Graph Isomorphism problem, Weisfeiler-Leman algortihm and coherent configurations

M. Muzychuk

Ben-Gurion University of the Negev,
Israel

10th PhD Summer School in Discrete Mathematics , Rogla
Slovenia, June 2022

Coherent algebras (D. Higman) = Cellular algebras (Weisfeiler-Leman)

Notation.

Let $A, B \in M_\Omega(\mathbb{F})$ be arbitrary matrices. We denote by

- AB (or $A \cdot B$) the usual matrix product;
- $A \circ B$ the Schur-Hadamard (component-wise) product, i.e. $(A \circ B)_{\alpha\beta} := A_{\alpha\beta} B_{\alpha\beta}$;
- $A^\top$ the transposed of A;
- I_Ω the identity matrix;
- J_Ω the all one matrix.

Coherent algebras (D. Higman) = Cellular algebras (Weisfeiler-Leman)

Notation.

Let $A, B \in M_\Omega(\mathbb{F})$ be arbitrary matrices. We denote by

- AB (or $A \cdot B$) the usual matrix product;
- $A \circ B$ the Schur-Hadamard (component-wise) product, i.e. $(A \circ B)_{\alpha\beta} := A_{\alpha\beta} B_{\alpha\beta}$;
- $A^\top$ the transposed of A;
- I_Ω the identity matrix;
- J_Ω the all one matrix.

Proposition

The algebra $(M_\Omega(\mathbb{F}), \circ)$ is a commutative associative algebra with identity J_Ω. It is isomorphic to $\mathbb{F}^n$ where $n = |\Omega|^2$.

Idempotents

Let $(\mathcal{A}, \star)$ be finite dimensional algebra over field $\mathbb{F}$.

Idempotents

Let $(\mathcal{A}, \star)$ be finite dimensional algebra over field $\mathbb{F}$.

- $e \in \mathcal{A}, e \neq 0$ is called $\star$-idempotent iff $e \star e = e$;

Idempotents

Let $(\mathcal{A}, \star)$ be finite dimensional algebra over field $\mathbb{F}$.

- $e \in \mathcal{A}, e \neq 0$ is called $\star$-idempotent iff $e \star e = e$;
- idempotents e, f are orthogonal if $e \star f = f \star e = 0$;

Idempotents

Let $(\mathcal{A}, \star)$ be finite dimensional algebra over field $\mathbb{F}$.

- $e \in \mathcal{A}, e \neq 0$ is called $\star$-idempotent iff $e \star e = e$;
- idempotents e, f are orthogonal if $e \star f = f \star e = 0$;
- idempotent e is minimal if it is not a sum $e = e_1 + e_2$ of pairwise orthogonal idempotents e_1, e_2;

Idempotents

Let $(\mathcal{A}, \star)$ be finite dimensional algebra over field $\mathbb{F}$.

- $e \in \mathcal{A}, e \neq 0$ is called $\star$-idempotent iff $e \star e = e$;
- idempotents e, f are orthogonal if $e \star f = f \star e = 0$;
- idempotent e is minimal if it is not a sum $e = e_1 + e_2$ of pairwise orthogonal idempotents e_1, e_2;
- a matrix $E \in M_\Omega(\mathbb{F})$ is $\cdot$-idempotent iff it's similar to a $(0, 1)$-diagonal matrix;

Idempotents

Let $(\mathcal{A}, \star)$ be finite dimensional algebra over field $\mathbb{F}$.

- $e \in \mathcal{A}, e \neq 0$ is called $\star$-idempotent iff $e \star e = e$;
- idempotents e, f are orthogonal if $e \star f = f \star e = 0$;
- idempotent e is minimal if it is not a sum $e = e_1 + e_2$ of pairwise orthogonal idempotents e_1, e_2;
- a matrix $E \in M_\Omega(\mathbb{F})$ is $\cdot$-idempotent iff it's similar to a $(0,1)$-diagonal matrix;
- a matrix $E \in M_\Omega(\mathbb{F})$ is $\circ$-idempotent iff it's $(0,1)$-matrix, that is $E = A(S)$ is the adjacency matrix of some $S \subseteq \Omega^2$, where

$$A(S)_{\alpha\beta} := \begin{cases} 1 & (\alpha, \beta) \in S; \\ 0 & (\alpha, \beta) \notin S \end{cases}$$

Idempotents

Let $(\mathcal{A}, \star)$ be finite dimensional algebra over field $\mathbb{F}$.

