Tree-layout based graph classes: the case of proper chordal graphs

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GROW 2022

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Tree-layouts

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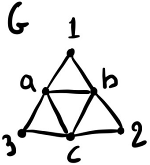

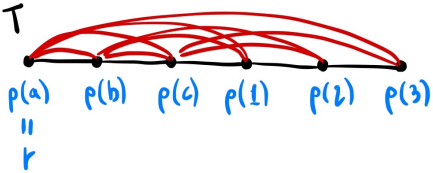

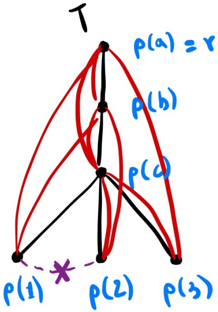

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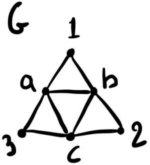

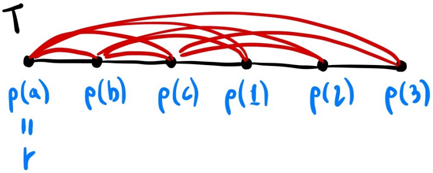

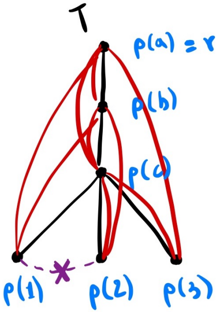

$u <_{\mathbf{T}} v$ iff $\rho(x)$ is an ancestor of $\rho(y)$ in T.
$\mathbf{T}$ is a **tree-layout** of G iff $\forall xy \in E(G)$: $x <_{\mathbf{T}} y$ or $y <_{\mathbf{T}} x$.

Tree-layout characterization of chordal graphs

G **chordal:** Admits a **simplicial elimination ordering**.

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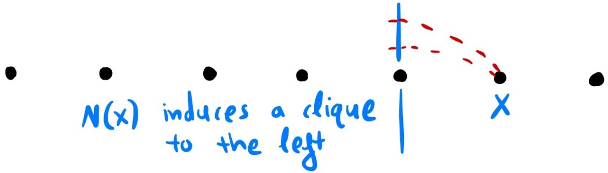

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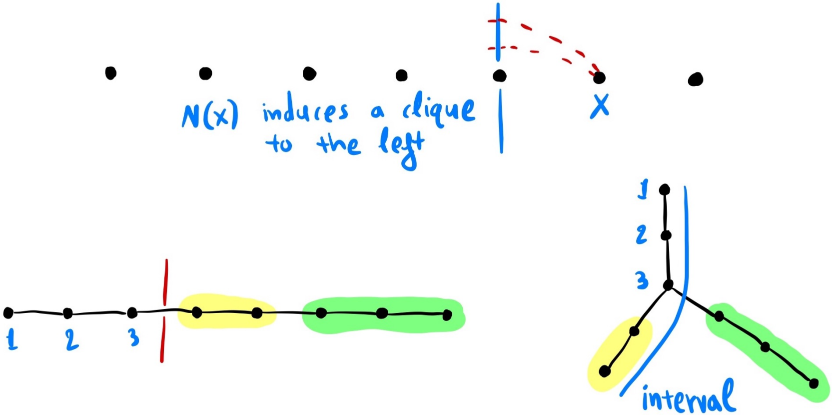

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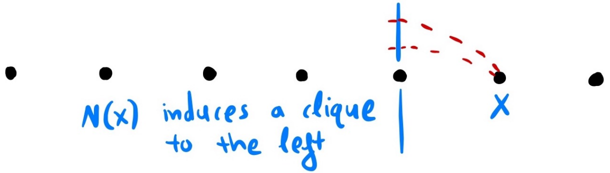

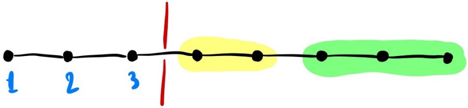

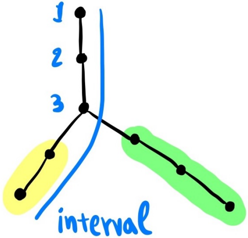

G **chordal:** Admits tree-layout where the graph induced by **every** root-leaf path is an **interval** graph.

Layout based classes

Specific **configuration** of layouts $\Rightarrow$ Graph class **characterization**.

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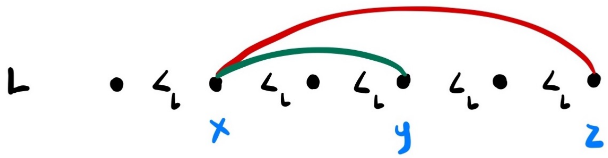

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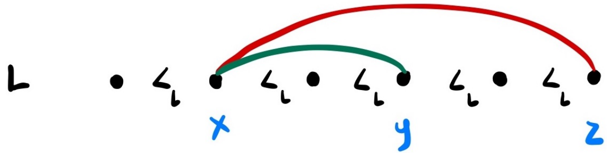

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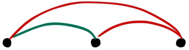
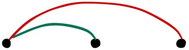
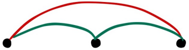
	Chordal	Chordal
	Interval	**Chordal**
	Proper Interval	???

