Critical $(P_3 + \ell P_1)$-free graphs

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(Joint work with Tala Abuadas, Ben Cameron and Joe Sawada)

k-critical graphs

Definition

A graph G is k-critical if G is k-chromatic but $G - x$ is $(k-1)$-colorable for any vertex x of G.

Problem

For a class $\mathcal{F}$ of graphs, is the number of k-critical graphs in $\mathcal{F}$ finite, for some given k?

If the answer is yes, then we can k-color the graphs in $\mathcal{F}$ in poly-time.

H-Free Graphs

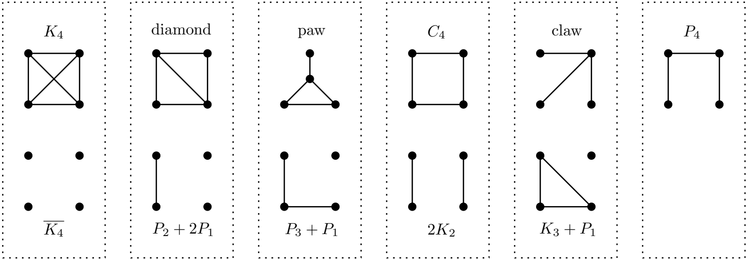
Definition

For a graph H, a graph G is H-free, when G **does not contain** H as an **induced** subgraph.

Definition

For a set $\mathcal{F}$ of graphs, a graph G is $\mathcal{F}$-free, when G **does not contain** any graph of $\mathcal{F}$ as an **induced** subgraph.

All graphs on four vertices



Complexity of coloring P_5 graphs

Theorem (Kral, Kratochvil, Tuza, Woeginger 2001)

Coloring ($2K_2$, co-diamond, $4K_1$, C_5)-free graphs is NP-hard

This implies it is NP-hard to color the following classes:

- graphs with no holes of length at least five
- odd-hole-free graphs
- P_5-free graphs

What about k-coloring P_5-free graphs for a fixed $k \geq 3$?

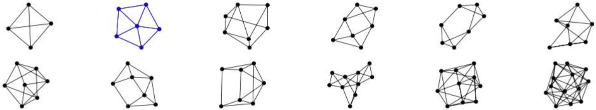
3-coloring P_5-free graphs

Theorem (Woeginger and Sgall 2001)

There is a polynomial time algorithm to 3-color a P_5-free graphs, or to show that no 3-coloring exists.

Theorem (Daniel Bruce, CTH, Joe Sawada 2009)

The number of 4-critical P_5-free graphs is finite.



What about 5-critical P_5-free graphs?

k-colorability of P_5-free graphs 1

Theorem (CTH, Kaminski, Lozin, Sawada, Shu 2010)

For every $k \geq 4$, k-colorability of P_5-free graphs can be solved in poly time.

k-critical P_5-free graphs

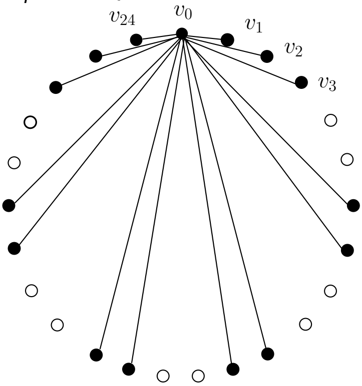
Theorem (CTH, Moore, Recoskie, Sawada, Vatshelle 2014)

For every $k \geq 5$, the number of k-critical $2K_2$-free graphs is **infinite***.*

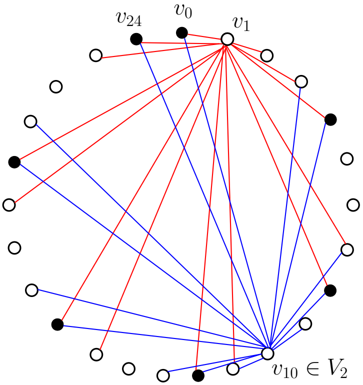
Note the k = 4 gap. We'll talk about it later.

Constructing 5-critical $2K_2$-free graphs

G_p has $4p + 1$ vertices.



(a)

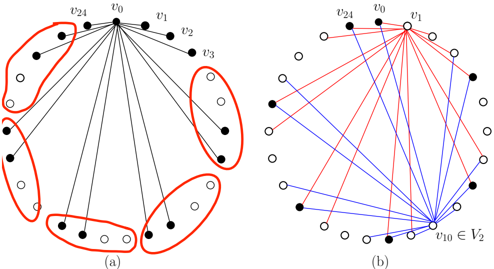


(b)

$p = 6$

Constructing 5-critical $2K_2$-free graphs

G_p has 4p+1 vertices
To increase p by 1, add one interval
We can make a graph with arbitrarily large α and $\omega = 4$.
Adding an universal vertex, we get arbirarily large ω



Some properties of k-critical graphs

Theorem (Erdos 1959)

If H contains a cycle, then for all $k \geq 3$, there are an infinite number of H-free k-critical graphs.

