Twin Buildings and Hypergroups II

Christopher P. French
Grinnell College
(joint work with Paul-Hermann Zieschang)

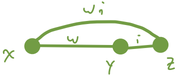
Combinatorics around q-Onsager algebra
Kranjska Gora, Slovenia
June 2025

Buildings

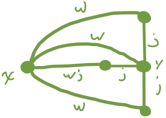
Let W be a Coxeter group with generating set I.
For $p \in W$, $l(p)$ is the *length* of p.

A building with Coxeter group W is a set X (of chambers) together with a Coxeter distance $\delta : X \times X \to W$.
Suppose $w \in W$, $i, j \in I$, $l(wi) = l(w) + 1$, $l(wj) = l(w) - 1$.

If $\delta(x, y) = w$, then



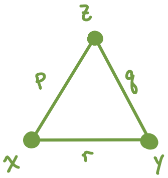
- $w = 1$ iff $x = y$;
- if $\delta(y, z) = i$, then $\delta(x, z) = wi$;
- there is one z with $\delta(y, z) = j$, $\delta(x, z) = wj$. For any other z with $\delta(y, z) = j$, $\delta(x, z) = w$.



A building is *thick* if for each $x \in X$, $i \in I$,

there are at least two chambers y, z with $\delta(x, y) = i = \delta(x, z)$.

We will assume all our buildings are *thick*.

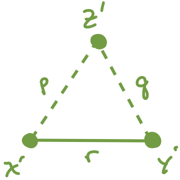


Then suppose $p, q, r \in W$, $x, y, z \in X$ with

$$\delta(x, y) = r,\ \delta(x, z) = p,\ \delta(y, z) = q.$$

Then for any $x', y' \in X$ with $\delta(x, y) = r$
there exists $z' \in X$ with

$$\delta(x', z') = p,\ \delta(y', z') = q$$



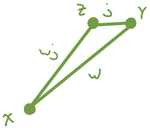
Twin Buildings

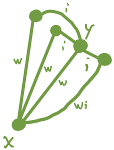
A *twin building* consists of two buildings (X_+, δ_+), (X_-, δ_-).
(Let $\delta_\pm(x, y) = \delta_+(x, y)$ when $x, y \in X_+$;
let $\delta_\pm(x, y) = \delta_-(x, y)$ when $x, y \in X_-$.)
We also need $\delta^0 : (X_+ \times X_-) \cup (X_- \times X_+) \to W$.

Suppose $w \in W$, $i, j \in I$, $l(wi) = l(w) + 1$, $l(wj) = l(w) - 1$.

If $\delta^0(x, y) = w$, we require

- $\delta^0(y, x) = w^{-1}$;
- if $\delta_\pm(y, z) = j$, then $\delta^0(x, z) = wj$;
- there is one z with $\delta_\pm(y, z) = i$, $\delta^0(x, z) = wi$.
  For any other z with $\delta_\pm(y, z) = i$, we have $\delta^0(x, z) = w$.





Conjecture 1

Let $p, q, r \in W$, $x, y \in X_-$ and $z \in X_+$ with

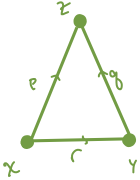
$$\delta_-(x, y) = r, \delta^0(x, z) = p, \delta^0(y, z) = q.$$

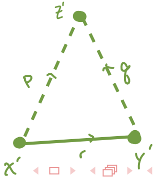
Given $x', y' \in X_-$ with $\delta_-(x', y') = r$,

must there be a $z' \in X_+$ with

$$\delta^0(x', z') = p, \delta^0(y', z') = q?$$

(If so, then twin buildings give rise to twin Coxeter hypergroups.)





Suppose $\delta^0(x, y) = w$, $i, j \in I$, $l(iw) = l(w) + 1, l(jw) = l(w) - 1$.

