Graph parameters, implicit representation and factorial properties

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Joint work with Bogdan Alecu, Vladimir Alekseev, Aistis Atminas, Viktor Zamaraev

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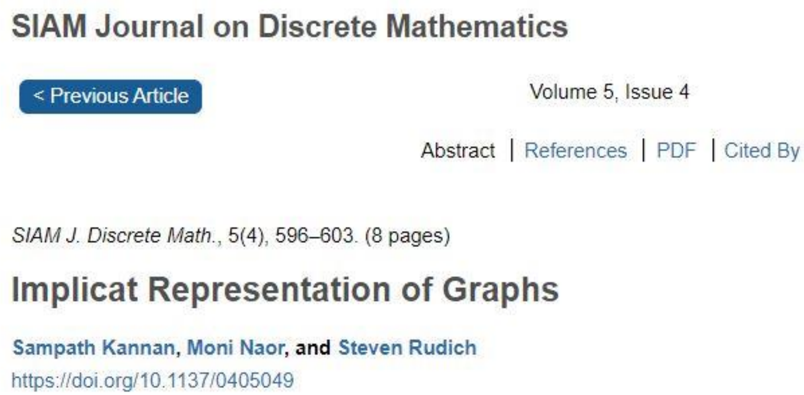

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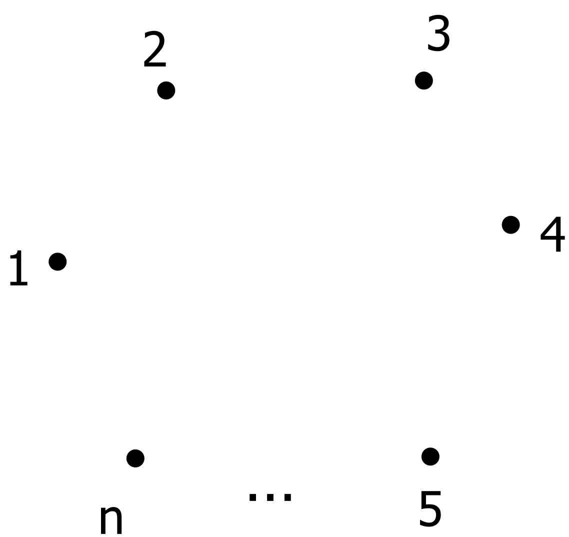

