Reconfiguration of vertex colouring and forbidden induced subgraphs

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Definition

1. All graphs considered here are simple, finite and undirected.
2. A vertex colouring of a graph G is a function $\phi : V(G) \longrightarrow \mathcal{C}$, where $\mathcal{C}$ is a set of colours, such that $\phi(u) \neq \phi(v)$ whenever $uv \in E(G)$.
3. A k-colouring of G is a colouring that uses at most k colours.
4. The *chromatic number* of G, denoted $\chi(G)$, is the smallest integer k such that G is k-colourable.

Reconfiguration of vertex colouring

The reconfiguration graph, $R_k(G)$, is the graph whose vertices are k-colourings of G and two vertices are adjacent if they differ exactly on one vertex.
The reconfiguration problem has been defined on other source problems, such as dominating set problem and independent set problem among others.

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2. The *connectivity problem* asks: For a fixed source problem and fixed definitions of feasible solutions and adjacency, is the reconfiguration graph connected?
3. If $R_k(G)$ is connected, then what is the diameter of $R_k(G)$?

Known results

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2. However, deciding whether there exists a path between any two 3-colourings of G in $R_3(G)$ can be solved in polynomial time (Cereceda, van den Heuvel, and Johnson, 2010).
3. Given a connected bipartite graph G, deciding whether $R_3(G)$ is connected is coNP-complete (Cereceda, van den Heuvel, and Johnson, 2009).

Induced subgraph

1. A graph G is said to be H-free, if G does not contain any induced subgraph isomorphic to H.
2. A graph G is said to be k-mixing if $R_k(G)$ is connected.
3. Let $\ell \geq \chi(G)+1$ be an integer.

Known results on H-free graphs

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4. For any $\ell \geq \chi(G)+1$, there exists a P_5-free graph G that is not ℓ-mixing (Feghali and Merkel, 2021).
5. For any $\ell \geq \chi(G)+1$, there exists a $2K_2$-free graph G that is not ℓ-mixing (Feghali and Merkel, 2021).

4-vertex graphs

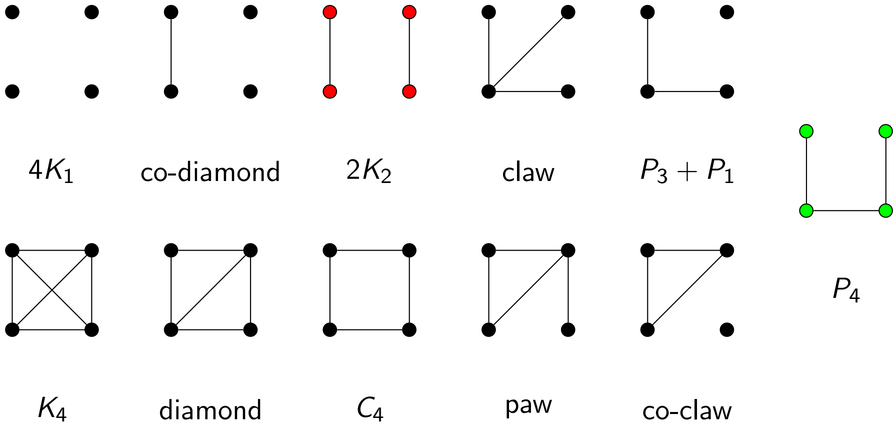


Figure: The 11 graphs on 4 vertices.

Frozen colourings

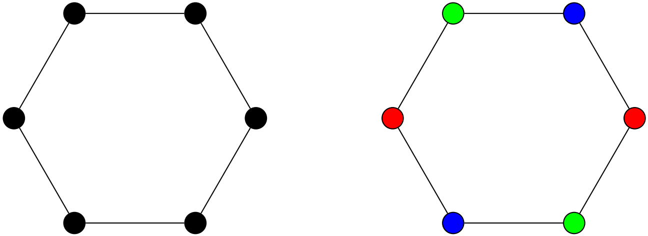


Figure: A frozen colouring of C_6.

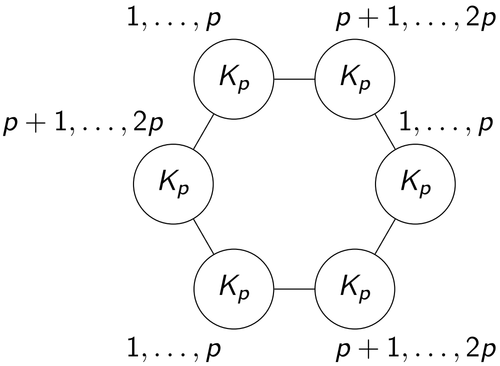


Figure: A $2p$-colouring of G_p.

