Pseudo-geometric strongly regular graphs with a regular point

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joint work with Krystal Guo

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Outline

- Definitions of relevant objects
- Regular points in generalized quadrangles
- A four-class association scheme
- Generalization to pseudo-geometric strongly regular graphs
- Characterization of regular points
- Exploiting the characterization to find new SRG from old SRG
- A combinatorial characterization of regular points
- Distance-regular graphs and spreads
- A problem

Strongly regular graphs

Definition

A graph Γ is strongly regular with parameters (v, k, λ, μ) if it is k-regular on v vertices, such that any two adjacent vertices have λ common neighbors, and any two non-adjacent vertices have μ common neighbors.

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Nice examples come from finite groups, finite geometry, codes, designs, Latin squares.

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Nice examples come from finite groups, finite geometry, codes, designs, Latin squares.

Wild constructions are available, e.g., Wallis, Fon-Der-Flaass.

Switching (generalized Godsil-McKay, Seidel) gives new SRG from old SRG.

Generalized quadrangles

Definition

A *generalized quadrangle of order* (s,t), denoted $\mathrm{GQ}(s,t)$, is a point-line incidence structure such that

(i) each point is incident with $t+1$ lines and any two points are incident with at most one line;

(ii) each line is incident with $s+1$ points and any two lines are incident with at most one point; and

(iii) for every point x and line L not incident with x, there is a unique line M incident to both x and a point y that is incident to L.

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Unique $\mathrm{GQ}(2, 2)$: points are 15 pairs of a 6-set X; lines are 15 partitions of X into 3 pairs.

The second subconstituent

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The collinearity graph of GQ(2, 2) is SRG with parameters $(15, 6, 1, 3)$.

The second subconstituent $\Gamma_2(x)$ (graph on the nonneighbors of a fixed vertex x) is the cube.

Cameron, Goethals, Seidel (1978), De Caen (1998): The spectra (set of eigenvalues) of $\Gamma_1(x)$ and $\Gamma_2(x)$ in a strongly regular graph are related.

Spectra of subconstituents

$$A^2 = (s-t-2)A + (t+1)(s-1)I + (t+1)J$$

$$A = \begin{bmatrix} 0 & j^\top & 0^\top \\ j & C & N \\ 0 & N^\top & B \end{bmatrix}$$

$$\begin{aligned} C^2 + NN^\top &= (s-t-2)C + (t+1)(s-1)I + tJ, \\ CN + NB &= (s-t-2)N + (t+1)J, \\ B^2 + N^\top N &= (s-t-2)B + (t+1)(s-1)I + (t+1)J \end{aligned}$$

Lemma

C is a disjoint union of $t+1$ s-cliques if and only if B has spectrum $\{(t+1)(s-1)^{[1]}, s-1^{[m_2]}, s-t-1^{[(t+1)(s-1)]}, [-t-1]^{[m_4]}\}$, with $m_2 = blabla$ and $m_4 = blablabla$.

Gardiner, Godsil, Hensel, Royle

The cube is a distance-regular antipodal 2-cover of a complete graph on 4 vertices. It is the unique graph with spectrum $\{3^1, 1^3, -1^3, -3^1\}$.

Gardiner, Godsil, Hensel, Royle (1992) determined all SRG for which EVERY second subconstituent is an antipodal distance-regular cover of a complete graph.

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GGHR problem (generalized)

Find SRG where at least one vertex has a subconstituent that is an imprimitive 3 or 4-class association scheme (with certain parameters).

Association schemes

Definition

A d-class association scheme is a partition of the edges of a complete graph into spanning subgraphs with adjacency matrices $B_1, B_2, \ldots, B_d$, such that $B_i B_j = \sum_{h=0}^{d} p_{ij}^h B_h$ for certain intersection numbers p_{ij}^h and $B_0 = I$.

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An SRG and its complement form a 2-class association scheme.

The distance-i graphs of a distance-regular graph with diameter d form a d-class association scheme.

Regular points in GQ

Goal

We will find new SRG and understand some old ones better!

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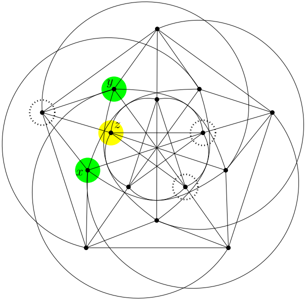
Definition

A point x in a GQ is regular if in the collinearity graph, for every vertex y not adjacent to x, its $t+1$ common neighbors have $t+1$ common neighbors (note: among which are x and y).