- $e \in \mathcal{A}, e \neq 0$ is called $\star$-idempotent iff $e \star e = e$;
- idempotents e, f are orthogonal if $e \star f = f \star e = 0$;
- idempotent e is minimal if it is not a sum $e = e_1 + e_2$ of pairwise orthogonal idempotents e_1, e_2;
- a matrix $E \in M_\Omega(\mathbb{F})$ is $\cdot$-idempotent iff it's similar to a $(0,1)$-diagonal matrix;
- a matrix $E \in M_\Omega(\mathbb{F})$ is $\circ$-idempotent iff it's $(0,1)$-matrix, that is $E = A(S)$ is the adjacency matrix of some $S \subseteq \Omega^2$, where

$$A(S)_{\alpha\beta} := \begin{cases} 1 & (\alpha, \beta) \in S; \\ 0 & (\alpha, \beta) \notin S \end{cases}$$

Exercise. Prove that a symmetric matrix is $\cdot$ and $\circ$ idempotent iff it's $(0,1)$-diagonal matrix.

Coherent (cellular) algebras.

Definition.

A subspace $\mathcal{A} \leq M_\Omega(\mathbb{F})$ is called a coherent (or cellular) algebra if it contains I_Ω, J_Ω and is closed with respect to $\cdot, \circ, ^\top$. The numbers $\dim(\mathcal{A})$ and $|\Omega|$ are called the rank and the degree of $\mathcal{A}$.

Coherent (cellular) algebras.

Definition.

A subspace $\mathcal{A} \leq M_\Omega(\mathbb{F})$ is called a coherent (or cellular) algebra if it contains I_Ω, J_Ω and is closed with respect to $\cdot, \circ, ^\top$. The numbers $\dim(\mathcal{A})$ and $|\Omega|$ are called the rank and the degree of $\mathcal{A}$.

Examples

- $\langle I_\Omega, J_\Omega \rangle$ - the trivial CA;
- $M_\Omega(\mathbb{F})$ - the discrete CA.

Coherent (cellular) algebras.

Definition.

A subspace $\mathcal{A} \leq M_\Omega(\mathbb{F})$ is called a coherent (or cellular) algebra if it contains I_Ω, J_Ω and is closed with respect to $\cdot, \circ, ^\top$. The numbers $\dim(\mathcal{A})$ and $|\Omega|$ are called the rank and the degree of $\mathcal{A}$.

Examples

- $\langle I_\Omega, J_\Omega \rangle$ - the trivial CA;
- $M_\Omega(\mathbb{F})$ - the discrete CA.

Proposition

Let $\mathcal{X} = (\Omega, \mathcal{C})$ be a coherent configuration. Then the linear span $\mathbb{F}[\mathcal{C}] := \langle A(C) \rangle_{C \in \mathcal{C}}$ is a coherent algebra of dimension $|\mathcal{C}|$. It is called the adjacency (or Bose-Mesner) algebra of $\mathcal{X}$.

Adjacency algebra of a coherent configuration.

The basis $A(C), C \in \mathcal{C}$ is called the standard basis of $\mathbb{F}[\mathcal{C}]$. The structure constants of the algebra $\mathbb{F}[\mathcal{C}]$ with respect to the standard basis coincide with the intersection numbers of $\mathcal{X}$, i.e.

$$A(S)A(T) = \sum_{R \in \mathcal{C}} c_{ST}^{R} A(R)$$

Notice that the standard basis matrices $A(S), S \in \mathcal{C}$ are $(0,1)$-matrices, or, equivalently, they are minimal $\circ$-idempotents.

Adjacency algebra of a coherent configuration.

The basis $A(C), C \in \mathcal{C}$ is called the standard basis of $\mathbb{F}[\mathcal{C}]$. The structure constants of the algebra $\mathbb{F}[\mathcal{C}]$ with respect to the standard basis coincide with the intersection numbers of $\mathcal{X}$, i.e.

$$A(S)A(T) = \sum_{R\in\mathcal{C}} c_{ST}^{R} A(R)$$

Notice that the standard basis matrices $A(S), S \in \mathcal{C}$ are $(0,1)$-matrices, or, equivalently, they are minimal $\circ$-idempotents.

Theorem

Every coherent algebra $\mathcal{A} \leq M_{\Omega}(\mathbb{F})$ has a unique basis consisting of minimal $\circ$-idempotents which are pairwise orthogonal. If $\mathrm{char}(\mathbb{F}) = 0$ then $\mathcal{A}$ is the adjacency algebra of a uniquely determined coherent configuration.

Proof of the Theorem

Since $\mathcal{A}$ is a $\circ$-subalgebra of $(M_\Omega(\mathbb{F}), \circ) \cong (\mathbb{F}^n, \circ)$, $n = |\Omega|^2$, we start with the following

Lemma

Let $\mathcal{A}$ be a k-dimensional subalgebra of $(\mathbb{F}^n, \circ)$. Then there exists a unique basis $A_1, ..., A_k$ of $\mathcal{A}$ consisting of minimal, pairwise orthogonal, $\circ$-idempotents. Each $\circ$-idemoptent of $\mathcal{A}$ is a $(0, 1)$-linear combination of $A_1, ..., A_k$ and $A_1 + ... + A_k$ is the unit of $\mathcal{A}$.

