&1&0&0&0&1 \\ 0&0&0&0&0&0&0&0&1&0&0&1&1&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&1&1 \end{pmatrix}$$

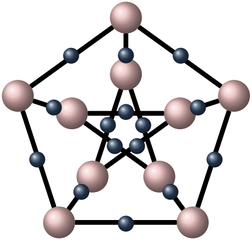


$$X_1 = NN^T - 3I$$

$$N_3 = NN^T N - 4N$$

$$X_2 = \left(NN^T\right)^2 - 6NN^T + 6I$$

$$N_5 = \frac{1}{2}\left(\left(NN^T\right)^2 N - 7NN^T N + 10N\right)$$



$$Y_1 = N^T N - 2I$$

$$N_3 = N^T N N - 4N^T$$

$$Y_2 = \left(N^T N\right)^2 - 5N^T N + 2I$$

$$N_5 = 1/2\left(\left(N^T N\right)^2 N^T - 7N^T N N^T + 10N^T\right)$$

$$Y_3 = 1/4\left(\left(N^T N\right)^3 - 9\left(N^T N\right)^2 + 20N^T N - 4I\right)$$

Distance-Biregular Graphs

Definition

Let $G = (\beta, \gamma)$ be a bipartite graph with diameter d. It is **distance-biregular** *if G has $d+1$ distinct eigenvalues and there exists two sequences of polynomials $F_0^\beta, \ldots, F_d^\beta$ and $F_0^\gamma, \ldots, F_d^\gamma$ such that*

$$F_{2i}^\beta\left(NN^T\right) = X_i,$$

$$F_{2i}^\gamma\left(N^T N\right) = Y_i,$$

$$F_{2i+1}^\beta\left(NN^T\right) N = N_{2i+1},$$

and

$$F_{2i+1}^\gamma\left(N^T N\right) N^T = N_{2i+1}^T.$$

Bipartite Coherent Configurations

Association Schemes

Definition

A set of 01-matrices $A_0, \ldots, A_d$ form a **symmetric association scheme** *if*

- $A_0 = I$;
- $\sum_{i=0}^{d} A_i = J$;
- $A_i^T = A_i$ *for* $0 \leq i \leq d$; *and*
- $A_i A_j \in \mathrm{Span}\{A_0, \ldots, A_d\}$ *for* $0 \leq i, j \leq d$.

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Definition

*An association scheme is P-***polynomial** *if for all* $0 \leq i \leq d$, *there is a polynomial* F_i *of degree* i *such that* $F_i(A_1) = A_i$.

Theorem (Bose-Mesner)

A symmetric association scheme has a dual basis $E_0, \ldots, E_d$ satisfying:

- $E_0 = \frac{1}{v} J$;
- $\sum_{i=0}^{d} E_i = I$;
- $E_i^T = E_i$ *for* $0 \leq i \leq d$; *and*
- $E_i \circ E_j \in \operatorname{span}\{E_0, \ldots, E_d\}$ *for* $0 \leq i, j \leq d$.
- $E_i E_j = \delta_{ij} E_i$

Coherent Configuration

Definition (Higman)

A set $\mathcal{C}$ of 01-matrices forms a coherent configuration if:

- *There exists $S \subseteq \mathcal{C}$ such that $\sum_{M \in S} M = I$;*
- $\sum_{M \in \mathcal{C}} M = J$;
- $M^T \in \mathcal{C}$; *and*
- *For all $M_1, M_2 \in \mathcal{C}$, we have $M_1 M_2 \in \operatorname{Span}(\mathcal{C})$.*

We can think of this in terms of $f \times f$ block matrices

$$\left\{ A_0^{ij}, A_1^{ij}, \ldots, A_{d_{ij}}^{ij} \right\}_{i,j=1}^{f}.$$

Suda studied fibre-symmetric coherent configurations where there exist matrices $\left\{E_r^{ij}\right\}$ satisfying:

- $$E_0^{ij} = \frac{1}{\sqrt{x_i x_j}} J;$$
- The set $\left\{E_r^{ij}\right\}$ is a basis of $\mathrm{Span}\left(\left\{A_r^{ij}\right\}\right)$ as a vector space;
- $\left(E_r^{ij}\right)^T = E_r^{ji}$; and
- $$E_r^{ij} E_s^{j'h} = \delta_{j,j'} \delta_{r,s} E_r^{ih}.$$

Definition (Lato)

