Group based constructions of k-regular graphs of large girth and small order

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June 29, 2022, Rogla, Slovenia
10th PhD Summer School in Discrete Mathematics

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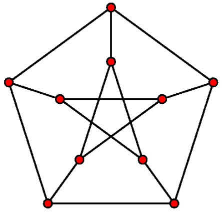
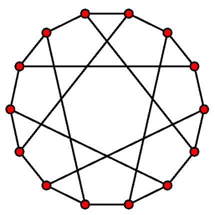
- The set of all (k, g)-graphs for a pair of parameters (k, g) might be infinite and we may not know how to efficiently construct all such graphs
- The order of $n(k, g)$ grows for a fixed k exponentially with respect to g

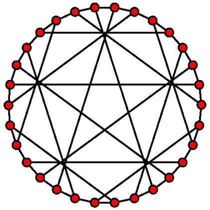
Applications of Cages

- Regular graphs have a wide range of applications in **Network Design.**
  - If all vertices of the graph are of the same degree, they can be mass produced.
  - Choosing a small graphs makes the network efficient and cheap.
- Small graphs of large girth are used in the construction of the **LDPC codes, Low Density Parity Check Codes**.
  - LDPC codes are linear error correcting codes whose generating matrices contain very few 1's.
  - The speed of the decoding is directly related to the girth of the related graph.

Examples of Cages







Petersen graph	**Heawood graph**	**Tutte graph**
$k = 3, g = 5$	$k = 3, g = 6$	$k = 3, g = 8$

Ancient History

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Theorem (Erdös and Sachs, 1963, Sachs 1963)

Infinitely many (k,g)-graphs exist for all pairs (k,g), $k \geq 2$, $g \geq 3$.

Future?

- Finding cages requires two steps:
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  - a good construction of a (k, g)-graph, and
  - a proof that there is no smaller (k, g)-graph
- Since proofs are hard to get by, one usually settles for

  Record Graphs,

  i.e., one finds a good construction and gives up on proving that the obtained graph is the smallest possible; just compares to the graphs obtained by others.

Record Cubic Cages

Girth g	Lower bound	Upper bound	Author(s)
13	202	272	McKay-Myrvold; Hoare
14	258	384	McKay; Exoo
15	384	620	Biggs
16	512	960	Exoo
17	768	2176	Exoo
18	1024	2560	Exoo
19	1536	4324	Hoare, H(47)
20	2048	5376	Exoo
21	3072	16028	Exoo
22	4096	16206	Biggs-Hoare, S(73)
23	6144	49326	Exoo
24	8192	49608	Bray-Parker-Rowley
25	12288	108906	Exoo

Record Cubic Cages

Girth g	Lower bound	Upper bound	Author(s)
26	16384	109200	Bray-Parker-Rowley
27	24576	285852	Bray-Parker-Rowley
28	32768	415104	Bray-Parker-Rowley
29	49152	1141484	Exoo-Jajcay
30	65536	1143408	Exoo-Jajcay
31	98304	3649794	Bray-Parker-Rowley
32	131072	3650304	Bray-Parker-Rowley

G. Exoo and R. Jajcay, Dynamic cage survey, Electron. J. Combin.

How to Become a Record Holder?

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- Highly symmetric graphs look all the same at each vertex
- Highly symmetric graphs usually have a compact description

Highly Symmetric Graphs

A graph is called **vertex-transitive** if there exists a graph automorphism mapping u to v for any pair of vertices u, v.

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Known Cubic Cages:

girth	**5**	**6**	7	**8**	9	10	11	**12**
order	**10**	**14**	24	**30**	58	70	112	**126**
# of cages	**1**	**1**	1	**1**	18	3	1	1
# of sym's	120	336	32	1440	≤ 24	≤ 120	64	12,096

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Theorem (Nedela, Škoviera (2001))

For every pair (k, g), there exists a vertex-transitive graph of degree k and girth g.

Current Record Holders - Infinite Families

- **Sextet Graphs, Hexagons**, Biggs and Hoare, based on finite fields $GF(q)$
- **The Lubotzky-Phillips-Sarnak Construction**, based on Cayley graphs of linear groups
- Loz, Miller, Šiagiová, Širáň, and Tomanová have shown that **all the smallest known vertex-transitive graphs of a given degree and girth** 6 are Cayley graphs.