Proof of the Theorem (the end)

- $A_i = A(R_i), R_i \subseteq \Omega^2$;

Proof of the Theorem (the end)

- $A_i = A(R_i), R_i \subseteq \Omega^2$;
- $i \neq j \implies A_i \circ A_j = 0 \implies R_i \cap R_j = \emptyset$;

Proof of the Theorem (the end)

- $A_i = A(R_i), R_i \subseteq \Omega^2$;
- $i \neq j \implies A_i \circ A_j = 0 \implies R_i \cap R_j = \emptyset$;
- $\sum_{i=1}^{k} A_i = J_\Omega \implies \bigcup_i R_i = \Omega^2$;

Proof of the Theorem (the end)

- $A_i = A(R_i), R_i \subseteq \Omega^2$;
- $i \neq j \implies A_i \circ A_j = 0 \implies R_i \cap R_j = \emptyset$;
- $\sum_{i=1}^{k} A_i = J_\Omega \implies \bigcup_i R_i = \Omega^2$;
- $I_\Omega = \sum_i A_i \implies 1_\Omega = \bigcup_i R_i$;

Proof of the Theorem (the end)

- $A_i = A(R_i), R_i \subseteq \Omega^2$;
- $i \neq j \implies A_i \circ A_j = 0 \implies R_i \cap R_j = \emptyset$;
- $\sum_{i=1}^{k} A_i = J_\Omega \implies \bigcup_i R_i = \Omega^2$;
- $I_\Omega = \sum_i A_i \implies 1_\Omega = \bigcup_i R_i$;
- $A_i^\top = A_j \implies R_i^* = R_j$;

Proof of the Theorem (the end)

- $A_i = A(R_i), R_i \subseteq \Omega^2$;
- $i \neq j \implies A_i \circ A_j = 0 \implies R_i \cap R_j = \emptyset$;
- $\sum_{i=1}^{k} A_i = J_\Omega \implies \bigcup_i R_i = \Omega^2$;
- $I_\Omega = \sum_i A_i \implies 1_\Omega = \bigcup_i R_i$;
- $A_i^\top = A_j \implies R_i^* = R_j$;
- $A_i A_j = \sum_k c_{ij}^k A_k$ for some $c_{ij}^k \in \mathbb{F}$;

Proof of the Theorem (the end)

- $A_i = A(R_i), R_i \subseteq \Omega^2$;
- $i \neq j \implies A_i \circ A_j = 0 \implies R_i \cap R_j = \emptyset$;
- $\sum_{i=1}^{k} A_i = J_\Omega \implies \bigcup_i R_i = \Omega^2$;
- $I_\Omega = \sum_i A_i \implies 1_\Omega = \bigcup_i R_i$;
- $A_i^\top = A_j \implies R_i^* = R_j$;
- $A_i A_j = \sum_k c_{ij}^k A_k$ for some $c_{ij}^k \in \mathbb{F}$;
- $(A_i A_j)_{\alpha\beta} = c_{ij}^k$ where k is defined by $(\alpha, \beta) \in R_k$;

Proof of the Theorem (the end)

- $A_i = A(R_i), R_i \subseteq \Omega^2$;
- $i \neq j \implies A_i \circ A_j = 0 \implies R_i \cap R_j = \emptyset$;
- $\sum_{i=1}^{k} A_i = J_\Omega \implies \bigcup_i R_i = \Omega^2$;
- $I_\Omega = \sum_i A_i \implies 1_\Omega = \bigcup_i R_i$;
- $A_i^\top = A_j \implies R_i^* = R_j$;
- $A_i A_j = \sum_k c_{ij}^k A_k$ for some $c_{ij}^k \in \mathbb{F}$;
- $(A_i A_j)_{\alpha\beta} = c_{ij}^k$ where k is defined by $(\alpha, \beta) \in R_k$;
- if $\mathrm{char}(\mathbb{F}) = 0$, then
  $(A_i A_j)_{\alpha\beta} = |\alpha R_i \cap R_j \beta| \implies |\alpha R_i \cap R_j \beta| = c_{ij}^k$.

Isomorphisms between coherent algebras

Definition

Given two coherent algebras $\mathcal{A} \leq M_\Omega(\mathbb{F}), \mathcal{A}' \leq M_{\Omega'}(\mathbb{F})$, a linear bijection $L : \mathcal{A} \to \mathcal{A}'$ is called an (algebraic) isomorphism iff

- $L(XY) = L(X)L(Y)$;
- $L(X \circ Y) = L(X) \circ L(Y)$;
- $L(X^\top) = L(X)^\top$.