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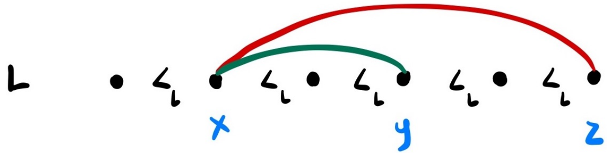

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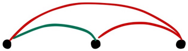
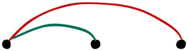
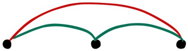
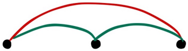
	Chordal	Chordal
	Interval	**Chordal**
	Proper Interval	???

: **indifference property**

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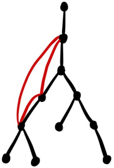

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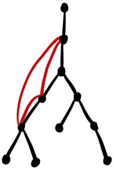

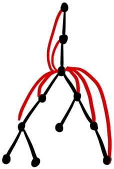

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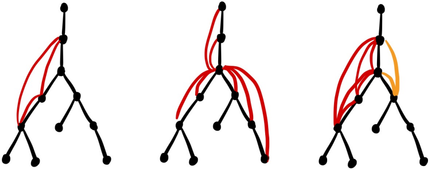

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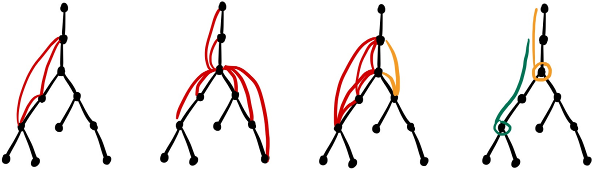

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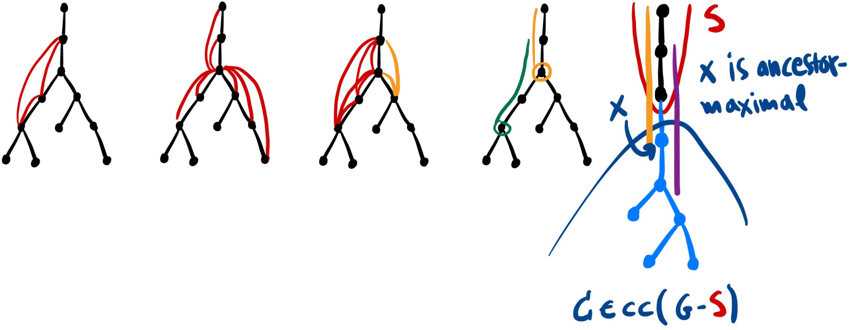

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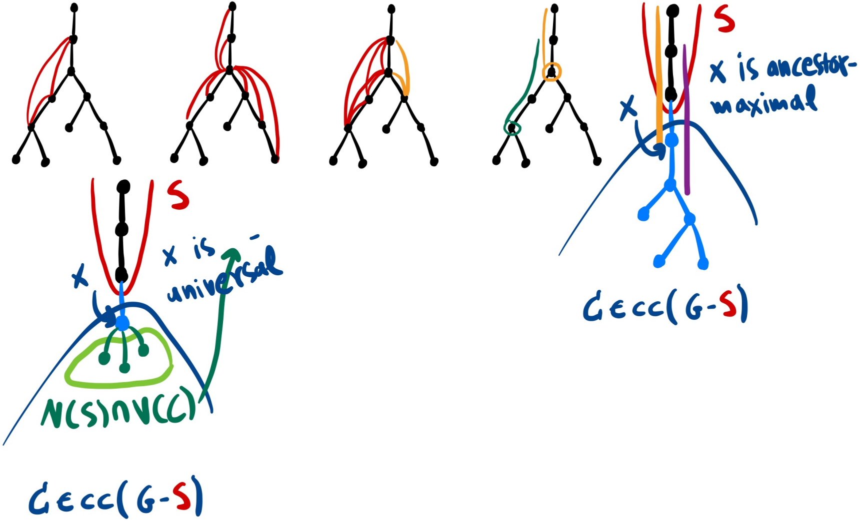