Notation: $\{x, y\}^{\perp\perp}$ has size $t+1$.

GQ(2,2) and regular points

In GQ$(2,2)$ every point is regular. $x = \{1,2\}, y = \{1,3\}$; common neighbors $\{4,5\}, \{4,6\}, \{5,6\}$; common neighbors $x, y, z = \{2,3\}$.



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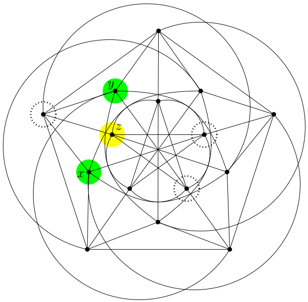

In the second subconstituent $\Gamma_2(x)$, vertices y and z have no common neighbors; they are antipodal in the cube.

A four-class association scheme

Theorem (Ghinelli and Löwe 1995)

If x is a regular point in a GQ, then the second subconstituent $\Gamma_2(x)$ generates an imprimitive 4-class association scheme with eigenmatrix

$$P = \begin{bmatrix} 1 & (s-1)(t+1) & (s-1)(t^2-1) & t-1 & (s-1)t(s-t) \\ 1 & s-1 & 1-s & -1 & 0 \\ 1 & s-t-1 & (t-1)(s-t-1) & t-1 & -t(s-t) \\ 1 & -t-1 & t+1 & -1 & 0 \\ 1 & -t-1 & 1-t^2 & t-1 & t^2 \end{bmatrix}.$$

Distance-3 (B_3) gives the imprimitivity classes (fibres). There are two kinds of distance-2 (B_2 and B_4).

Note: if $s = t$, then this is a 3-class scheme ($B_4 = 0$), and $\Gamma_2(x)$ is an antipodal distance-regular cover of a complete graph.

Regular points in pseudo-geometric SRG

Definition (not a very intuitive one, though)

A vertex x is a regular point in a pseudo-geometric SRG with parameters $((s+1)(st+1), s(t+1), s-1, t+1)$ if the second subconstituent $\Gamma_2(x)$ generates an imprimitive 4-class association scheme with eigenmatrix

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Note: all intersection numbers p_{ij}^h follow from P.

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Note: all intersection numbers p_{ij}^h follow from P.

In particular, for $B = B_1$, we have

$$B^2 = (s-1)(t+1)I + (s-2)B + tB_2 + (t+1)B_4.$$

The first subconstituent?

Fix a regular point x in a pseudo-geometric graph

Lemma

The first subconstituent $\Gamma_1(x)$ *is a disjoint union of* $t+1$ *cliques of size* s*.*

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Recall

$$A = \begin{bmatrix} 0 & j^\top & 0^\top \\ j & C & N \\ 0 & N^\top & B \end{bmatrix}.$$

Now

$$C^2 = (s-1)I + (s-2)C.$$

What about incidence matrix N ?

Between the first and second subconstituent?

Recall

$$C^2 + NN^\top = (s-t-2)C + (t+1)(s-1)I + tJ,$$
$$B^2 + N^\top N = (s-t-2)B + (t+1)(s-1)I + (t+1)J$$

Now

$$NN^\top = tsI + t(J - I - C),$$
$$N^\top N = (t+1)(B_3 + I) + B + B_2$$

Lemma

Vertices in the same fibre of $\Gamma_2(x)$ are adjacent to the same $t+1$ vertices in $\Gamma_1(x)$.

A net

Take the quotient in $\Gamma_2(x)$ with respect to the fibres.

Lemma

The incidence relation $\mathcal{I}$ between the first subconstituent and the fibres of the second subconstituent is a $(t+1)$-net of order s, and its collinearity graph is the quotient of B.

A $(t+1)$-net of order s is the same as $t-1$ MOLS of side s.

The collinearity graph is a strongly regular Latin square graph with parameters $(s^2, (t+1)(s-1), t^2-t+s-2, t(t+1))$.

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Note: if $s=t$, then the net is an affine plane, and the quotient is a complete graph.