Isomorphisms between coherent algebras

Definition

Given two coherent algebras $\mathcal{A} \leq M_{\Omega}(\mathbb{F}), \mathcal{A}' \leq M_{\Omega'}(\mathbb{F})$, a linear bijection $L : \mathcal{A} \to \mathcal{A}'$ is called an (algebraic) isomorphism iff

- $L(XY) = L(X)L(Y)$;
- $L(X \circ Y) = L(X) \circ L(Y)$;
- $L(X^{\top}) = L(X)^{\top}$.

Proposition

Let $L : \mathbb{F}[\mathcal{C}] \to \mathbb{F}[\widetilde{\mathcal{C}}]$ be an algebraic isomorphism between the adjacency algebras of co.co.s $\mathcal{C}$ and $\widetilde{\mathcal{C}}$. Then there exists a bijection $\varphi : \mathcal{C} \to \widetilde{\mathcal{C}}$ such that $L(A(C)) = A(C^{\varphi})$ and $c_{RS}^{T} = \widetilde{c}_{R^{\varphi}S^{\varphi}}^{T^{\varphi}}$. Vice versa, any bijection $\varphi : \mathcal{C} \to \widetilde{\mathcal{C}}$ satisfying the above equations extends uniquely up to an algebraic isomorphism between $\mathbb{F}[\mathcal{C}]$ and $\mathbb{F}[\widetilde{\mathcal{C}}]$. We'll call it an algebraic isomorphism between the co.co.s.

Properties of algebraic isomorphisms

Proposition

Let $\varphi : \mathcal{C} \to \widetilde{\mathcal{C}}$ be an algebraic isomorphism between the co.co.s $\mathcal{X} = (\Omega, \mathcal{C})$ and $\widetilde{\mathcal{X}} = (\widetilde{\Omega}, \widetilde{\mathcal{C}})$. Then

- $(RS)^\varphi = R^\varphi S^\varphi$ for any $R, S \in \mathcal{C}^\cup$
- for each fiber Δ of $\mathcal{X}$ there exists a unique fiber Δ' of $\mathcal{X}'$ such that $(1_\Delta)^\varphi = 1_{\Delta'}$, that is φ induces a bijection between fibres;
- $S \in \mathcal{C} \implies D(S^\varphi) = D(S)^\varphi, R(S^\varphi) = R(S)^\varphi$;
- $|n_{C^\varphi}| = |n_C|$ for each $C \in \mathcal{C}$;
- $|\Delta^\varphi| = |\Delta|$ for any $\Delta \in \Phi(\mathcal{X})$;

Isomorphisms between coherent algebras

Proposition

For each $f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}')$ the mapping f^* is an algebraic isomorphism between the configurations. In this case we say that f^* is an algebraic isomorphism induced by a combinatorial one.

All algebraic automorphisms of a co.co. $\mathcal{X} = (\Omega, \mathcal{C})$ form a group (a subgroup of $\mathrm{Sym}(\mathcal{C})$) denoted as $\mathrm{Alg}(\mathcal{X})$ or $\mathrm{Alg}(\mathcal{C})$. Notice that $\mathrm{Iso}(\mathcal{X})/\,\mathrm{Aut}(\mathcal{X}) \hookrightarrow \mathrm{Alg}(\mathcal{X})$.

Proposition

Let $A \leq \mathrm{Alg}(\mathcal{X})$. Then the subspace

$$\mathbb{Q}[\mathcal{C}]^A := \{x \in \mathbb{Q}[\mathcal{C}] \mid \forall_{a \in A}\ x^a = x\}$$

is a coherent algebra. The corresponding coherent conguration is denoted as $\mathcal{C}^A$. It is called an algebraic fusion of $\mathcal{C}$.

Coherent closure.

Proposition. Let $\mathcal{X} = (\Omega, \mathcal{C})$ and $\mathcal{X}' = (\Omega, \mathcal{C}')$ be coherent configurations. Then

- $\mathbb{F}[\mathcal{C}] \subseteq \mathbb{F}[\mathcal{C}'] \iff \mathcal{C} \sqsubseteq \mathcal{C}'$;
- $\mathbb{F}[\mathcal{C}] \cap \mathbb{F}[\mathcal{C}'] = \mathbb{F}[\mathcal{C} \wedge \mathcal{C}'] \implies \mathcal{C} \wedge \mathcal{C}'$ is a coherent configuration.

Notice that the sum of coherent algebras is not necessarily coherent algebra.