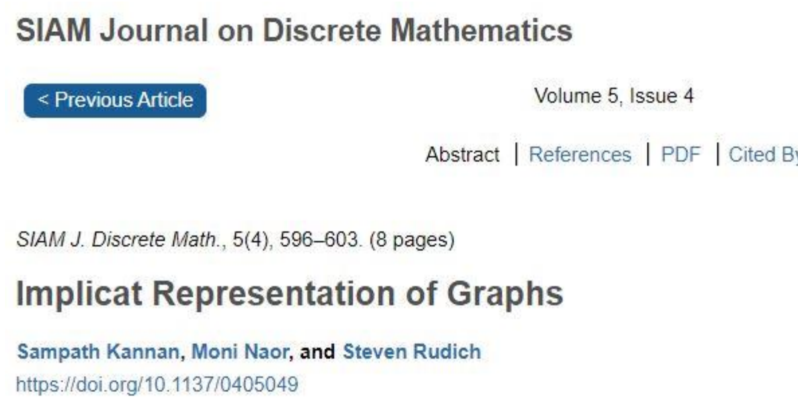
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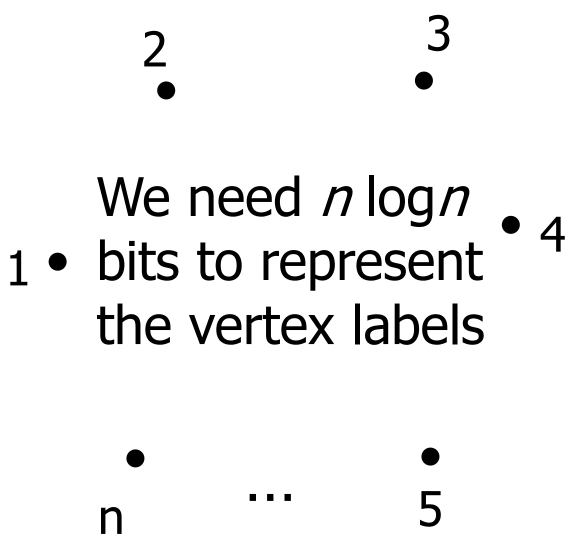
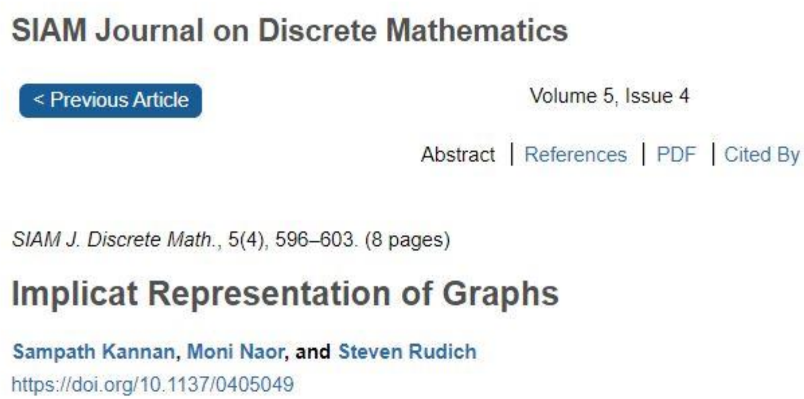


We need *n* log *n* bits to represent the vertex labels



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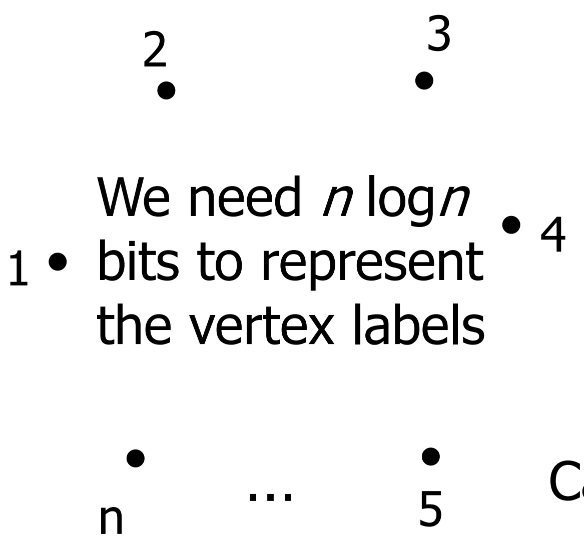
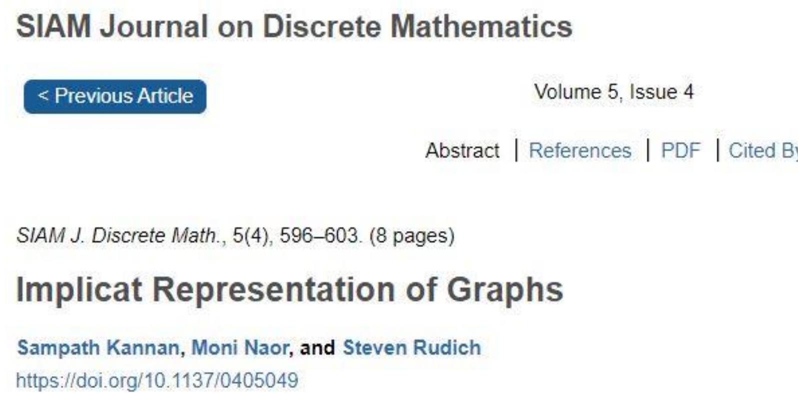


We need $n \log n$ bits to represent the vertex labels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Implicat Representation of Graphs

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Can a labelled graph be represented with $n \log n$ bits?