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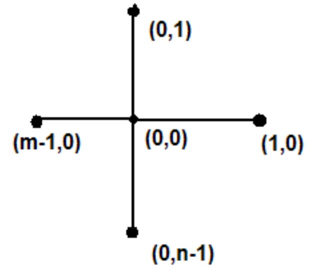
Note: Vertex-transitive graphs are the same locally around each vertex, and each vertex lies on the same system of cycles, and so if there are no short cycles in the neighborhood of one vertex, then there are no short cycles at all.

Cayley Graphs

Γ, a finite group
$X = \{x_1, x_2, \dots, x_k\}$, a generating set,
$X = X^{-1}$, $1_\Gamma \notin X$.
$Cay(\Gamma, X)$ has Γ for its set of vertices and each g is adjacent to all the vertices $gx_1, gx_2, \dots, gx_k$.

Cayley Graph - Example

$G = \mathbb{Z}_m \times \mathbb{Z}_n$, $X = \{(1,0),(0,1),(m-1,0),(0,n-1)\}$



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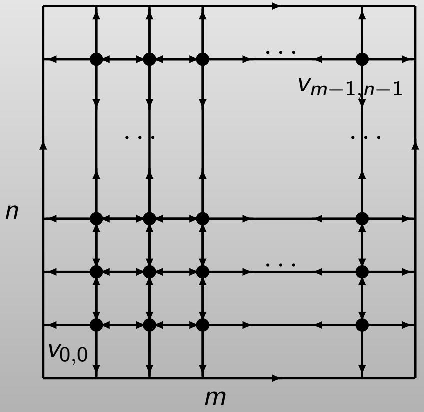

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Theorem (Biggs)

Given any $k, g \geq 3$, there is k-regular graph G whose girth is **at least** g.

Proof.

- $k, g \geq 3$, $r = \lfloor \frac{g}{2} \rfloor$,
- $T_{k,r}$, the finite tree of radius r and degree k,
- color the edges of $T_{r,k}$ by k colors; no two adjacent edges of the same color,
- for each color i, let α_i denote the involutory permutation of the vertices of $T_{k,r}$:

$$\alpha_i(u) = v \text{ if and only if the edge } \{u, v\} \text{ is colored by } i.$$

- $\Gamma = \langle X \rangle = \langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$
- the k-regular graph $C(\Gamma, X)$ has girth **at least** g. □

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Theorem (Širáň, RJ (2009))

For every pair (k, g), there exists a Cayley graph of degree k and girth g.

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We have three proofs by now:

- factorization of infinite maps (taking advantage of residual finiteness)
- adding edges to Biggs' construction
- constructing Cayley graphs from permutations obtained from ordinary (k, g)-graphs

Girth of a Cayley Graph

The oriented edges of a Cayley graph are 'colored' by the elements in X

$$\Longrightarrow$$

each closed walk in a Cayley graph is associated with a reduced word $w(x_1, x_2, \ldots, x_k) \in X^*$ satisfying the condition

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Example: Girth of a Cayley Graph of an Abelian Group
Since X is closed under taking inverses, for any pair of distinct generators $x_i, x_j \in X$, there exists a path in the Cayley graph labeled by the non-trivial reduced word

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any Caley graph of an abelian group Γ is of girth at most 4.

General Bounds for the Girth of the Cayley Graphs of Nilpotent Groups

The class of **nilpotent groups** is the class of groups that is widely considered to be 'the closest' to the abelian groups.

Theorem (Conder, Exoo, RJ (2009))

If N is a finite nilpotent group of nilpotency ν, generated by a set of X of size at least 3, then

$$\textit{girth of } Cay(N, X) \leq (\nu + 1)^2$$

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Theorem (Exoo,Conder, RJ (2009))

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- It was improved by a **voltage graph construction** by Exoo, $rec(3, 14) = 384$.

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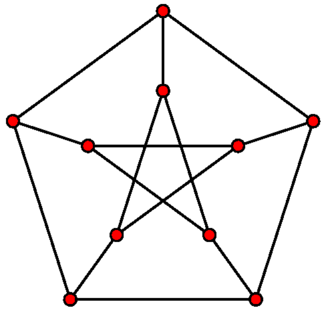
The **derived regular cover (lift)** of G with respect to the voltage assignment α is the graph denoted by G^α.