Theorem (Lazebnik Ustimenko 1995)

If H contains a claw, then for all $k \geq 3$, there are an infinite number of H-free k-critical graphs.

Theorem (Chudnovsky,Goedgebeur, Schaudt and Zhong 2016)

Let H be a graph. There are only finitely many H-free 4-vertex-critical graphs if and only if H is an induced subgraph of $P_6, 2P_3$, or $P_4 + kP_1$ for some $k \in N$.

A question on H-free k-critical graphs

Problem

For which graphs H are there a finite number of k-critical H-free graphs for all $k \geq 5$?

H cannot contain a cycle, or a claw, or a $2K_2$.
H must be, for a constant ℓ

- $P_2 + \ell K_1$, or
- $P_3 + \ell K_1$, or
- $P_4 + \ell K_1$, or

The first two cases were solved.

The graphs $P_2 + \ell P_1$ and $P_3 + \ell P_1$

Theorem (Ben Cameron, CTH, Joe Sawada 2020)

All all $\ell \geq 0$ and k, the number of k-critical $(P_2 + \ell P_1)$-free graph is finite.

Theorem (Ben Cameron, CTH, Joe Sawada 2022)

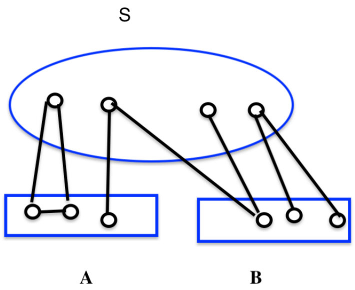
All all $\ell \geq 0$ and k, the number of k-critical $(P_3 + \ell P_1)$-free graph is finite.

Proof of $(P_3 + \ell P_1)$-free graphs

Outline of the proof:

- Let G be a k-critical graph.
- We know $\omega(G) \leq k$ because $\chi(G) \geq \omega(G)$.
- We will prove $\alpha(G)$ is bounded.
- By Ramsey theorem we are done, because $|V(G)| \leq R(\alpha + 1, \omega + 1)$.

Proof



Take S to be max stable set

Want to prove |S| < k^2 (l + 3)

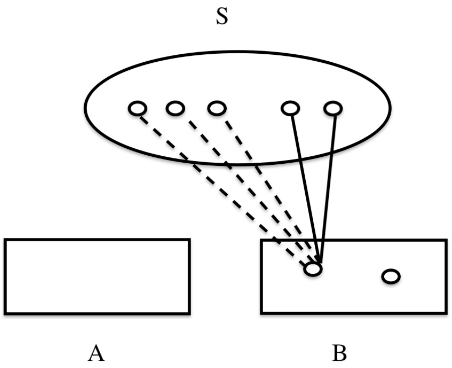
Assume to contrary

A = vertices in G - S with exactly one neighbor in S

B = vertices in G - S with at least two neighbors in S

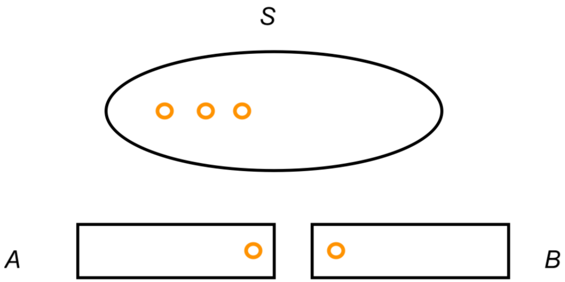
A + B + S = V(G)

Observation 1



If some v in B is non-adjacent to at least l vertices in S then we have a P3+ l P1