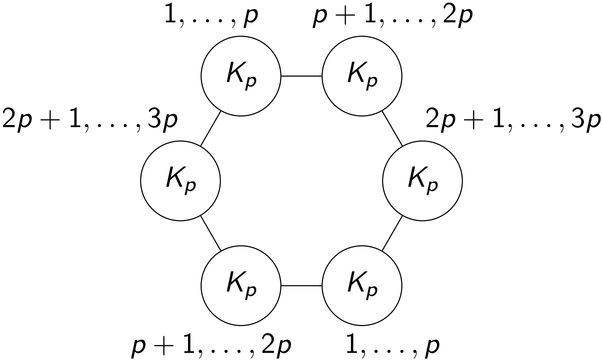


Figure: A frozen $3p$-colouring of G_p.

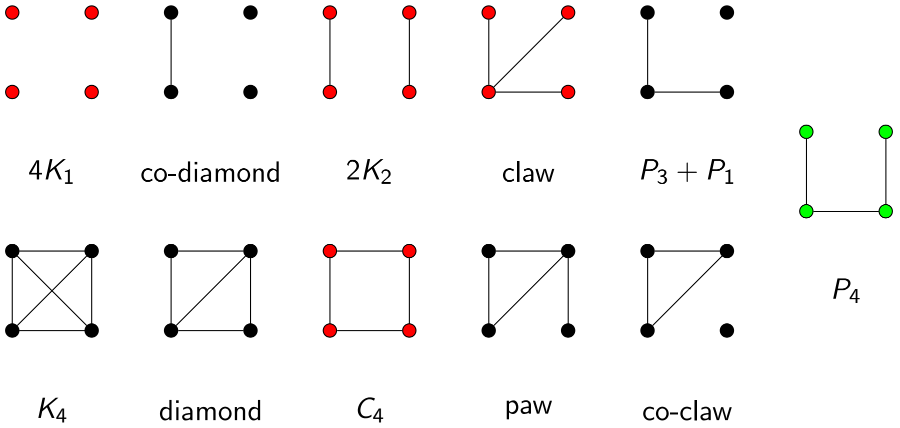
Results

Lemma

For $p \geq 1$, let G_p be the graph obtained from C_6 by substituting the complete graph K_p into each vertex. Then G_p is $(4K_1, C_4, \text{claw})$-free, is $2p$-colourable, and has a frozen $3p$-colouring.

Theorem

For any $p \geq 1$, there exists a $(4K_1, C_4, \text{claw})$-free graph G that is not $(k+p)$-mixing.


$4K_1$
co-diamond
$2K_2$
claw
$P_3 + P_1$
P_4
K_4
diamond
C_4
paw
co-claw

Let B_p be the graph obtained from the complete bipartite graph $K_{p,p}$ by deleting the edges of a perfect matching. Cereceda, van den Heuvel, and Johnson showed that B_p has a frozen colouring.

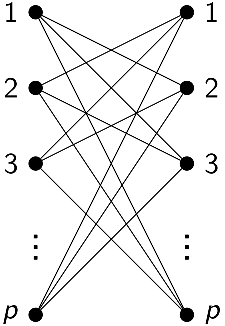


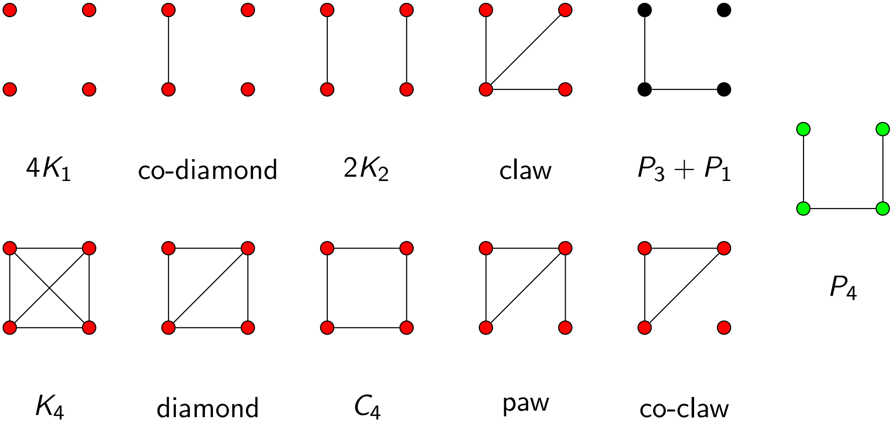
Figure: A frozen p-colouring of the graph B_p.