- $V(G^\alpha) = V(G) \times \Gamma$,
- u_g and v_f are adjacent iff $e = (u, v) \in D(G)$ and $f = g \cdot \alpha(e)$.

Voltage Graph Construction - Example



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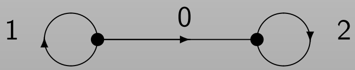

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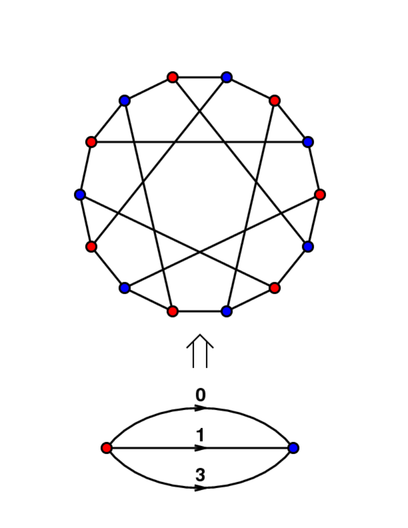

Girth of a Voltage Graph

Once again, the arcs of the base graph are colored by the elements in Γ

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each **“non-path-reversing”** closed walk in the base graph is associated with a **reduced** word $w(\alpha(D(G)))$.

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If $w(\alpha(D(G)))$ happens to be equal to 1_Γ, the lift G^α contains a cycle of length at most $|w(\alpha(D(G)))|$.

Note, however, that (unlike the case of Cayley graphs) not every reduced word $w(\alpha(D(G)))$ satisfying $w(\alpha(D(G))) = 1_\Gamma$ gives rise to a cycle in the lift.

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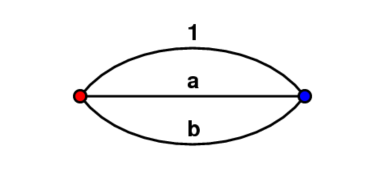

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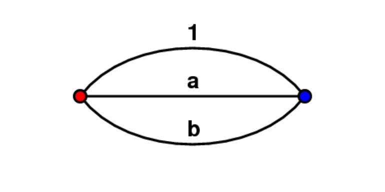

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- Nevertheless, entering identities allows us to travel in the opposite direction, and so entering just two identities into $aba^{-1}b^{-1}$ makes reading the expanded word possible:

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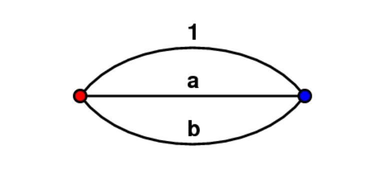

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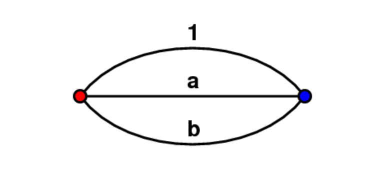

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- **Voltage graph constructions generally outperform Cayley graphs in record constructions**

Girth of Voltage Graphs with Nilpotent Voltages

Theorem (Exoo, RJ)

Let Γ *be a* **nilpotent group** *of nilpotency* ν. *The girth* g *of any lift of a base graph containing* $\theta(\ell_1, \ell_2, \ell_3)$ *using voltages from* Γ *is bounded above by*

$$[\ell_1 + \ell_3](\nu + 1)^2 \tag{1}$$

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$$[\max(j_1, j_3) + j_2)](\nu + 1)^2 \tag{2}$$

The corresponding bound for Cayley graphs was simply $(\nu + 1)^2$, while $\ell_1 + \ell_3 \geq 2$ and $\max(j_1, j_3) + j_2 \geq 2$.

Girth of Voltage Graphs with Solvable Voltages

Theorem (Exoo, RJ)

Let Γ *be a* **solvable group** *of derived length* δ*. The girth* g *of any lift of a base graph containing* $\theta(\ell_1, \ell_2, \ell_3)$ *using voltages from* Γ *is bounded above by*

$$[\ell_1 + \ell_3]4^\delta \tag{3}$$

The girth g *of any lift of a base graph containing* $DB(j_1, j_2, j_3)$ *using voltages from* Γ *is bounded above by*

$$[\max(j_1, j_3) + j_2)]4^\delta \tag{4}$$

The corresponding bound for Cayley graphs was simply 4^n, while $\ell_1 + \ell_3 \geq 2$ and $\max(j_1, j_3) + j_2 \geq 2$.