Observation 2

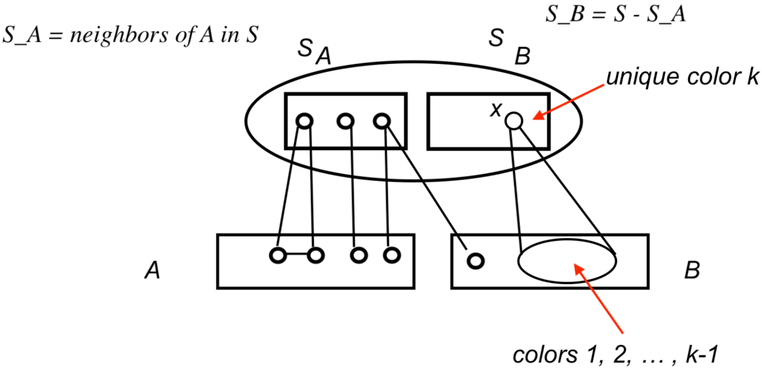


A color class is big if it has >= k (l + 3) vertices in S

In a k-coloring, some color class is big

If a color appears in B, it cannot belong to a big color class

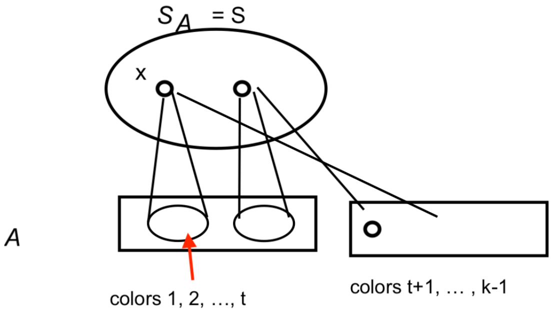
Step 3



Claim: S_B is empty

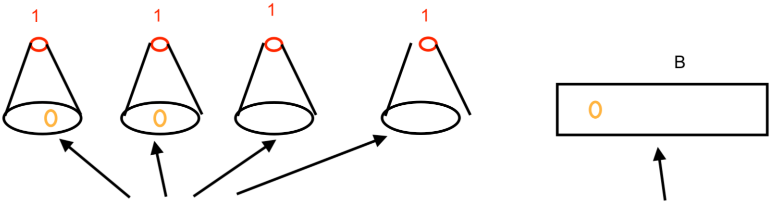
Proof:
Remove x in S_B, color G-x with k-1 colors, these appear in B, one of the color is big

Step 4



Take x in S_A with smallest degree in A
x_A is a clique (neighbors of x in A)
There is a k-coloring of G such that x is the only vertex with color k
and colors 1, 2, …, k-1 appear in N(x)
Colors t+1, …, k-1 are small
WLOG, assume color 1 is big

Step 5



A color not in {1, 2, …., t}, because the clique has order t+1

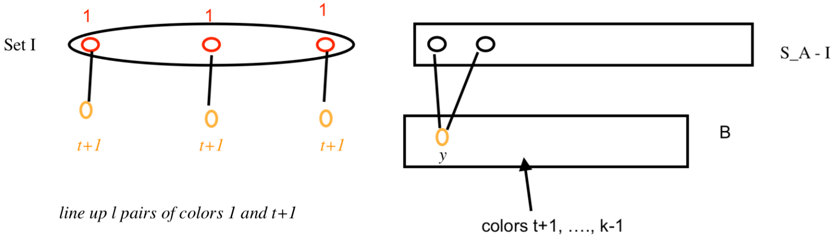
colors t+1, …., k-1

A color in {t+1, …., k-1} appears in each of the cliques

There are k (l +3) cliques

Some color, say t+1, appears at least l times in the cliques

Step 6



The set S_A - I has k^2 (l+3) - l > l vertices

Vertex y cannot miss l vertices in S_A - I so it sees at least two vertices

There is a P3 + l P1

Open problems

Problem

Are there a finite number of k-critical $(P_4 + P_1)$-free graphs, for all $k \geq 5$?

Problem

For which values of $k \geq 5$ and $\ell \geq 2$ is there a finite number of k-critical $(P_4 + \ell P_1)$-free graphs?

co-gem-free graphs

The graph $P_4 + P_1$ is called a co-gem.
The problem is open for co-gem-free graphs.

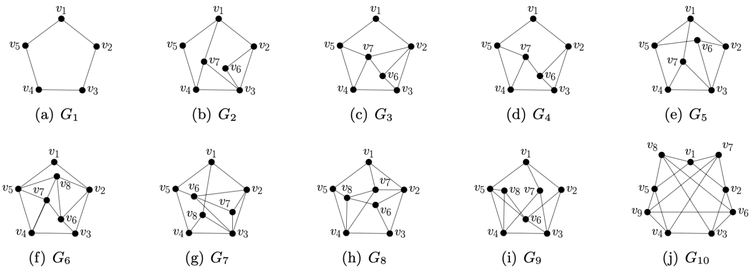
Theorem (Ben Cameron, CTH, Joe Sawada)

Therer is a finite number of k-critical (gem, co-gem)-free graphs for $k \geq 5$.

This uses a theorem of Karthick and Maffray.

Theorem (T. Karthick, F. Maffray)

If G is (gem, co-gem)-free, then G is perfect or a P4- or clique-expansion of a finite number of finite graphs.



(a) G_1 (b) G_2 (c) G_3 (d) G_4 (e) G_5

(f) G_6 (g) G_7 (h) G_8 (i) G_9 (j) G_{10}

Theorem (T. Karthick, F. Maffray)

If G is (gem, co-gem)-free, then G is perfect, or a P4- or clique-expansion of 10 graphs, or G belongs to a family of special graphs.

Thank you for your attention