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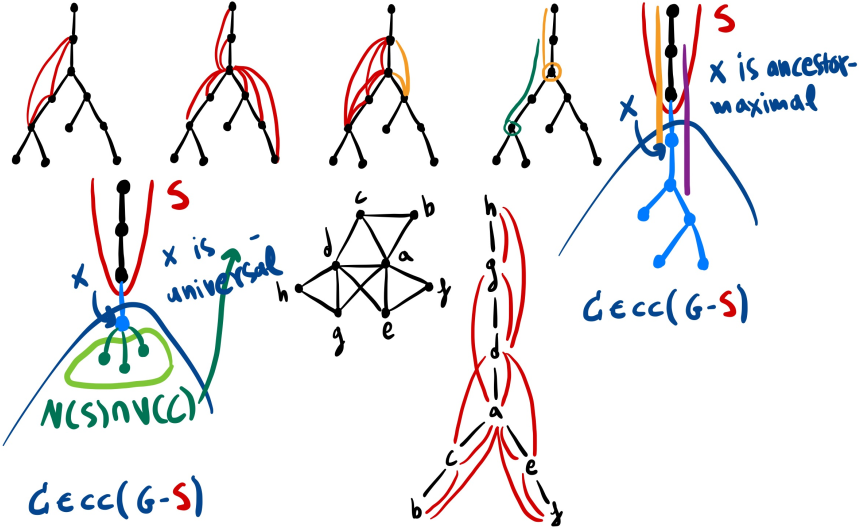

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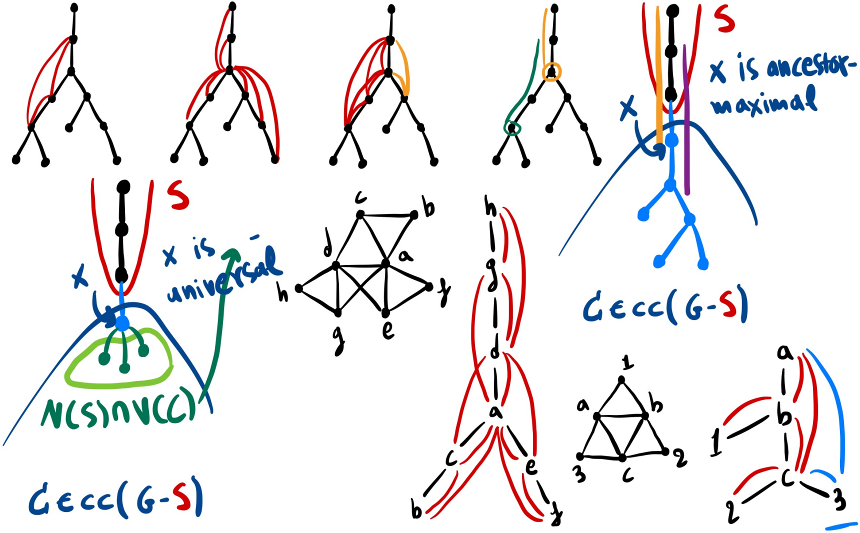


Positioning of Proper Chordal graphs

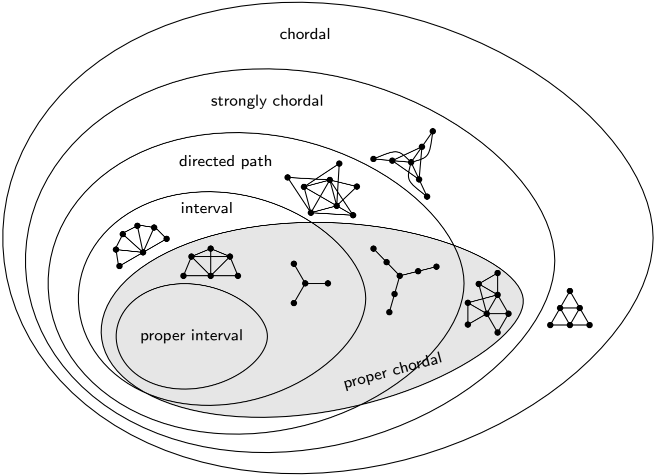
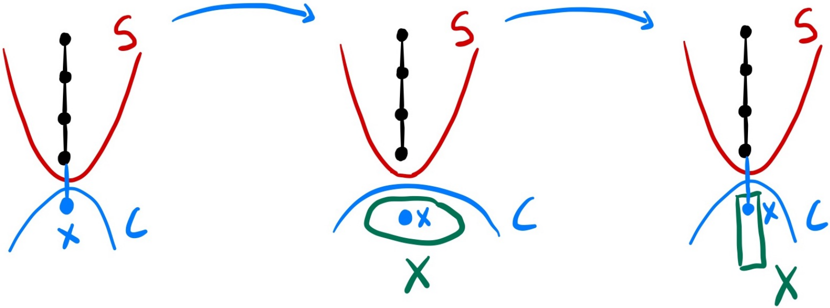


Figure: Relationship between proper chordal graphs and subclasses of chordal graphs.