A collection

$$\mathcal{C} = \left\{ \begin{pmatrix} X_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \bigcup \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Y_j \end{pmatrix} \bigcup \begin{pmatrix} \mathbf{0} & N_h \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \bigcup \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ N_h^T & \mathbf{0} \end{pmatrix} \right\}$$

is a bipartite coherent configuration if it satisfies

- $X_0 = I$ *and* $Y_0 = I$;
- $\sum_{M \in \mathcal{C}} M = J$;
- *For all* $M_1, M_2 \in \mathcal{C}$, *we have* $M_1 M_2 \in \mathrm{Span}\,(\mathcal{C})$;
- $N_i N_j^T = N_j N_i^T$ *and* $N_i^T N_j = N_j^T N_i$;
- $\left\{N_i N_j^T\right\} \cup \{I\}$ *spans the span of* $\{X_i\}$ *and* $\left\{N_i^T N_j\right\} \cup \{I\}$ *spans the span of* $\{Y_i\}$.

Theorem (Lato)

A bipartite coherent configuration $\mathcal{C}$ has a dual basis

$$\mathcal{B} = \left\{ \begin{pmatrix} L_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \bigcup \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & R_j \end{pmatrix} \bigcup \begin{pmatrix} \mathbf{0} & D_h \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \bigcup \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ D_h^T & \mathbf{0} \end{pmatrix} \right\}$$

satisfying

- $L_0 = \frac{1}{|\beta|}$, $R_0 = \frac{1}{|\gamma|}J$, *and* $D_0 = \frac{1}{\sqrt{|\beta\gamma|}}J$;
- $\sum_i L_i = I$ *and* $\sum_i R_i = I$;
- *For all* $M_1, M_2 \in \mathcal{C}$, *we have* $M_1 \circ M_2 \in \mathrm{Span}\left(\mathcal{B}\right)$;
- $L_i L_j = \delta_{ij} L_i$ *and* $R_i R_j = \delta_{ij} R_i$; *and*
- $D_i D_j^T = \delta_{ij} L_i$ *and* $D_i^T D_j = \delta_{ij} R_i$.

Examples

- Distance-biregular graphs
- Quasi-symmetric designs
- Strongly-regular designs (certain partial geometric designs or $1\frac{1}{2}$-designs

P- and Q-Polynomial

Definition

Let $\mathcal{A}$ be an association scheme. Then there exist eigenvalues $\{P_{rs}\}_{r,s=0}^{d}$ such that for $0 \leq r \leq d$, we have

$$A_r = \sum_{s=0}^{d} P_{rs} E_s.$$

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Definition

*An association scheme is P-**polynomial** if there exists an ordering $A_0, \ldots, A_d$ such that for all $0 \leq r \leq d$, there is a polynomial F_r of degree r such that for $0 \leq s \leq d$, we have $F_r(P_{1s}) = P_{rs}$.*

Definition

Let $\mathcal{A}$ be an association scheme. Then there exist dual eigenvalues $\{Q_{rs}\}_{r,s=0}^{d}$ such that for $0 \leq r \leq d$, we have

$$E_r = \sum_{s=0}^{d} Q_{rs} A_s.$$

Definition

*An association scheme is Q-**polynomial** if there exists an ordering $E_0, \ldots, E_d$ such that for all $0 \leq r \leq d$, there is a polynomial G_r of degree r such that for $0 \leq s \leq d$, we have $G_r(Q_{1s}) = Q_{rs}$.*

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Definition (Higman)

A set $\mathcal{C}$ of 01-matrices forms a coherent configuration if:

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- $\sum_{M \in \mathcal{C}} M = J$;
- $M^T \in \mathcal{C}$; *and*
- *For all $M_1, M_2 \in \mathcal{C}$, we have $M_1 M_2 \in \mathrm{Span}\left(\mathcal{C}\right)$.*

We can think of this in terms of $f \times f$ block matrices

$$\left\{A_0^{ij}, A_1^{ij}, \ldots, A_{d_{ij}}^{ij}\right\}_{i,j=1}^{f}.$$

Suppose there exist matrices $\left\{E_r^{ij}\right\}$ satisfying:

- $$E_0^{ij} = \frac{1}{\sqrt{x_i x_j}} J;$$
- The set $\left\{E_r^{ij}\right\}$ is a basis of $\mathrm{Span}\left(\left\{A_r^{ij}\right\}\right)$ as a vector space;
- $\left(E_r^{ij}\right)^T = E_r^{ji}$; and
- $$E_r^{ij} E_s^{j'h} = \delta_{j,j'} \delta_{r,s} E_r^{ih}.$$