Lemma

For $p \geq 3$, B_p is a (K_4, diamond, paw, co-claw, co-diamond)-free, 2-colourable graph that admits a frozen p-colouring.

Theorem

For any $p \geq 1$, there exists a (K_4, diamond, paw, co-claw, co-diamond)-free graph G that is not $(k+p)$-mixing.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We use the following results in the proof of the above theorem.

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Lemma (Renaming Lemma by Bonamy and Bousquet, 2018)

Let α' and β' be two χ-colourings of G that induce the same partition of vertices into colour classes and let $\ell \geq \chi(G)+1$. Then α' can be recoloured into β' in $R_\ell(G)$ by recolouring each vertex at most 2 times.

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Theorem (By a result of Olariu, 1988)

Each anticomponent of a $(P_3 + P_1)$-free graph is either $3K_1$-free or P_3-free.

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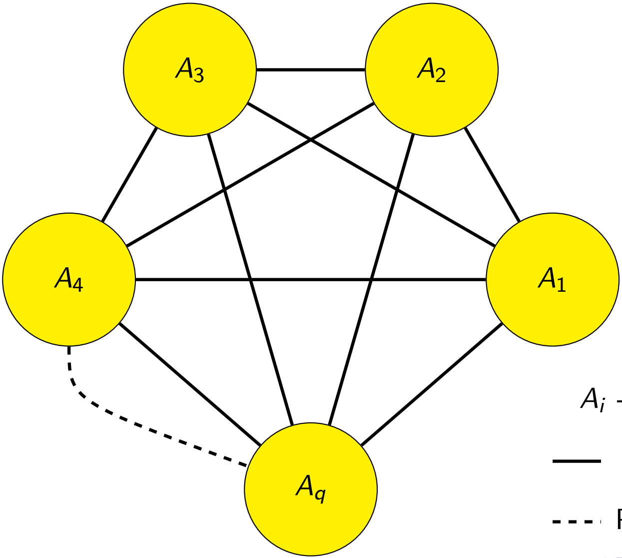
Lemma (Merkel, 2021)

Let γ be a $\chi(G)$ colouring of a $3K_1$-free graph G. Any ℓ-colouring of G can be recoloured into $\chi(G)$-colouring γ' such that the colour classes of γ' matches the colour classes of γ by recolouring each vertex at most once.

Structure of (P_3+P_1)-free graphs



A_i - anticomponents

All possible edges

Possible anticomponents

Recolouring (P_3+P_1)-free graphs

Proof.

Let α and β be two arbitrary ℓ-colourings of G. Let γ denote some χ-colouring of G. Let A_1, A_2,...,A_q denote anticomponents of G.

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Proof.

Let α and β be two arbitrary ℓ-colourings of G. Let γ denote some χ-colouring of G. Let A_1, A_2,...,A_q denote anticomponents of G. We prove that there exists a path of length at most $2n$ between α and $\alpha^{'}$ in $R_\ell(G)$, where $\alpha^{'}$ is a χ-colouring of G that induces same colour classes as γ.

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Claim

In any ℓ-colouring α of G, either some colour does not appear in α or there is some anticomponent A of G such that α uses at least $\chi(A)+1$ colours on A.

Proof of Claim omitted.

Recolouring (P_3+P_1)-free graphs

Proof (Contd.).

Without loss of generality, let A_1 be the anticomponent such that α uses at least $\chi(A_1)+1$ colours on A_1 or some colour does not appear in α.

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Proof (Contd.).

Without loss of generality, let A_1 be the anticomponent such that α uses at least $\chi(A_1)+1$ colours on A_1 or some colour does not appear in α.
If A_1 is $3K_1$-free then using Lemma, given by Merkel, we recolour A_1 to match colour classes of γ on A_1 in at most $|A_1|$ steps.

Recoluring (P_3+P_1)-free graphs

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If A_1 is P_3-free then using the Renaming Lemma we recolour A_1 to match the colour classes of γ on A_1 in at most $2|A_1|$ steps.

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Continue this process to recolour all anticomponents and obtain a χ-colouring $\alpha^{'}$ of G that induces the same colour classes as γ. We can do this by recolouring every vertex at most twice.

Recolouring (P_3+P_1)-free graphs

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By repeating the same process with the colouring β, we obtain a χ-colouring $\beta^{'}$ that induces the same colour classes as γ.