In reverse

Given an association scheme $\mathcal{H}$ with eigenmatrix P, a $(t+1)$-net $\mathcal{I} = (\mathcal{P}, \mathcal{L})$ of order s, and a bijection ϕ from the fibres of $\mathcal{H}$ to $\mathcal{P}$, we can define a graph $\Gamma = \Gamma(\mathcal{H}, \mathcal{I}, \phi)$.

$$V(\Gamma) = \{x\} \cup \mathcal{L} \cup V(\mathcal{H});$$

- x is adjacent to all blocks in $\mathcal{L}$;
- two blocks in $\mathcal{L}$ are adjacent if they are parallel in $\mathcal{I}$;
- a block $\ell \in \mathcal{L}$ is adjacent to a vertex in $V(\mathcal{H})$ if the latter is in a fibre W such that $\phi(W)$ is incident to ℓ in $\mathcal{I}$; and
- two vertices of $V(\mathcal{H})$ are adjacent if they are adjacent in the first relation B of $\mathcal{H}$.

The characterization

Theorem

Γ is a strongly regular graph with a regular point if and only if $\Gamma = \Gamma(\mathcal{H}, \mathcal{I}, \phi)$, where

(i) *$\mathcal{H}$ is an association scheme with eigenmatrix P;*

(ii) *$\mathcal{I}$ is a $(t+1)$-net of order s;*

(iii) *ϕ is an isomorphism from the quotient graph of B to the collinearity graph of $\mathcal{I}$.*

New graphs from old ones

Starting from a strongly regular graph with a regular point, we can apply any automorphism of the collinearity graph of $\mathcal{I}$ (to get a different ϕ) to construct new strongly regular graphs with a regular point.

Theorem

The Hermitian generalized quadrangle $H(3, q^2)$ has a cospectral strongly regular graph that is not geometric.

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This is a $\mathrm{GQ}(q^2, q)$ with regular points for prime powers q. The quotient graph is the bilinear forms graph $H_q(2, 2)$, which has two kinds of maximal cliques. Applying an automorphism that interchanges the two kinds of cliques gives rise to the new graph.

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$q = 2$: There are 78 strongly regular graphs with parameters $(45, 12, 3, 3)$, of which 6 have a regular point. They come in 3 pairs through our construction.

The symplectic generalized quadrangle

Theorem

For $q \geq 3$, the symplectic generalized quadrangle $W(q)$ has at least q cospectral strongly regular graphs that are not geometric.

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Fon-Der-Flaass (2002): prolific construction of such graphs, with regular points.

Our method: $q = 4$: 133718 "new" strongly regular graphs with parameters $(85, 20, 3, 5)$.

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Our method: $q = 4$: 133718 "new" strongly regular graphs with parameters $(85, 20, 3, 5)$.

$q = 3$: 17 of the 28 strongly regular graphs with parameters $(40, 12, 2, 4)$ have a regular point.

A second characterization

Theorem

A vertex x in a pseudo-geometric strongly regular graph with parameters $((ts+1)(s+1), (t+1)s, s-1, t+1)$ is regular if and only if the first subconstituent $\Gamma_1(x)$ is a disjoint union of cliques and the sets $\{x, y\}^{\perp\perp}$ are cocliques of size $t+1$ for all y not adjacent to x.

Distance-regular graphs and spreads

Payne (1971) introduced the concept of regular points in a GQ(q, q) to construct GQ$(q-1, q+1)$ with a spread (on the second subconstituent).

Brouwer (1984) showed that removing a spread from a pseudo-geometric SRG gives a distance-regular antipodal cover of a complete graph.

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Our generalization of regular points in pseudo-geometric SRG with parameters $((q+1)(q^2+1), q(q+1), q-1, q+1))$ gives distance-regular antipodal q-covers of a complete graph on q^2 vertices.

Adding the spread (on the fibres) gives pseudo-geometric SRG with parameters $(q^3, (q-1)(q+2), q-2, q+2)$.

Example: a new SRG with parameters $(125, 28, 3, 7)$.

A problem

Cameron, Goethals, Seidel (1978): every pseudo-geometric SRG for $\mathrm{GQ}(q, q^2)$ is geometric.

Pseudo $\mathrm{GQ}(q^2, q^3)$: Cossidente and Pavese (2016)

Pseudo $\mathrm{GQ}(q-1, q+1)$: Fon-Der-Flaass (2002),...., EvD and Guo

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Pseudo $\mathrm{GQ}(q+1, q-1)$: Wallis (1971)

Pseudo $\mathrm{GQ}(q^2, q)$: EvD and Guo

Problem

Find pseudo-geometric SRG for $\mathrm{GQ}(q^3, q^2)$.

The end

Many thanks to the organizers!

`https://arxiv.org/abs/2204.04755`

The end