Spectral method in extremal combinatorics

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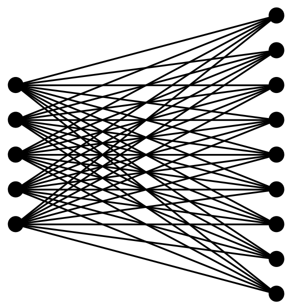
European Research Council
Established by the European Commission

June 26, 2022

Spectral method

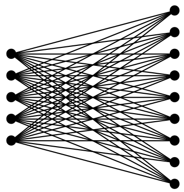
- spectral properties of the adjacency matrix
  rows and columns indexed by vertices
  $0 =$ non-edge and $1 =$ edge
- two applications
  common graphs (Ramsey multiplicity)
  extremal problems in tournaments



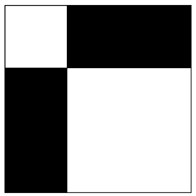
$$\begin{matrix}
0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&1&1&1&1&1&1&1&1\\
1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&1&0&0&0&0&0&0&0&0
\end{matrix}$$

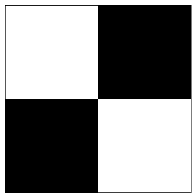
Combinatorial limits

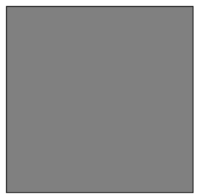
- adjacency matrix $\to$ function $W : [0,1]^2 \to [0,1]$
  the values of W may be non-integral
- graphon: $W(x,y) = W(y,x)$
  tournamenton: $W(x,y) = 1 - W(y,x)$
- possible to define a density $t(H,W)$ of H in W
  summation $\to$ integration
  $t(K_3,W) = \int W(x,y)W(y,z)W(x,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

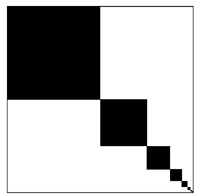


$$\begin{matrix} 0&0&0&0&1&1&1&1&1&1&1&1\\ 0&0&0&0&1&1&1&1&1&1&1&1\\ 0&0&0&0&1&1&1&1&1&1&1&1\\ 0&0&0&0&1&1&1&1&1&1&1&1\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0 \end{matrix}$$









Ramsey multiplicity

- Ramsey's Theorem (1930)
  For every graph H, there exists N such that every 2-edge-coloring of K_N has a monochromatic copy of H.
- How many monochromatic copies must be there?
  What is the minimum value of $t(H, W) + t(H, 1 - W)$?
- Goodman's Bound
  Copies of K_3 minimized by the random 2-edge-coloring.
  $t(K_3, W) + t(K_3, 1 - W) \geq 1/4$

Common graphs

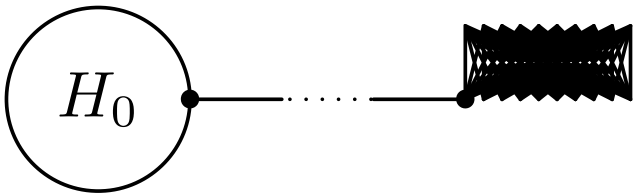
- Conjecture (Erdős, 1962)
  The random 2-edge-coloring is minimizing for $H = K_n$.
- H is common if the random 2-edge-coloring minimizes the number of monochromatic copies.
  $t(H,W) + t(H,1-W) \geq 2^{-(\|H\|-1)}$ for every W
- Conjecture (Burr and Rosta, 1980)
  Every graph is common.
- triangle+edge is not common (Sidorenko, 1989)
  no complete graph is common (Thomason, 1989)
  no graph containing K_4 is common (Thomason, 1989)

(Non-)bipartite graphs

- Sidorenko's Conjecture $\Rightarrow$ bipartite graphs are common
- Theorem (Sidorenko, 1989)
  Every odd cycle is common.
- Theorem (Jagger, Šťovíček, Thomason, 1996)
  Every even wheel is common.
- Theorem (Hatami, Hladký, K., Norin, Razborov, 2012)
  The 5-wheel is common.
- What about chromatic number at least five?

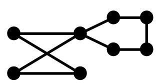
Our results

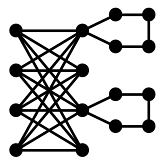
- Common graph with chromatic number at least five?
- Theorem (K., Volec and Wei, 2022+)
  A connected ℓ-chromatic common graph for every ℓ.
- construction: $H := H_0 \oplus P_m \oplus K_{2n,2n}$, girth of $H_0 \geq 30$
- $f, g : [0,1] \to [0,1] \approx$ rooted copies of H_0 and $K_{2n,2n}$
  $t(H,W) = \langle f, W^m g \rangle$, spectral decomposition of f and g
  various properties of f, g, e.g., $\langle j, g \rangle \geq (\lambda_1^4 + \lambda_2^4 + \cdots)^n$