Proposition

Let $A_1, \ldots, A_m \in M_\Omega(\mathbb{F})$ be an arbitrary sequence of matrices. The intersection of all coherent algebras containing $A_1, \ldots, A_m$ is a coherent algebra too, called the coherent closure of $A_1, \ldots, A_m$ and denoted as $\langle\langle A_1, \ldots, A_m \rangle\rangle$.

Computing coherent closure by WL-algortihm.

Definition

Given a sequence $M_1, \ldots, M_k \in M_\Omega(\mathbb{F})$ of matrices. We define a partition $\mathcal{P}(M_1, \ldots, M_k)$ of Ω^2 via the following equivalence relation

$$(\alpha, \beta) \sim (\gamma, \delta) \iff \forall_{1 \leq i \leq k} \ (M_i)_{\alpha\beta} = (M_i)_{\gamma\delta}$$

Computing coherent closure by WL-algortihm.

Definition

Given a sequence $M_1, ..., M_k \in M_\Omega(\mathbb{F})$ of matrices. We define a partition $\mathcal{P}(M_1, ..., M_k)$ of Ω^2 via the following equivalence relation

$$(\alpha, \beta) \sim (\gamma, \delta) \iff \forall_{1 \leq i \leq k} \ (M_i)_{\alpha\beta} = (M_i)_{\gamma\delta}$$

Proposition

If $\mathcal{X} = (\Omega, \mathcal{C})$ is a coherent configuration, then $\mathcal{P}(M_1, ..., M_k) \sqsubseteq \mathcal{C}$ for any tuple $M_1, ..., M_k \in \mathbb{F}[\mathcal{C}]$.

Computing coherent closure by WL-algortihm.

Definition

Given a sequence $M_1, ..., M_k \in M_\Omega(\mathbb{F})$ of matrices. We define a partition $\mathcal{P}(M_1, ..., M_k)$ of Ω^2 via the following equivalence relation

$$(\alpha, \beta) \sim (\gamma, \delta) \iff \forall_{1 \le i \le k} \; (M_i)_{\alpha\beta} = (M_i)_{\gamma\delta}$$

Proposition

If $\mathcal{X} = (\Omega, \mathcal{C})$ is a coherent configuration, then $\mathcal{P}(M_1, ..., M_k) \sqsubseteq \mathcal{C}$ for any tuple $M_1, ..., M_k \in \mathbb{F}[\mathcal{C}]$.

Proposition. Let $\mathcal{X} = (\Omega, \mathcal{C})$ be a co.co. and $\mathcal{S} \vdash \Omega^2$. Then

$\mathcal{S} \sqsubseteq \mathcal{C} \implies$ Ы$(\mathcal{S}) \sqsubseteq \mathcal{C}$

Computing coherent closure by WL-algortihm.

Definition

Given a sequence $M_1, ..., M_k \in M_\Omega(\mathbb{F})$ of matrices. We define a partition $\mathcal{P}(M_1, ..., M_k)$ of Ω^2 via the following equivalence relation

$$(\alpha, \beta) \sim (\gamma, \delta) \iff \forall_{1 \le i \le k}\ (M_i)_{\alpha\beta} = (M_i)_{\gamma\delta}$$

Proposition

If $\mathcal{X} = (\Omega, \mathcal{C})$ is a coherent configuration, then $\mathcal{P}(M_1, ..., M_k) \sqsubseteq \mathcal{C}$ for any tuple $M_1, ..., M_k \in \mathbb{F}[\mathcal{C}]$.

Proposition. Let $\mathcal{X} = (\Omega, \mathcal{C})$ be a co.co. and $\mathcal{S} \vdash \Omega^2$. Then

$\mathcal{S} \sqsubseteq \mathcal{C} \implies$ Ы$(\mathcal{S}) \sqsubseteq \mathcal{C}$ (follows from Ы$(\mathcal{S}) \sqsubseteq$ Ы$(\mathcal{C})$ and Ы$(\mathcal{C}) = \mathcal{C}$)

Computing coherent closure by WL algortihm.

- **Input:** $A_1, ..., A_k \in M_\Omega(\mathbb{F})$,
- **Output:** the coherent closure $\langle\langle A_1, ..., A_k \rangle\rangle$;
- Compute $\mathcal{S}_0 := \mathcal{P}(A_1, ..., A_k, A_1^\top, ..., A_k^\top, I_\Omega)$;
- Starting from $\mathcal{S} := \mathcal{S}_0$ apply WL-stabilization procedure $\mathcal{S} \to \text{Ы}(\mathcal{S})$ until $\mathcal{S} = \text{Ы}(\mathcal{S})$.

Theorem

The WL-algorithm produces the coherent closure of the matrices $A_1, ..., A_k$.