The axioms imply:

- If $\delta_{\pm}(z, x) = j$, then $\delta_{\pm}(z, y) = jw$
- There is one z with $\delta_{\pm}(z, x) = i$, $\delta^0(z, y) = iw$.

  For any other z with $\delta_{\pm}(z, x) = i$, we have $\delta^0(z, y) = w$.

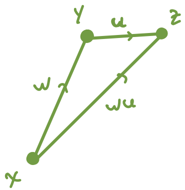
properties of twin buildings

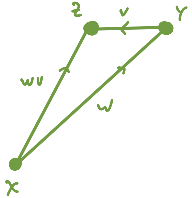
Also, suppose $u, v, w \in W$, $x \in X_-$, $y \in X_+$, $\delta^0(x, y) = w$, and

$$l(wu) = l(w) + l(u),\ l(wv) = l(w) - l(v), \text{ and}$$

Then

- There is a unique $z \in X_+$ with $\delta_+(y, z) = u$, $\delta^0(x, z) = wu$
- If $\delta_+(y, z) = v$ for some $z \in X_+$, then $\delta^0(x, z) = wv$.





opposites

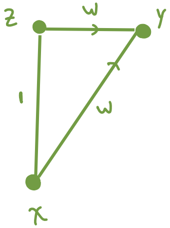
Suppose $x \in X_-, y \in X_+, \delta^0(x, y) = w$.

Then we can find z with $\delta(y, z) = w^{-1}$ (so $\delta(z, y) = w$.)

Then since $l(ww^{-1}) = l(w) - l(w^{-1})$, $\delta^0(x, z) = 1$.

When $\delta^0(x, z) = 1$, we say x, z are *opposite chambers*.

So, for any $x \in X_-$, we can find an opposite chamber $z \in X_+$.

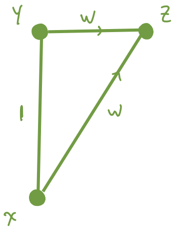


Let $x \in X_-$ and $w \in W$.

Choose $y \in X_+$ with $\delta^0(x, y) = 1$.

Since $l(1w) = l(w) = l(1) + l(w)$,

there is one $z \in X_+$ with $\delta^0(x, z) = w$.

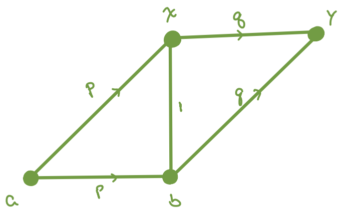


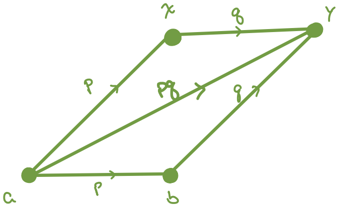
Twin Apartment Lemma Suppose

- $a, b \in X_-$, $x, y \in X_+$,
- $\delta_-(a, b) = p$, $\delta_+(x, y) = q$,
- $\delta^0(a, x) = p$, $\delta^0(b, x) = 1$, $\delta^0(b, y) = q$

We claim $\delta^0(a, y) = pq$.

This is true if $q = 1$, since then $x = y$.

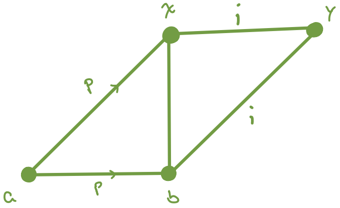




Consider the case $l(q) = 1$, so $q = i \in I$.

- $a, b \in X_-$, $x, y \in X_+$,
- $\delta_-(a, b) = p$, $\delta_+(x, y) = i$,
- $\delta^0(a, x) = p$, $\delta^0(b, x) = 1$, $\delta^0(b, y) = i$

If $l(pi) = l(p) - 1$, we must have $\delta^0(a, y) = pi$.