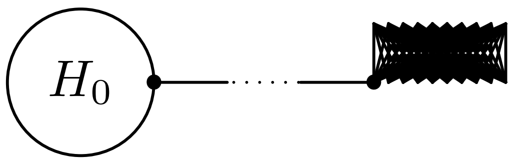


Generalization to more colors

- H is k-common if the random k-edge-coloring minimizes the number of monochromatic copies.
- Sidorenko: bipartite $H \Rightarrow t(H,W) \geq t(K_2,W)^{-\|H\|}$
  $t(H,W) \geq t(K_2,W)^{-\|H\|} \Leftrightarrow k$-common for all k
- Theorem (K., Noel, Norine, Volec and Wei, 2022)
  A connected non-bipartite k-common graph.
- Theorem (K., Volec and Wei, 2022+)
  A connected ℓ-chromatic k-common graph for all k, ℓ.



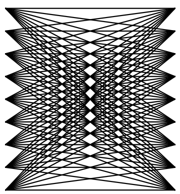


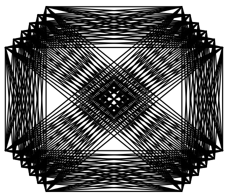


Time for questions :-)

Turán problems

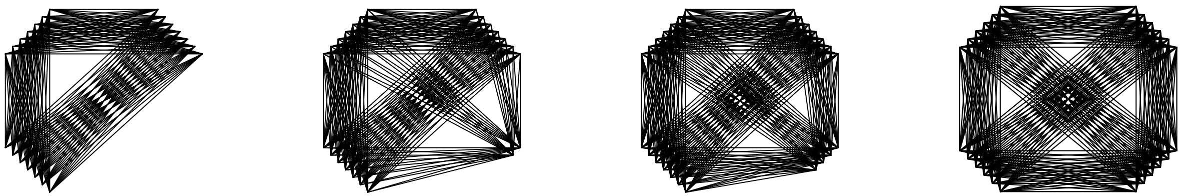
- Maximum edge-density of H-free graph
- Mantel's Theorem (1907): $\frac{1}{2}$ for $H = K_3$ $(K_{\frac{n}{2},\frac{n}{2}})$
- Turán's Theorem (1941): $\frac{\ell-2}{\ell-1}$ for $H = K_\ell$ $(K_{\frac{n}{\ell-1},\dots,\frac{n}{\ell-1}})$
- Erdős-Stone Theorem (1946): $\frac{\chi(H)-2}{\chi(H)-1}$
- extremal examples unique up to $o(n^2)$ edges





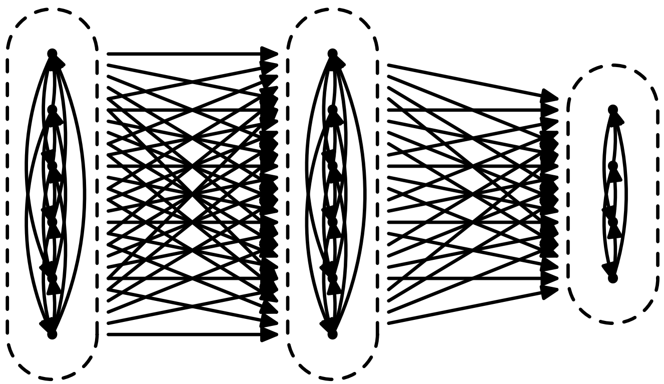
Erdős-Rademacher problem

- Turán's Theorem:
  edge-density $\leq 1/2 \Leftrightarrow$ minimum triangle density $= 0$
- What happens if edge-density $> 1/2$?
- minimum attained by $K_{n,\ldots,n}$ for edge-density $\frac{k-1}{k}$
- smooth transformation from $K_{n,n}$ for $K_{n,n,n}$, etc.
- resolved by Razborov in 2008