Record Cubic Cages using Voltage Graphs

Girth g	Lower Bound	Upper Bound	Due to
25	12288	108906	Exoo
26	16384	109200	Bray-Parker-Rowley
27	24576	285852	Bray-Parker-Rowley
28	32768	415104	Bray-Parker-Rowley
29	49152	1141484	**Exoo-Jajcay**
30	65536	1143408	**Exoo-Jajcay**
31	98304	3649794	Bray-Parker-Rowley
32	131072	3650304	Bray-Parker-Rowley

Record Cubic Cages Obtained Using Bi-Coset Graphs

Girth g	Lower Bound	Upper Bound	Due to
25	12288	1089	Exoo
26	16384	109200	Bray-Parker-Rowley
27	24576	285852	Bray-Parker-Rowley
28	32768	368640	**Erskine-Tuite**
29	49152	805746	**Erskine-Tuite**
30	65536	806736	**Erskine-Tuite**
31	98304	1440338	**Erskine-Tuite**
32	131072	1441440	**Erskine-Tuite**

Coset graphs

Definition (J.Tits, 1956)

Let G be a finite group and $H_1, H_2, \ldots, H_k$ be subgroups of G with a core-free intersection, $\bigcap_{g \in G} g^{-1} H_i g = \langle 1_G \rangle$.
The **coset graph** $\Gamma_{(G;H_1,H_2,\ldots,H_k)}$ is the graph whose vertices are the cosets of the subgroups H_i, $1 \leq i \leq k$, with the adjacency determined by non-empty intersection.

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Observation. If $k \geq 3$, the girth of $\Gamma_{(G;H_1,H_2,\ldots,H_k)}$ is 3:

The subgraph induced by the cosets H_1, H_2, H_3 is a triangle.

Bi-coset graphs

Definition

Let G be a finite group and H, K be subgroups of G with a core-free intersection, $\bigcap_{g \in G} g^{-1}(H \cap K)g = \langle 1_G \rangle$.
The **bi-coset graph** $\Gamma_{(G;H,K)}$ is the bipartite graph whose vertices $V(\Gamma_{(G;H,K)}) = V_1 \cup V_2$ are the cosets of H and K in G,

$$V_1 = \{1_G H, g_2 H, g_3 H, \ldots g_{|G|/|H|} H\},$$
$$V_2 = \{1_G K, f_2 K, f_3 K, \ldots f_{|G|/|K|} K\},$$

and the adjacency relation is defined via non-empty intersection: $g_i H$ is adjacent to $f_j K$ if and only if $g_i H \cap f_j K \neq \emptyset$.

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- Equivalently, $g_i H$ and $f_j K$ are adjacent if and only if $g_i^{-1} f_j \in HK$ and $f_j^{-1} g_i \in KH$
- The vertices in V_1 are of degree $|H|/|H\cap K|$, the vertices in V_2 are of degree $|K|/|H\cap K|$. Thus, if $|H| = |K|$, then $\Gamma_{(G;H,K)}$ is regular of degree $|H|/|H\cap K|$

Fundamental properties of bi-coset graphs

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- The full automorphism group of $\Gamma_{(G;H,K)}$ acts transitively on the edges of $\Gamma_{(G;H,K)}$, and thus $\Gamma_{(G;H,K)}$ is either **semi-symmetric** (when not vertex-transitive) or **symmetric** (i.e., arc- and vertex-transitive)

The girth of a bi-coset graph

Lemma (Gyürki, RJ, Jánoš, Širáň, Wang, 2022)

Let G be a finite group and $K, H \leq G$ be two non-trivial subgroups of G with a core-free intersection. Then the bi-coset graph $\Gamma_{(G;H,K)}$ has a cycle of length $2r$, $r \geq 2$, if and only if there exist sequences of non-identity elements $h_1, h_2, \ldots, h_r \in H$ and $k_1, k_2, \ldots, k_r \in K$ such that