Prüfer code establishes a bijection between the set of labelled trees with n vertices and the set of all sequences of length $n-2$ composed of the elements of $\{1,2,...,n\}$.

Every labelled tree with n vertices can be represented by a binary word of length $n \log n$.

The number of labelled trees with n vertices is n^{n-2} (Cayley's Formula)

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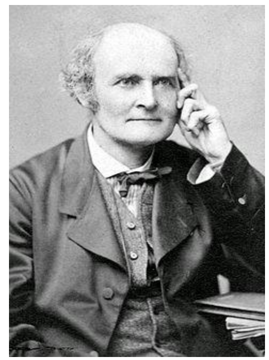
Carl Borchardt
(1817-1880)



Arthur Cayley
(1821-1895)

Ernst Paul Heinz Prüfer
(1896 – 1934)

The formula was first discovered by Carl Borchardt in 1860, and proved via a determinant. In a short 1889 note, Cayley extended the formula in several directions, by taking into account the degrees of the vertices. Although he referred to Borchardt's original paper, the name "Cayley's formula" became standard in the field.

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Prüfer code does not allow this, but a tree can be represented in this way by assigning to each vertex a label of length $2\log n$: choose a root (arbitrarily) and assign to each vertex its name (a number from 1 to n) and the name of its parent (a number from 1 to n).

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A representation of an *n*-vertex graph G is *implicit* if it assigns to each vertex of G a binary code of length $O(\log n)$ so that the adjacency of two vertices is a function of their codes.

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A representation of an n-vertex graph G is *implicit* if it assigns to each vertex of G a binary code of length $O(\log n)$ so that the adjacency of two vertices is a function of their codes.

Every class of graphs that admits an implicit representation contains $2^{O(n \log n)}$ labelled graphs

Factorial properties of graphs

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Definition. *A class of graphs is hereditary if it is closed under taking induced subgraphs.*

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Definition. *A class of graphs is hereditary if it is closed under taking induced subgraphs.*

Every hereditary class of graphs that admits an implicit representation has (at most) factorial speed of growth.

Implicit representation conjecture

Do graphs in any hereditary class with at most factorial speed of growth admit an implicit representation?

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If the speed of a hereditary class is not responsible for implicit representation, then what is responsible for it?

Classes of graphs admitting an implicit representation

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- Graphs of bounded vertex degree

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- Graphs of bounded degeneracy (the minimum k such that every induced subgraph has a vertex of degree at most k), which includes all proper minor closed classes, all classes of bounded tree-width, etc.

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- Graphs of bounded boxicity, including interval graphs

- Graphs of bounded clique-width

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- Graphs of bounded twin-width

  Bonnet, Édouard; Geniet, Colin; Kim, Eun Jung; Thomassé, Stéphan; Watrigant, Rémi Twin-width II: small classes. *Proc. the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA),* 1977–1996.

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bounded clique-width → bounded twin-width → bounded functionality

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bounded clique-width → bounded twin-width → bounded symmetric difference → bounded functionality

The *symmetric difference of two vertices x and y* in a graph G is the number of vertices in G-{x,y} adjacent to exactly one of x and y.

The *symmetric difference of G* as the smallest number *k* such that every induced subgraph of G has a pair of vertices with symmetric difference at most *k*.

Classes of graphs

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We study *hereditary* classes of *bipartite* graphs

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  - Every such class can be described by a set M of minimal forbidden induced *bipartite* subgraphs

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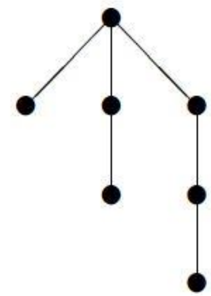
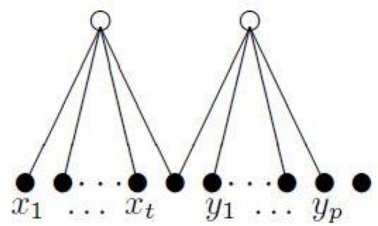
Theorem. *For a bipartite graph H, the class of H-free bipartite graphs has at most factorial speed of growth if and only if H is an induced subgraph of one of the following graphs:* P_7, $S_{1,2,3}$ *and* $F_{t,p}$.