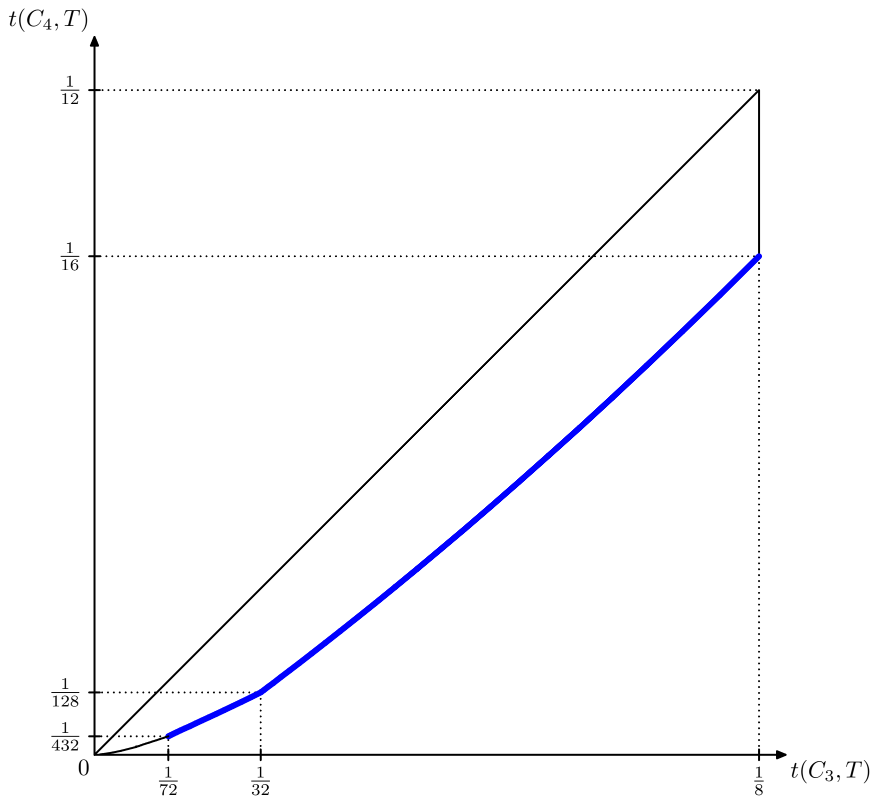
Erdős-Rademacher in tournaments

- minimum density of C_4 for a fixed density of C_3
- Conjecture of Linial and Morgenstern (2014)
  blow-up of a transitive tournament (random inside)
  with all but one equal parts and a smaller part



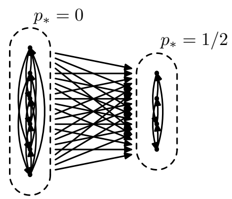
Our result

Theorem (Chan, Grzesik, K. and Noel, 2020)



Sketch of the proof

- spectrum of W such that $W = \mathbb{J} - W^T$
  $\mathrm{Tr}\, W = \lambda_1 + \lambda_2 + \cdots = 1/2$, $\mathrm{Re}\, \lambda_i \geq 0$
  Perron–Frobenius $\Rightarrow \exists \rho \in \mathbb{R} : \rho = \lambda_1$ and $|\lambda_i| \leq \lambda_1$
- fix $t(C_3, W) = \lambda_1^3 + \lambda_2^3 + \cdots \in [1/36, 1/8]$
  minimize $t(C_4, W) = \lambda_1^4 + \lambda_2^4 + \cdots$
  optimum $\lambda_{\leq k-1} = \rho$ and $\lambda_k = 1/2 - (k-1)\rho$, $k \in \{2, 3\}$
- structure for $k = 2$: assign $p_v \in [0, 1/2]$
  orient from v to w with probability $1/2 + (p_v - p_w)$



Maximum density of cycles

- What is maximum density of cycles of length k?
  $c(k)$ = maximum density / random tournament
- Kendall, Babington Smith; Szele (1940's): $c(3) = 1$
  Beineke, Harary (1965), Colombo (1964): $c(4) = 4/3$
  Komarov, Mackey (2017): $c(5) = 1$
- Conjecture (Bartley 2018, Day 2017):
  $c(k) = 1$ if and only if k is not divisible by four
  $c(k) = 1 + 2\sum_{i=1}^{\infty}\left(\frac{2}{(2i-1)\pi}\right)^k$ if $4|k$

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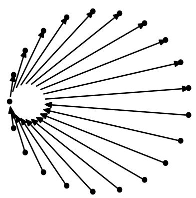

Maximum density of cycles

- Theorem (Grzesik, K., Lovász Jr., Volec, 2020+)
  $c(k) = 1 \Leftrightarrow k$ not divisible by four
  $1 + 2 \cdot (2/\pi)^k \leq c(k) \leq 1 + (2/\pi + o(1))^k$ if $4|k$
  $c(8) = 332/315$

- extremal constructions:
  $k \equiv 1 \mod 4 \Leftrightarrow \lambda_1 = 1/2$, $\mathrm{Re}\,\lambda_{\geq 2} = 0$
  $k \equiv 2 \mod 4 \Leftrightarrow \lambda_1 = 1/2$, $\lambda_{\geq 2} = 0$
  $k \equiv 3 \mod 4 \Leftrightarrow \lambda_1 = 1/2$, $\mathrm{Re}\,\lambda_{\geq 2} = 0$
  $k = 4$ or $k = 8 \Leftrightarrow W$ derived from Volterra operator
  $k \equiv 0 \mod 4 \Rightarrow \lambda_1 = 1/2$, $\mathrm{Re}\,\lambda_{\geq 2} = 0$, $\lambda_{2,3} = \pm 2i/\pi$

Maximum density of cycles

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  $c(k) = 1 \Leftrightarrow k$ not divisible by four
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  $c(8) = 332/315$
- extremal constructions:
  $k \equiv 1 \mod 4 \Leftrightarrow$ regular tournament
  $k \equiv 2 \mod 4 \Leftrightarrow$ quasirandom tournament
  $k \equiv 3 \mod 4 \Leftrightarrow$ regular tournament
  $k = 4$ or $k = 8 \Leftrightarrow$ carousel tournament

Thank you for your attention!