Definition

Let $\mathcal{C}$ be a coherent configuration satisfying Suda's conditions. Then for $1 \leq i, j \leq f$, there exist dual eigenvalues $\left\{ Q_{rs}^{ij} \right\}_{r,s=0}^{d}$ such that for $0 \leq r \leq d$, we have

$$E_r^{ij} = \sum_{s=0}^{d} Q_{rs}^{ij} A_s^{ij}.$$

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Definition (Suda)

*A coherent configuration is Q-**polynomial** if for all $1 \leq i, j \leq f$, there exists an ordering $E_0^{ij}, \ldots, E_d^{ij}$ such that for all $0 \leq r \leq d$, there is a polynomial G_r^{ij} of degree r such that for $0 \leq s \leq d$, we have $G_r^{ij}\left(Q_{1s}^{ij}\right) = Q_{rs}^{ij}$.*

Definition

Let $\mathcal{C}$ be a coherent configuration satisfying Suda's conditions. Then for $1 \leq i, j \leq f$, there exist eigenvalues $\left\{P_{rs}^{ij}\right\}_{r,s=0}^{d}$ such that for $0 \leq r \leq d$, we have

$$A_r^{ij} = \sum_{s=0}^{d} P_{rs}^{ij} E_s^{ij}.$$

Definition

Let $\mathcal{C}$ be a coherent configuration satisfying Suda's conditions. Then for $1 \leq i, j \leq f$, there exist eigenvalues $\left\{P_{rs}^{ij}\right\}_{r,s=0}^{d}$ such that for $0 \leq r \leq d$, we have

$$A_r^{ij} = \sum_{s=0}^{d} P_{rs}^{ij} E_s^{ij}.$$

Definition (Naive Attempt)

*A coherent configuration is P-***polynomial** *if for all $1 \leq i, j \leq f$, there exists an ordering $A_0^{ij}, \ldots, A_d^{ij}$ such that for all $0 \leq r \leq d$, there is a polynomial F_r^{ij} of degree r such that for $0 \leq s \leq d$, we have $F_r^{ij}\left(P_{1s}^{ij}\right) = P_{rs}^{ij}$.*

Definition

Let $G = (\beta, \gamma)$ *be a bipartite graph with diameter* d*. It is* **distance-biregular** *if* G *has* $d + 1$ *distinct eigenvalues and there exists two sequences of polynomials* $F_0^\beta, \dots, F_d^\beta$ *and* $F_0^\gamma, \dots, F_d^\gamma$ *such that*

$$F_{2i}^\beta \left(NN^T\right) = X_i,$$

$$F_{2i}^\gamma \left(N^T N\right) = Y_i,$$

$$F_{2i+1}^\beta \left(NN^T\right) N = N_{2i+1},$$

and

$$F_{2i+1}^\gamma \left(N^T N\right) N^T = N_{2i+1}^T.$$

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$$E_r^{ij}E_s^{j'h} = \delta_{j,j'}\delta_{r,s}E_r^{ih}.$$

$$E_r^{ij}E_s^{ji} = \delta_{r,s}E_r^{ii}.$$

P-polynomial coherent configuration

Definition (Lato)

*A coherent configuration is P-**polynomial** if for all $1 \leq i \leq f$, there exists an ordering*

$$\left\{A_0^{i1}, \ldots, A_{d_{i1}}^{i1}, A_0^{i2}, \ldots, A_{d_{if}}^{if}\right\}$$

such that for all $0 \leq r \leq d$, there is a polynomial F_r^i of degree r such that for $0 \leq s \leq d$, we have $F_r^i\left(P_{1s}^{ij}\right) = P_{rs}^{ij}$.

Theorem (Lato)

If $\mathcal{C}$ is a P-polynomial coherent configuration, then it is formed from the distance adjacency matrices of a distance-regular or distance-biregular graph.

Definition

A collection

$$\mathcal{C} = \left\{ \begin{pmatrix} X_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \bigcup \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Y_j \end{pmatrix} \bigcup \begin{pmatrix} \mathbf{0} & N_h \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \bigcup \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ N_h^T & \mathbf{0} \end{pmatrix} \right\}$$

is a bipartite coherent configuration if it satisfies

- $X_0 = I$ *and* $Y_0 = I$;
- $\sum_{M \in \mathcal{C}} M = J$;
- *For all* $M_1, M_2 \in \mathcal{C}$, *we have* $M_1 M_2 \in \mathrm{Span}(\mathcal{C})$;
- $N_i N_j^T = N_j N_i^T$ *and* $N_i^T N_j = N_j^T N_i$;
- $\left\{ N_i N_j^T \right\} \cup \{I\}$ *spans the span of* $\{X_i\}$ *and* $\left\{ N_i^T N_j \right\} \cup \{I\}$ *spans the span of* $\{Y_i\}$.