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Corollary

The girth of the bi-coset graph $\Gamma_{(G;H,K)}$ is equal to $2r$, with $r \geq 1$ being the smallest positive integer for which there exist sequences of non-identity elements h_1, h_2, ..., $h_r \in H$ and $k_1, k_2, \ldots, k_r \in K$ such that

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Easy examples of bi-coset graphs of girth 4

Example

$G = \mathbb{Z}_k \times \mathbb{Z}_k$, $k > 2$, $H = \langle(1,0)\rangle$, and $K = \langle(0,1)\rangle$, gives rise to $\Gamma_{(G;H,K)}$ isomorphic to the k-regular complete bipartite graph $K_{k,k}$ of girth 4

$$(1,0) + (0,1) + (k-1,0) + (0,k-1) = (0,0)$$

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Lemma (Gyürki, RJ, Jánoš, Širáň, Wang, 2022)

The girth of any $\Gamma_{(G;H,K)}$ in which the elements from H and K commute is less than or equal to 4, and if $G = HK$, $|H| = |K| = n$, the graph $\Gamma_{(G;H,K)}$ is $K_{n,n}$.

Example of girth 6

Example

- Let G be the group $\mathbb{Z}_3 \times \mathbb{Z}_3$ with permutation representation $\langle(1,2,3),(4,5,6)\rangle$. If $H = \langle(1,2,3)\rangle$ and $K = \langle(4,5,6)\rangle$, then $\Gamma_{(G;H,K)} \cong K_{3,3}$ of order 6 and girth 4.

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- Adding $\varphi = (1,2,3)$, constructing $\hat{G} = G \wr \varphi$, and using subgroups $\hat{H} = \langle(7,8,9)(13,15,14)\rangle$ and $\hat{K} = \langle(1,7,13)(2,8,14)(3,9,15)(4,11,18)(5,12,16)(6,10,17)\rangle$ yields a connected component of a bi-coset graph of order 18, valency 3, and girth 6.

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  The full group of automorphisms of this graph is of order 216 and it is the well-known **Pappus graph**.

Regular bi-coset graphs of fixed degree and unbounded girths

Theorem (Gyürki, RJ, Jánoš, Širáň, Wang, 2022)

For every $k \geq 3$, there exists a bi-coset (k, g)-graph with arbitrarily large girth g.

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Comparing the Three Constructions with Regard to Constructions of Cages

- 'Evidence' suggests that voltage lifts are better than Cayley graphs, and the bi-coset graphs are better than voltage lifts
- Bi-coset graphs are bipartite, and so one cannot use bi-coset graphs to construct odd girth graphs (at least not directly)
- In all three constructions, it appears to be the case, that 'more complicated' groups produce better results; so it is hard to deduce conclusions based on the computational evidence
- Gyürki, Jánoš and Širáň have results on when a bi-coset graph is a lift; so sometimes we are talking about the same graphs

Canonical Double Cover

Definition

Let Γ be a finite graph. We say that Γ^{α} is a **canonical double cover** of Γ if the voltage group is $\mathbb{Z}_2$ and each dart of Γ receives the voltage assignment $1 \in \mathbb{Z}_2$.

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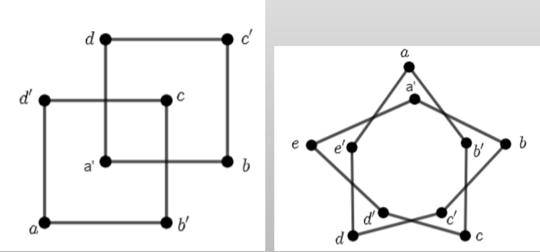
▶ 

Figure: Canonical Double Covers of C_4 and C_5

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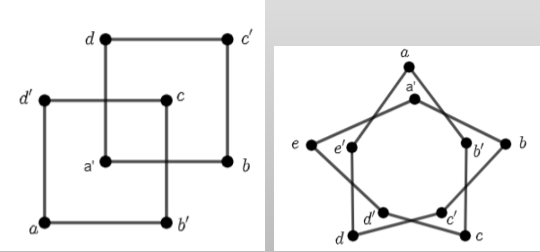
- 

Figure: Canonical Double Covers of C_4 and C_5

- The canonical double cover of the Petersen graph of girth 5 is the Desargues graph, which has 20 vertices and girth 6.