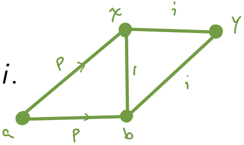


If $l(pi) = l(p) + 1$, $\exists! z \in X_+$ with $\delta_+(x, z) = i$, $\delta^0(a, z) = pi$.

Then $\delta^0(z, a) = (pi)^{-1} = ip^{-1}$, $\delta_-(a, b) = p$.

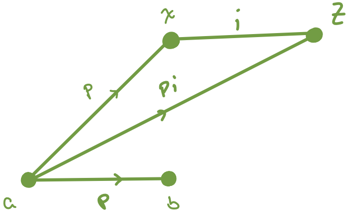
Since $l(ip^{-1}p) = 1 = l(ip^{-1}) - l(p)$, $\delta^0(z, b) = i$.

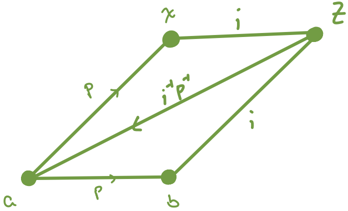
So $\delta^0(b, z) = i$.



But there is only one $y \in X_+$ with $\delta_+(x, y) = i$, $\delta^0(b, y) = i$.

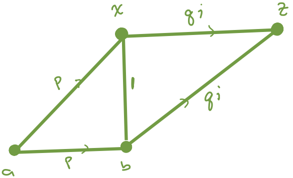
So, $z = y$, so $\delta^0(a, z) = pi$.





Assume the lemma when $l(q) \leq n$. Suppose $l(qi) = l(q) + 1$ and

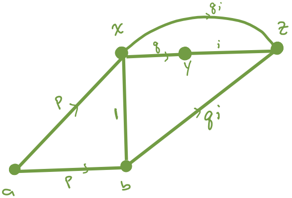
- $a, b \in X_-$, $x, z \in X_+$,
- $\delta_-(a, b) = p$, $\delta_+(x, z) = qi$,
- $\delta^0(a, x) = p$, $\delta^0(b, x) = 1$, $\delta^0(b, z) = qi$

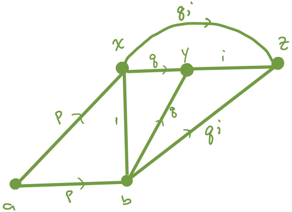


There is one y with $\delta_+(x, y) = q$, $\delta_+(y, z) = i$.

Since $\delta^0(b, z) = qi$ and $\delta_+(z, y) = i$, $\delta^0(b, y) = q$.

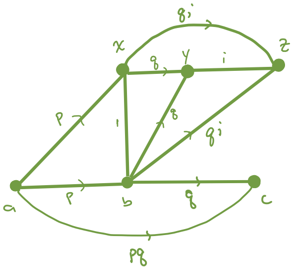
By induction, $\delta^0(a, y) = pq$.

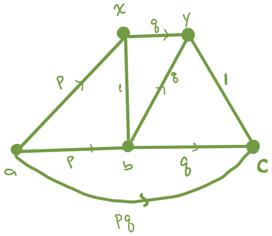




We can find c with $\delta_-(b,c) = q$, $\delta_-(a,c) = pq$.

We have $\delta^0(y,b) = q^{-1}$, and $l(qq^{-1}) = l(q) - l(q^{-1})$
so $\delta^0(y,c) = 1$, so $\delta^0(c,y) = 1$.

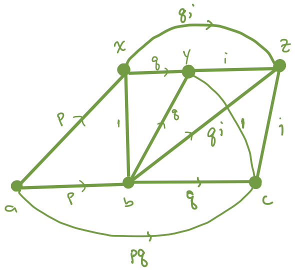




Also, $\delta^0(z, b) = (qi)^{-1} = iq^{-1}$ and $\delta_-(b, c) = q$.