$S_{1,2,3}$



$F_{t,p}$

Classes of graphs

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Every such class can be described by a set M of minimal forbidden induced *bipartite* subgraphs

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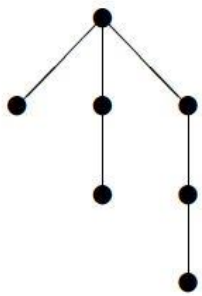

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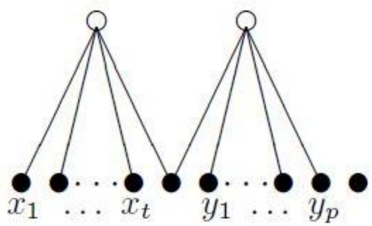
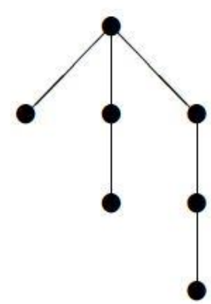


$F_{t,p}$

Three extremal classes of graphs



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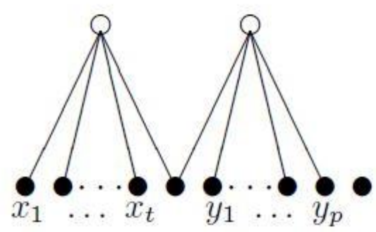

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$S_{1,2,3}$-free bipartite graphs have bounded clique-width → $S_{1,2,3}$-free bipartite graphs admit an implicit representation and have bounded symmetric difference

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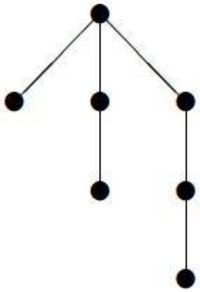

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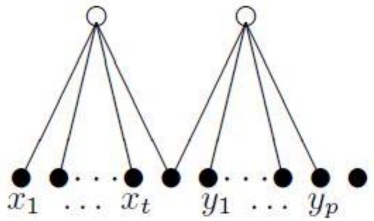

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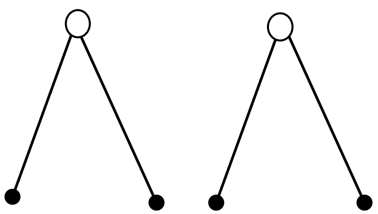
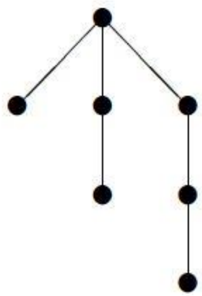


$2P_3$



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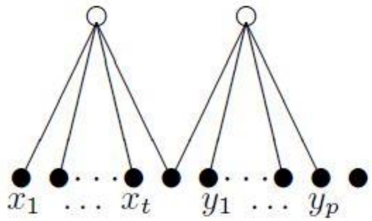

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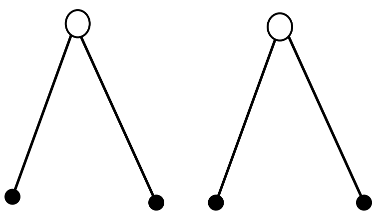

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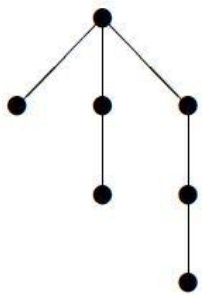
Theorem. *$F_{t,p}$-free bipartite graphs have bounded symmetric difference.*