Proof. Let $\mathcal{C}$ denote the underlying coherent configuration of $\langle\langle A_1, ..., A_k \rangle\rangle$, i.e. $\langle\langle A_1, ..., A_k \rangle\rangle = \mathbb{Q}[\mathcal{C}]$. Then

- $A_1, \ldots, A_k, A_1^\top, \ldots, A_k^\top, I_\Omega \in \mathbb{Q}[\mathcal{C}] \implies \mathcal{S}_0 \sqsubseteq \mathcal{C}$
- Let $\mathcal{S}_i := \text{Ы}(\mathcal{S}_{i-1}), i = 1, \ldots, m$ be the sequence of parititions generated by WL-algorithm (thus $|\mathcal{S}_{m+1}| = |\mathcal{S}_m|$);
- since $\mathcal{S}_{i-1}^* = \mathcal{S}_{i-1}$ and $1_\Omega \in \mathcal{S}_{i-1}^\cup$, we obtain

$$\mathcal{S}_i^* = \mathcal{S}_i,\ \mathcal{S}_{i-1} \sqsubseteq \mathcal{S}_i \implies 1_\Omega \in \mathcal{S}_i^\cup, \mathcal{S}_{m+1} = \mathcal{S}_m;$$

- $\mathcal{S}_{i-1} \sqsubseteq \mathcal{C} \implies \mathcal{S}_i \sqsubseteq \mathcal{C}$;
- $\mathcal{S}_m \sqsubseteq \mathcal{C} \implies \mathcal{S}_m = \mathcal{C}$. $\square$

Canonical ordering

Proposition

Let $\vec{\mathcal{S}} = (S_1, ..., S_m)$ and $\vec{\mathcal{T}} = (T_1, ..., T_m)$ be ordered partitions of Ω^2 and Δ^2 resp. Let $\langle\langle\vec{\mathcal{S}}\rangle\rangle = (P_1, ..., P_k)$ and $\langle\langle\vec{\mathcal{T}}\rangle\rangle = (Q_1, ..., Q_\ell)$ be the canonical ordering of the coherent closures produced by WL-algorithm. If there exists an isomorphism $f : \Omega \to \Delta$ such that $S_i^f = T_i$, then $k = \ell$ and $P_i^f = Q_i, i = 1, ..., k$. In particular, the mapping $P_i \mapsto Q_i$ is an algebraic isomorphism between the co.co.s $\langle\langle\mathcal{S}\rangle\rangle$ and $\langle\langle\mathcal{T}\rangle\rangle$.

Reformulation of the GI

Given an algebraic isomoprhism between the coherent configurations $\mathcal{S}$ and $\mathcal{T}$. Find whether it is induced by a combinatorial one.

Association schemes.

Association scheme.

A pair (Ω, S) where $\mathcal{S} \vdash \Omega^2$ is an association scheme (=homogeneous coherent configuration) iff

- $1_\Omega \in \mathcal{S}$;
- $\mathcal{S}^* = \mathcal{S}$;
- ЫI$(\mathcal{S}) = \mathcal{S}$

Recall that the intersection numbers are

$$\forall_{S,R,T\in\mathcal{S}}\ \forall_{(\alpha,\beta)\in T}\ |\alpha S \cap R\beta| = c_{RS}^{T}.$$

The number $n_S := c_{SS^*}^{1_\Omega} = |\omega S|$ is called the valency of S.

Elementary properties of schemes.

Proposition

- the set $\mathcal{S}^{\cup}$ is closed w.r.t. boolean operations;

Elementary properties of schemes.

Proposition

- the set $\mathcal{S}^{\cup}$ is closed w.r.t. boolean operations;
- $1_{\Omega}, \Omega^2 \in \mathcal{S}^{\cup}$;

Elementary properties of schemes.

Proposition

- the set $\mathcal{S}^{\cup}$ is closed w.r.t. boolean operations;
- $1_{\Omega}, \Omega^2 \in \mathcal{S}^{\cup}$;
- $(\mathcal{S}^{\cup})^* = \mathcal{S}^{\cup}$;

Elementary properties of schemes.

Proposition

- the set $\mathcal{S}^{\cup}$ is closed w.r.t. boolean operations;
- $1_{\Omega}, \Omega^2 \in \mathcal{S}^{\cup}$;
- $(\mathcal{S}^{\cup})^* = \mathcal{S}^{\cup}$;
- $\mathcal{S}$ is closed w.r.t. relational product

Elementary properties of schemes.