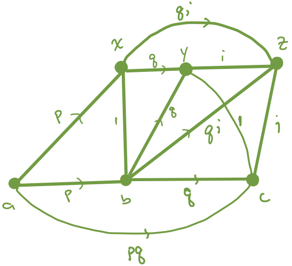
Since $l(iq^{-1}q) = 1 = l(iq^{-1}) - l(q)$,

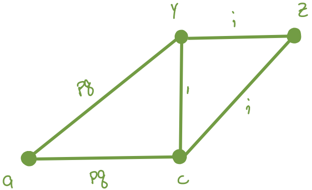
$\delta^0(z, c) = i$, so $\delta^0(c, z) = i$.



Now $\delta_-(a, c) = pq$, $\delta_+(y, z) = i$, $\delta^0(a, y) = pq$, $\delta^0(c, z) = i$, and $\delta^0(c, y) = 1$.

By the case when $l(q) = 1$, we get $\delta^0(a, z) = pqi$

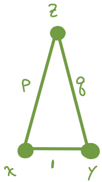




Suppose $l(r) = 0$. Then $r = 1$.

Let $p, q \in W$, $x, y \in X_-$ and $z \in X_+$ with

$$\delta_-(x, y) = r, \quad \delta^0(x, z) = p, \quad \delta^0(y, z) = q.$$

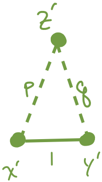


Since $r = 1$, $x = y$, so $q = p$.

Let $x', y' \in X_-$ with $\delta_-(x', y') = r = 1$. (So $x' = y'$).

We know there exists $z' \in X_+$ with $\delta^0(x', z') = p$.

$$\text{Then } \delta^0(y', z') = \delta^0(x', z') = p = q.$$



Conjecture for $l(r) = 1$

Let $p, q \in W$, $x, y \in X_-$ and $z \in X_+$ with

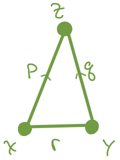
$$\delta_-(x, y) = r, \quad \delta^0(x, z) = p, \quad \delta^0(y, z) = q.$$

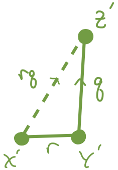
Suppose $x', y' \in X_-$ with $\delta_-(x', y') = r$.

Case 1: $l(rq) = l(q) - 1$. Since $\delta^0(y, z) = q$ and $\delta_-(x, y) = r$, we must have $p = \delta^0(x, z) = rq$.

Now just choose z' with $\delta^0(y', z') = q$. Since $\delta_-(x', y') = r$, we must have $\delta^0(x', z') = rq$.

So, $\delta^0(x', z') = p$.





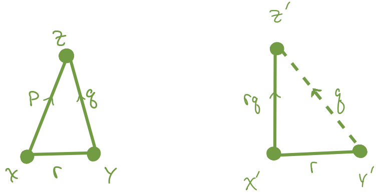
$l(rq) = l(q) + 1$

Case 2: $l(rq) = l(q) + 1$

Then $p = \delta^0(z, x) \in \{q, rq\}$

Case 2a: $p = rq$. Choose z' with $\delta^0(x', z') = p = rq$.

Since $\delta_-(y', x') = r$ and $l(q) = l(rq) - 1$, $\delta^0(y', z') = q$.



Case 2b: $l(rq) = l(q) + 1$ and $p = q$.

Since X_- is thick, choose $x'' \in X_-$ with $x'' \neq x'$, $\delta_-(y', x'') = r$.

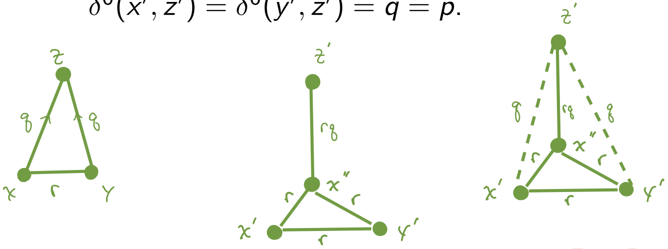
Then $\delta_-(x'', y') = r^{-1} = r$ and $\delta_-(y', x') = r^{-1} = r$, and $x' \neq x''$

so we must have $\delta_-(x'', x') = r$.