Theorem. *$F_{t,p}$-free bipartite graphs admit an implicit representation.*



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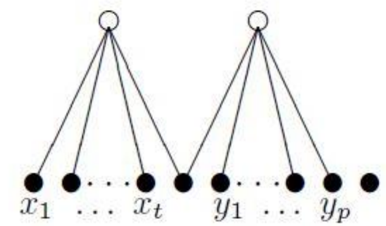

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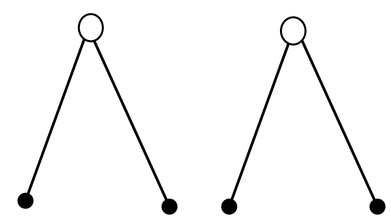

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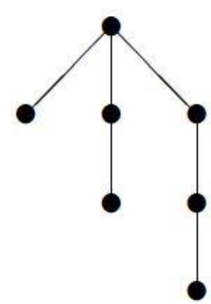
Theorem. *$F_{t,p}$-free bipartite graphs have bounded symmetric difference.*

Theorem. *$F_{t,p}$-free bipartite graphs admit an implicit representation.*

Conjecture. *P_7-free bipartite graphs have bounded symmetric difference.*

Conjecture. *Hereditary classes of graphs of bounded symmetric difference admit an implicit representation.*



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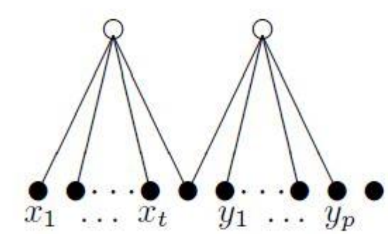

$F_{t,p}$

Back to factorial properties

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Theorem. *Every class of graphs of bounded functionality is (at most) factorial.*

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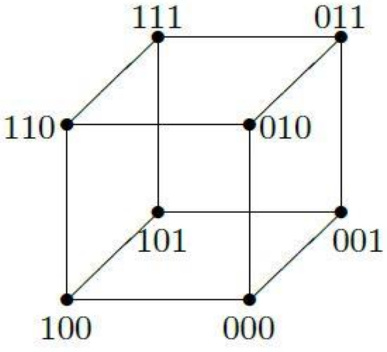
Theorem. *Hypercubes can have arbitrarily large functionality.*



Let Q be the class of induced subgraphs of hypercubes.

What is the speed of Q?

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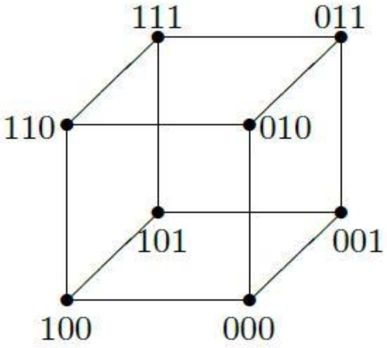
Theorem. There are at most n^{2n} n-vertex labelled graphs in Q.



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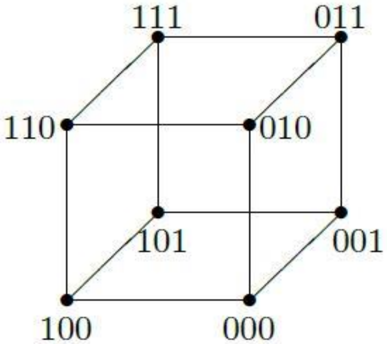

Q_3

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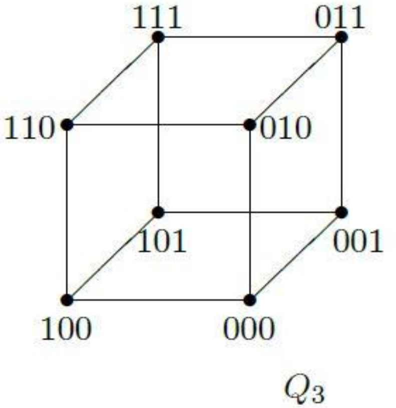
As a preliminary, let $G \in Q$ be a connected graph on n vertices. By definition of Q, G embeds into Q_m for some m. We claim that, in fact, G embeds into Q_{n-1} (by induction on the number of vertices in G). Additionally, by symmetry, if G has a distinguished vertex r, we remark that we may find an embedding sending r to (0,0,…,0).