Canonical Double Cover

Lemma

Let Γ^α be the canonical double cover of a graph Γ, then

1. $|V(\Gamma^\alpha)| = 2 \times |V(\Gamma)|$, $|E(\Gamma^\alpha)| = 2 \times |E(\Gamma)|$;
2. Γ^α *is a bipartite graph;*
3. Γ^α *is connected if and only if* Γ *is connected and non-bipartite;*
4. *If C is a cycle in Γ of odd length $2r+1$, the preimage of C in Γ^α (the lift of C in Γ^α) is a cycle of the double length $4r+2$;*
5. *If C is a cycle in Γ of even length $2r$, the lift of C in Γ^α is a pair of cycles of length $2r$;*
6. *If Γ is k-regular, Γ^α is also k-regular.*

Canonical Double Cover

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Let Γ^α be the canonical double cover of a graph Γ, then

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If g is odd,

$$n(k, g+1) \leq 2n(k, g)$$

Canonical Double Covers and Cayley Graphs

Theorem

Let $\Gamma = \mathcal{C}(G,S)$, $S = \{s_1, s_2, \ldots, s_k\}$. *The Cayley graph* $\Gamma = \mathcal{C}(G \times \mathbb{Z}_2, \{(s_1,1),(s_2,1),\ldots,(s_k,1)\})$ *is the canonical double cover of* Γ.

Lifts of Cayley Graphs and Canonical Double Covers

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Lifts of Cayley Graphs and Canonical Double Covers

- Let $\mathcal{C}(G, S)$ be a Cayley graph.
- Consider the base graph $\Gamma = (V, E)$ which is a dipole consisting of two vertices and $|S|$ multiple parallel edges. Take G to be the voltage group, and let α assign to each edge of Γ a unique element of S.

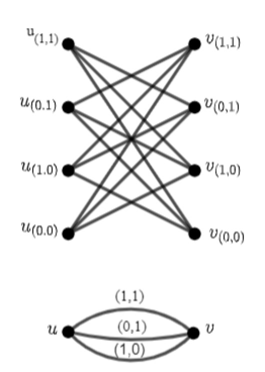

Figure: Lift Graph of the Cayley Graph $G = \mathbb{Z}_2^2$

Lifts of Cayley Graphs and Canonical Double Covers

Theorem

Let $\mathcal{C}(G, S)$ be a Cayley graph. The lift of the dipole graph with $|S|$ parallel edges is isomorphic to the canonical double cover of $\mathcal{C}(G, S)$.

Odd Girth Graphs from Even Girth Graphs

Theorem (L. Eze, RJ)

There is no $\alpha \in \mathbb{R}$ *such that for any* $k \geq 3$ *and even* $g \geq 4$*,* $n(k, g+1) \leq \alpha n(k, g)$*.*

Proof: Based on an analysis of the Moore bound.

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Lemma (L. Eze, RJ)

- $\tilde{\Gamma}^\alpha$ *is a* $(k+1)$*-regular bipartite graph.*
- *If* Γ *is a bipartite* k*-regular graph of girth* 6*, then* $\tilde{\Gamma}^\alpha$ *is a* $(k+1)$*-regular graph of girth* 6 *and order the* 3*-multiple of the order of* Γ*.*
- $n(k+1,6) \leq 3n(k,6)$

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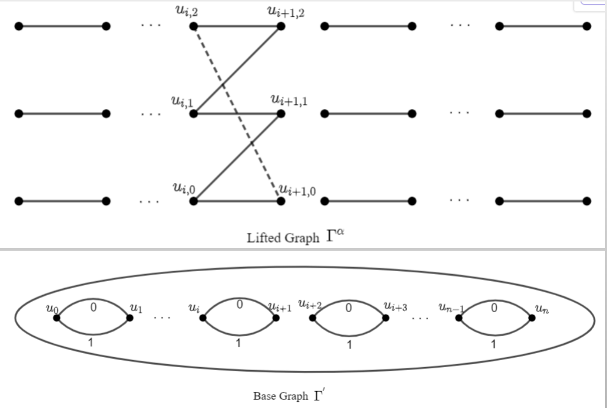

Figure: Base graph $\tilde{\Gamma}$ and lift graph $\tilde{\Gamma}^{\alpha}$


Thank you