Choose $z' \in X_+$ so $\delta^0(x'', z') = rq$.

Since $\delta_-(x'', x') = r = \delta_-(x'', y')$ and $l(q) = l(rq) - 1$,

$$\delta^0(x', z') = \delta^0(y', z') = q = p.$$



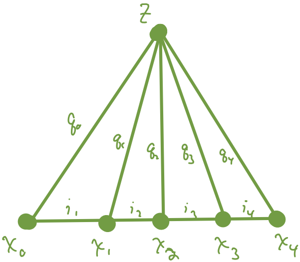
Let $i_1, i_2, \ldots, i_k \in I$ such that $i_1 i_2 \ldots i_k$ has length k.

A sequence $q_0, q_1, \ldots, q_t \in W$ is a *harp* over $i_1, k_2, \ldots, i_k$ if

there exists $x_0, x_1, x_2, \ldots, x_k \in X_-, z \in X_+$ such that

$$\delta_-(x_{t-1}, x_t) = i_t \text{ for } 0 \leq t \leq k$$

$$\text{and } \delta^0(x_t, z) = q_t \text{ for } 0 \leq t \leq k.$$



Universal harps

We say a harp $q_0, q_1, \ldots, q_t$ over $i_1, i_2, \ldots, i_k$ is *universal* if

for any sequence $x_0, x_1, \ldots, x_k \in X_-$ with

$$\delta_-(x_{t-1}, x_t) = i_t \text{ for } 1 \leq t \leq k$$

there exists $z \in X$ such that

$$\delta^0(x_t, z) = q_t \text{ for } 0 \leq t \leq k.$$

Harp Conjecture: Every harp is universal.

Harp conjecture $\implies$ Conjecture 1

Suppose given $x, y, \in X_-, z \in X_+$ with

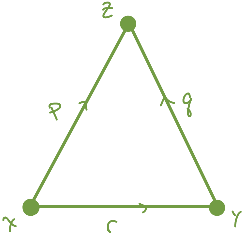
$$\delta_-(x, y) = r, \delta^0(x, z) = p, \delta^0(y, z) = q.$$

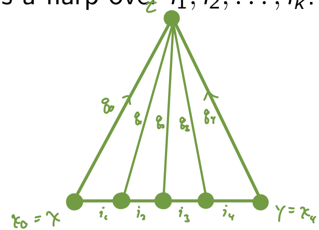
Let $k = l(r)$. Fix $i_1, i_2, \ldots, i_k$ with $r = i_1 i_2 \ldots i_k$.

Choose $x = x_0, x_1, \ldots, x_k = y$ such that $\delta(x_{t-1}, x_t) = i_t$ for each t.

Let $q_t = \delta^0(x_t, r)$ for $1 \leq t \leq k$,

so $q_0, q_1, \ldots, q_k$ is a harp over $i_1, i_2, \ldots, i_k$.





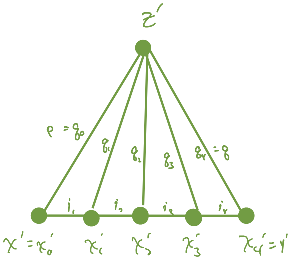
Harp conjecture $\implies$ Conjecture 1

Now suppose $x', y' \in X_-$ with $\delta_-(x', y') = r$.

Choose $x' = x'_0, x'_1, \ldots, x'_k = y'$ such that $\delta(x'_{t-1}, x'_t) = i_t$ for each t.

Since every harp is universal,
we can choose z' such that $\delta^0(x'_t, z') = q_t$ for all t.

Then $\delta^0(x', z') = q_0 = p$ and $\delta^0(y', z') = q_k = q$.