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Let $G \in Q$ be any labelled graph with vertex set $\{x_1,\ldots,x_n\}$. We start by choosing, for each connected component C of G:
- a spanning tree T_C of C;
- a root r_C of T_C;
- an embedding φ_C of T_C into Q_{n-1} sending r_C to $(0,0,\ldots,0)$.

We define two functions $p,d : V(G) \to \{1,\ldots,n\}$:

$p(x_i)=i$ if x_i is the root
$P(x_i)=j$ if x_j is the parent of x_i

$d(x_i)=i$ if x_i is the root
$d(x_i)=j$ if $\varphi(x_i)$ and $\varphi(p(x_j))$ differ in coordinate j.

Back to factorial properties

Let Q be the class of induced subgraphs of hypercubes.

Theorem. There are at most n^{2n} n-vertex labelled graphs in Q.

We now claim that G can be restored from the sequence $p(x_1), d(x_1),\ldots,p(x_n), d(x_n)$.
To do so, we first note that this sequence allows us to easily determine the partition of G into connected components. Moreover, for each connected component, we may then determine its embedding $\varphi(C)$ into Q_{n-1}: $\varphi_C(r_C)$ is by assumption $(0,0,\ldots,0)$; we may then identify its children using p, then compute their embeddings using d; we may then proceed inductively. This information allows us to determine the adjacency in G as claimed, and the encoding uses 2n integers between 1 and n as required.

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Open questions

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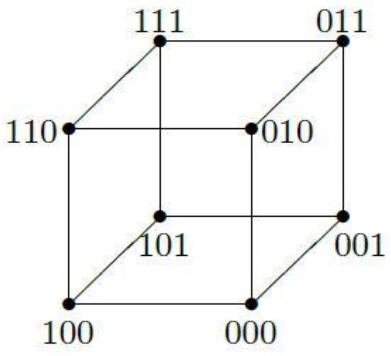
Do graphs in Q admit an implicit representation?

Open questions



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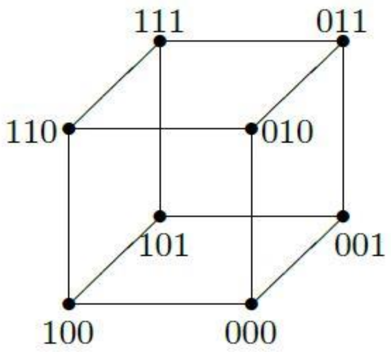

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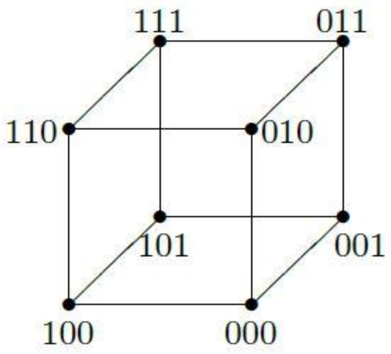
N. Harms, S. Wild, V. Zamaraev, Randomized communication and implicit graph representations. STOC 2022, 1220-1233.

Non-constructive proof with no explicit labelling scheme

Open questions



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Find specific implicit representation for the class Q

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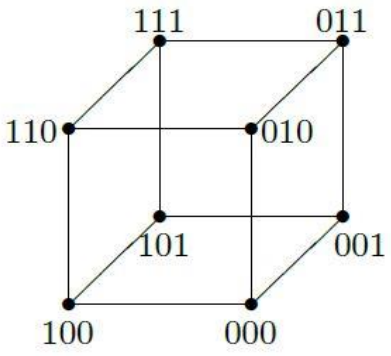

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Find specific implicit representation for the class Q

Do P_7-free (chordal) bipartite graphs admit an implicit representation?

Thank you