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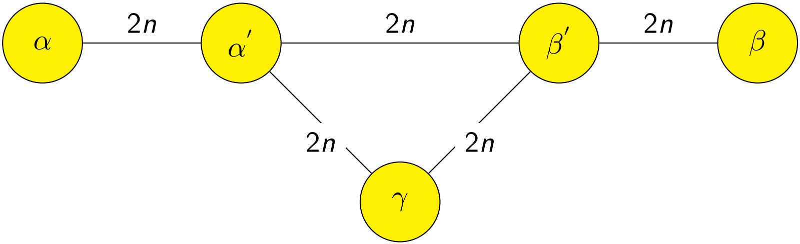
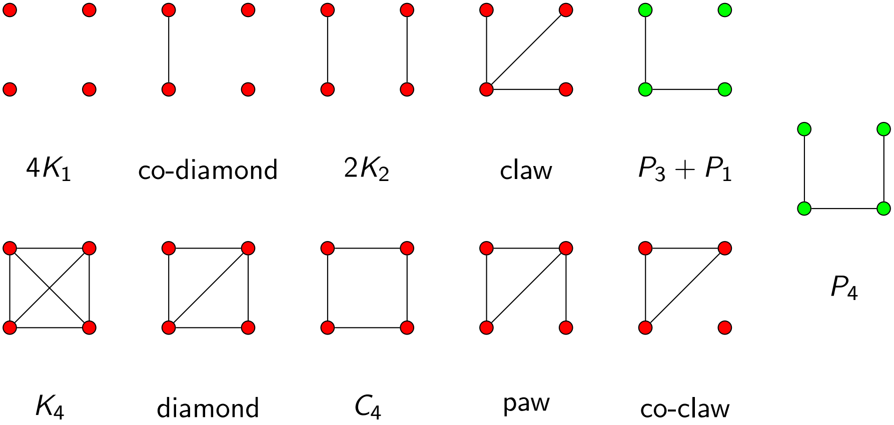
By repeating the same process with the colouring β, we obtain a χ-colouring $\beta^{'}$ that induces the same colour classes as γ.
Then by the Renaming Lemma there exists a path between $\alpha^{'}$ and $\beta^{'}$ in $R_\ell(G)$ of length at most $2n$. $\square$



α
$2n$
α'
$2n$
β'
$2n$
β
$2n$
$2n$
γ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Main result

Theorem (Main result)

Every H-free graph is ℓ-mixing if and only if H is an induced subgraph of P_4 or $P_3 + P_1$.

These are the same class of graphs for which deciding if they admit a k-colouring can be answered in polynomial time.

(H_1, H_2)-free graphs

Let H_1 and H_2 be any 4-vertex graphs other than P_4 and P_3+P_1.

Theorem

If $R_{k+1}(G)$ is connected for every k-colourable (H_1, H_2)-free graph G, then either H_1 or H_2 is isomorphic to $2K_2$.

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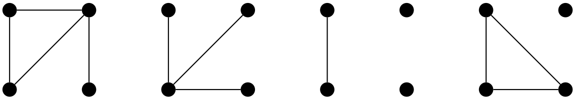
Theorem

Every $(2K_2, C_4)$-free graph is ℓ-mixing and the ℓ-recolouring diameter is at most $4n$.

Theorem

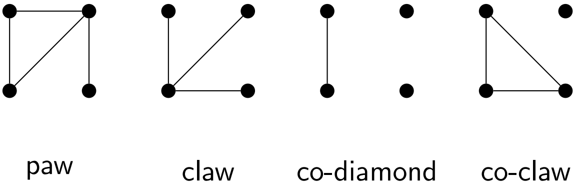
Every (P_5, C_4)-free graph is ℓ-mixing.



paw claw co-diamond co-claw

Theorem

Every $(2K_2, H)$-free graph is ℓ-mixing, where H is an induced subgraph of either a paw or a claw.



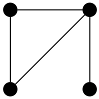
Theorem

Every $(2K_2, H)$-free graph is ℓ-mixing, where H is an induced subgraph of either a paw or a claw.

Theorem

For all $p \geq 1$, there exists a k-colourable $(2K_2, 4K_1, \textit{co-diamond}, \textit{co-claw})$-free graph that is not $(k+p)$-mixing.

Important result



paw

Theorem

Every (triangle, H)-free graph G_1 is $\chi(G_1)+p$-mixing if and only if every (paw, H)-free graph G_2 is $\chi(G_2)+p$-mixing, for any $p \geq 1$.

Future work

1. All H-free and (H_1, H_2)-free graph classes that are proved here to be ℓ-mixing are also polynomial-time colourable. It is interesting to check if it is true in general.
2. Characterise $(2K_2$, diamond$)$-free graphs and $(2K_2,\ K_4)$-free graphs relative to ℓ-mixing.
3. Find a dichotomy theorem for $(H_1,\ H_2)$-free graphs relative to ℓ-mixing.

Thank You