Proposition

- the set $\mathcal{S}^{\cup}$ is closed w.r.t. boolean operations;
- $1_{\Omega}, \Omega^2 \in \mathcal{S}^{\cup}$;
- $(\mathcal{S}^{\cup})^* = \mathcal{S}^{\cup}$;
- $\mathcal{S}$ is closed w.r.t. relational product $\Longrightarrow \forall_{\ell \in \mathbb{N}} \forall_{S \in \mathcal{S}}\ S^{\ell} \in \mathcal{S}^{\cup}$;

Elementary properties of schemes.

Proposition

- the set $\mathcal{S}^{\cup}$ is closed w.r.t. boolean operations;
- $1_{\Omega}, \Omega^2 \in \mathcal{S}^{\cup}$;
- $(\mathcal{S}^{\cup})^* = \mathcal{S}^{\cup}$;
- $\mathcal{S}$ is closed w.r.t. relational product $\Longrightarrow \forall_{\ell \in \mathbb{N}} \forall_{S \in \mathcal{S}}\ S^{\ell} \in \mathcal{S}^{\cup}$;
- each relation $R \in \mathcal{S}^{\cup}$ is regular;

Elementary properties of schemes.

Proposition

- the set $\mathcal{S}^{\cup}$ is closed w.r.t. boolean operations;
- $1_{\Omega}, \Omega^2 \in \mathcal{S}^{\cup}$;
- $(\mathcal{S}^{\cup})^* = \mathcal{S}^{\cup}$;
- $\mathcal{S}$ is closed w.r.t. relational product $\Longrightarrow$ $\forall_{\ell \in \mathbb{N}} \forall_{S \in \mathcal{S}}\ S^{\ell} \in \mathcal{S}^{\cup}$;
- each relation $R \in \mathcal{S}^{\cup}$ is regular;
- for any $S \in \mathcal{S}$ there exists m s.t. $1_{\Omega} \subseteq S^m$;

Elementary properties of schemes.

Proposition

- the set $\mathcal{S}^{\cup}$ is closed w.r.t. boolean operations;
- $1_{\Omega}, \Omega^2 \in \mathcal{S}^{\cup}$;
- $(\mathcal{S}^{\cup})^* = \mathcal{S}^{\cup}$;
- $\mathcal{S}$ is closed w.r.t. relational product $\Longrightarrow$ $\forall_{\ell \in \mathbb{N}} \forall_{S \in \mathcal{S}}\ S^{\ell} \in \mathcal{S}^{\cup}$;
- each relation $R \in \mathcal{S}^{\cup}$ is regular;
- for any $S \in \mathcal{S}$ there exists m s.t. $1_{\Omega} \subseteq S^m$;
- for any $S \in \mathcal{S}$ there exists m s.t. $S^* \subseteq S^m$;

Elementary properties of schemes.

Proposition

- the set $\mathcal{S}^{\cup}$ is closed w.r.t. boolean operations;
- $1_{\Omega}, \Omega^2 \in \mathcal{S}^{\cup}$;
- $(\mathcal{S}^{\cup})^* = \mathcal{S}^{\cup}$;
- $\mathcal{S}$ is closed w.r.t. relational product $\implies \forall_{\ell \in \mathbb{N}} \forall_{S \in \mathcal{S}}\ S^{\ell} \in \mathcal{S}^{\cup}$;
- each relation $R \in \mathcal{S}^{\cup}$ is regular;
- for any $S \in \mathcal{S}$ there exists m s.t. $1_{\Omega} \subseteq S^m$;
- for any $S \in \mathcal{S}$ there exists m s.t. $S^* \subseteq S^m$;
- $S \in \mathcal{S}^{\cup} \implies S^+ = \cup_{i=1}^{\infty} S^i \in \mathcal{S}^{\cup}$;

Elementary properties of schemes.

Proposition

- the set $\mathcal{S}^{\cup}$ is closed w.r.t. boolean operations;
- $1_\Omega, \Omega^2 \in \mathcal{S}^{\cup}$;
- $(\mathcal{S}^{\cup})^* = \mathcal{S}^{\cup}$;
- $\mathcal{S}$ is closed w.r.t. relational product $\implies \forall_{\ell \in \mathbb{N}} \forall_{S \in \mathcal{S}}\ S^\ell \in \mathcal{S}^{\cup}$;
- each relation $R \in \mathcal{S}^{\cup}$ is regular;
- for any $S \in \mathcal{S}$ there exists m s.t. $1_\Omega \subseteq S^m$;
- for any $S \in \mathcal{S}$ there exists m s.t. $S^* \subseteq S^m$;
- $S \in \mathcal{S}^{\cup} \implies S^+ = \cup_{i=1}^{\infty} S^i \in \mathcal{S}^{\cup}$;
- $S \in \mathcal{S}^{\cup} \implies S^+$ is an equivalence relation on Ω

Elementary properties of schemes.