Non-repeating harps

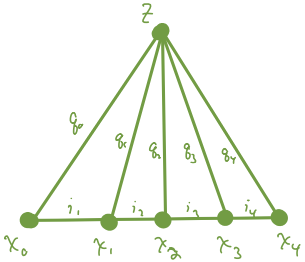
Suppose $q_0, q_1, \ldots, q_k$ is a harp over $i_1, i_2, \ldots, i_k$.

For each t, $\delta^0(x_t, z) = q_t$ and $\delta_-(x_{t-1}, x_t) = i_t$,
so we have $q_t \in \{q_{t-1}, i_t q_{t-1}\}$,

A harp is *non-repeating* if $q_t \neq q_{t-1}$ for $1 \leq t \leq k$.

so $q_t = i_t q_{t-1}$ for $1 \leq t \leq k$.

so $q_{t-1} = i_t q_t$ for $1 \leq t \leq k$.



Non-repeating harps

Let $i_1, i_2, \ldots, i_k \in I$, and suppose $r := i_1 i_2 \ldots i_k$ has length k.

Let $q_0, q_1, \ldots, q_k$ be a non-repeating harp over $i_1, i_2, \ldots, i_k$. Then

$$q_{k-1} = i_k q_k,$$
$$q_{k-2} = i_{k-1} q_{k-1} = i_{k-1} i_k q_k,$$
$$q_t = i_{t+1} i_{t+2} \cdots i_k q_k \textit{ for all } t$$

Harp conjecture for non-repeating harps

Theorem: Suppose given $i_1, i_2, \ldots, i_k \in I$ with $l(i_1 i_2 \ldots i_k) = k$. Suppose $q \in W$.

Let $q_k = q$, and for $t < k$, let $q_t = i_{t+1} i_{t+2} \cdots i_k q$.

Let $x_0, x_1, \ldots, x_k \in X_-$, where $\delta_-(x_{t-1}, x_t) = i_t$ for $1 \leq t \leq k$.

Then there exists $z \in X_+$ such that $\delta^0(x_t, z) = q_t$ for each t.

So, every non-repeating harp is universal.

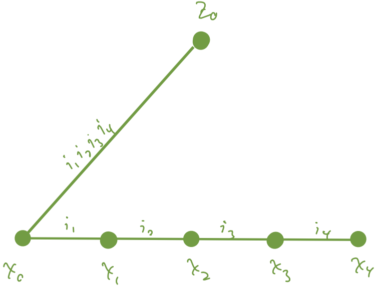
Proof of harp conjecture for non-repeating harps

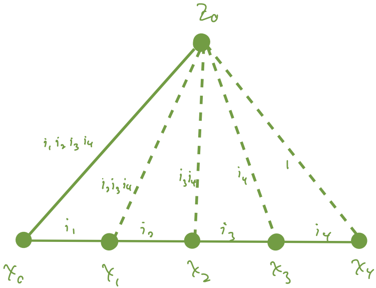
Choose $z_0 \in X_+$ so $\delta^0(x_0, z_0) = i_1 i_2 \dots i_k$.

Then since $\delta_-(x_1, x_0) = i_1$ and $\delta^0(x_0, z_0) = i_1 i_2 \dots i_k$,

$$\delta^0(x_1, z_0) = i_2 \cdots i_k.$$

Similarly, $\delta^0(x_t, z_0) = i_{t+1} \dots i_k$ for each t.





Proof of harp conjecture for non-repeating harps

We have $z_0 \in X_+$ with $\delta^0(x_t, z_0) = i_{t+1} \cdots i_k$ for each t.

In particular, $\delta^0(x_k, z_0) = 1$.

There is a unique z with $\delta(z_0, z) = q$ and $\delta^0(x_k, z) = q$.

By the twin apartment lemma,

$$\delta^0(x_t, z) = i_{t+1} \cdots i_k q = q_t \text{ for each } t.$$