Definition

A scheme is called symmetric (antisymmetric) if every $S \neq 1_\Omega$ is symmetric (anti-symmetric, resp.). A scheme is called commutative if its adjacency algebra is commutative.

Elementary properties of schemes.

Definition

A scheme is called symmetric (antisymmetric) if every $S \neq 1_\Omega$ is symmetric (anti-symmetric, resp.). A scheme is called commutative if its adjacency algebra is commutative.

Proposition

A symmetric scheme is always commutative.

Elementary properties of schemes.

Definition

A scheme is called symmetric (antisymmetric) if every $S \neq 1_\Omega$ is symmetric (anti-symmetric, resp.). A scheme is called commutative if its adjacency algebra is commutative.

Proposition

A symmetric scheme is always commutative.

Proposition

Let $(\Omega, \mathcal{S})$ be a scheme. Then

- $\sum_{S \in \mathcal{S}} n_S = |\Omega|$;

Elementary properties of schemes.

Definition

A scheme is called symmetric (antisymmetric) if every $S \neq 1_\Omega$ is symmetric (anti-symmetric, resp.). A scheme is called commutative if its adjacency algebra is commutative.

Proposition

A symmetric scheme is always commutative.

Proposition

Let $(\Omega, \mathcal{S})$ be a scheme. Then

- $\sum_{S \in \mathcal{S}} n_S = |\Omega|$;
- $2 \leq |\mathcal{S}| \leq |\Omega|$;

Elementary properties of schemes.

Definition

A scheme is called symmetric (antisymmetric) if every $S \neq 1_\Omega$ is symmetric (anti-symmetric, resp.). A scheme is called commutative if its adjacency algebra is commutative.

Proposition

A symmetric scheme is always commutative.

Proposition

Let $(\Omega, \mathcal{S})$ be a scheme. Then

- $\sum_{S \in \mathcal{S}} n_S = |\Omega|$;
- $2 \leq |\mathcal{S}| \leq |\Omega|$;
- $|\mathcal{S}| = 2 \iff \mathcal{S} = \{1_\Omega, \Omega^2 \setminus 1_\Omega\}$;

Elementary properties of schemes.

Definition

A scheme is called symmetric (antisymmetric) if every $S \neq 1_\Omega$ is symmetric (anti-symmetric, resp.). A scheme is called commutative if its adjacency algebra is commutative.

Proposition

A symmetric scheme is always commutative.

Proposition

Let $(\Omega, \mathcal{S})$ be a scheme. Then

- $\sum_{S \in \mathcal{S}} n_S = |\Omega|$;
- $2 \leq |\mathcal{S}| \leq |\Omega|$;
- $|\mathcal{S}| = 2 \iff \mathcal{S} = \{1_\Omega, \Omega^2 \setminus 1_\Omega\}$;
- $|\mathcal{S}| = |\Omega| \iff \forall_{S \in \mathcal{S}} n_S = 1$

Primitive and Imprimitive schemes

Definition

A scheme $(\Omega, \mathcal{S})$ is called imprimitive if $\mathcal{S}^{\cup}$ contains a non-trivial and non-discrete equivalence relation E. The equivalence classes $\omega E, \omega \in \Omega$ form a partition of Ω called imprimitivity system of $\mathcal{S}$.

Proposition

If $E \in \mathcal{S}^{\cup}$ is an equivalence. Then $|\Omega/E| \cdot n_E = |\Omega|$.

Proposition

The following are equivalent

- $\mathcal{S}$ is imprimitive;
- $\exists\ S \in \mathcal{S}, S \neq 1_\Omega$ s.t $S^+ \neq \Omega^2$;
- $\exists\ \mathcal{T} \subset \mathcal{S}, 1 < |\mathcal{T}| < |\mathcal{S}|$ s. t. $\langle A(T) \rangle_{T \in \mathcal{T}}$ is a subalgebra of $\mathbb{Q}[\mathcal{S}]$

Schurian schemes

Recall that a scheme $(\Omega, \mathcal{S})$ is schurian iff there exists $G \leq \mathrm{Sym}(\Omega)$ s.t. $\mathcal{S} = \mathrm{Inv}(G)$.

- G is transitive on Ω;
- ωS is an orbit of G_ω for each $\omega \in \Omega$ and $S \in \mathcal{S}$;
- the mapping $S \mapsto \{g \in G \,|\, \omega^g \in \omega S\}$ is a bijection between the basic relations of $\mathcal{S}$ and double cosets of G_ω.
- a rescaling of the above mapping is an isomorphism between the adjacency algebra $\mathbb{F}[\mathcal{S}]$ and the Hecke algebra $\mathbb{F}(G_\omega \backslash G / G_\omega)$;
- G is primitive iff $\mathrm{Inv}(G)$ is primitive.