Blocks



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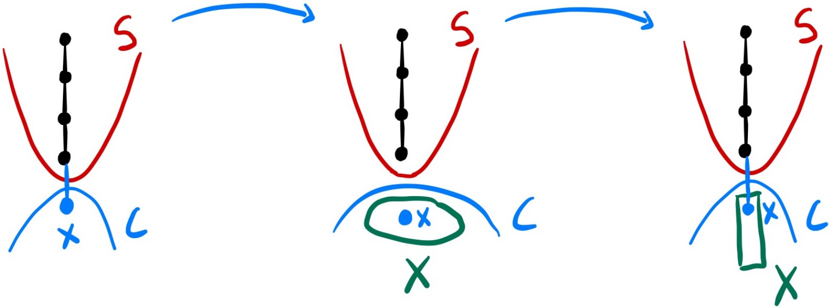

- S is the set of ancestors of x on some indiff. tl. $\mathbf{T}$; $C \in \mathrm{cc}(G - S)$;
- X is the set of ancestor-maximal and $(N(S) \cap V(C))$-universal vertices in C.

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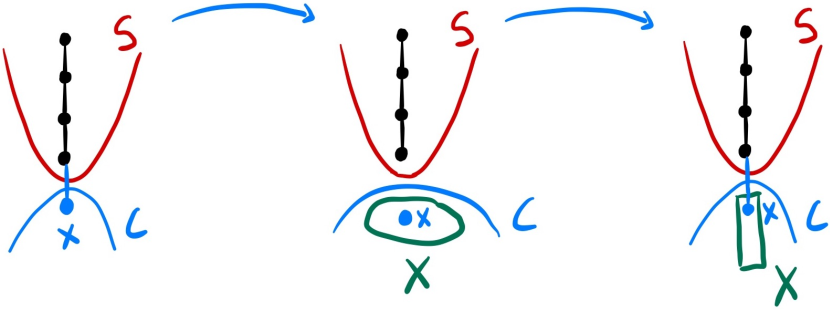

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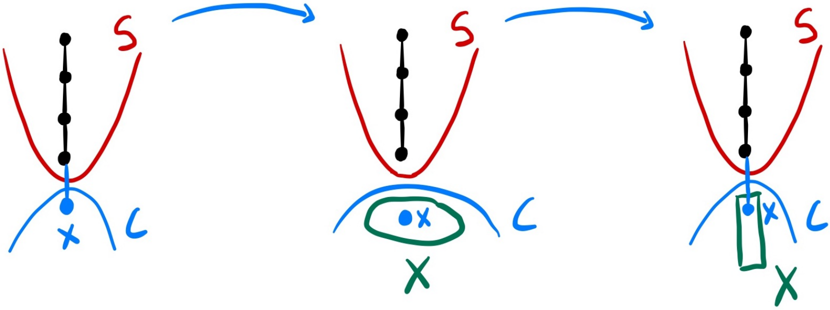

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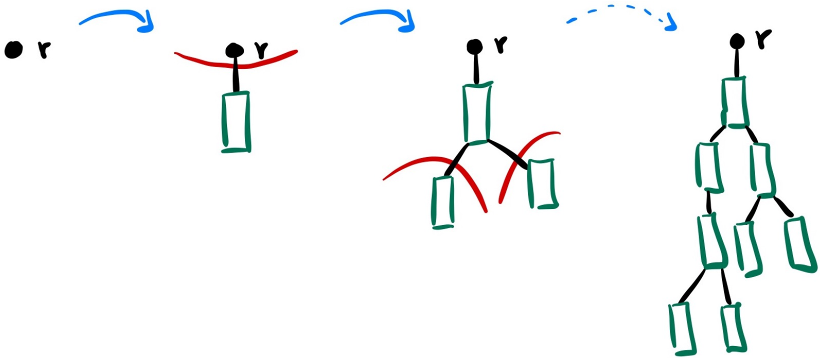
We call X a **block**.

Block Tree



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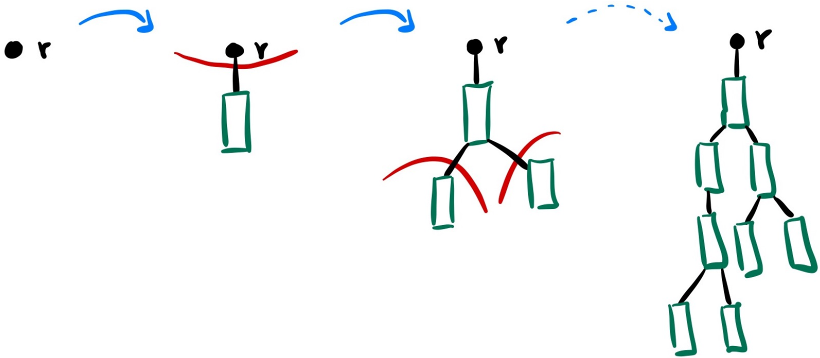

There is a **unique** block tree rooted at each $r \in V(G)$.

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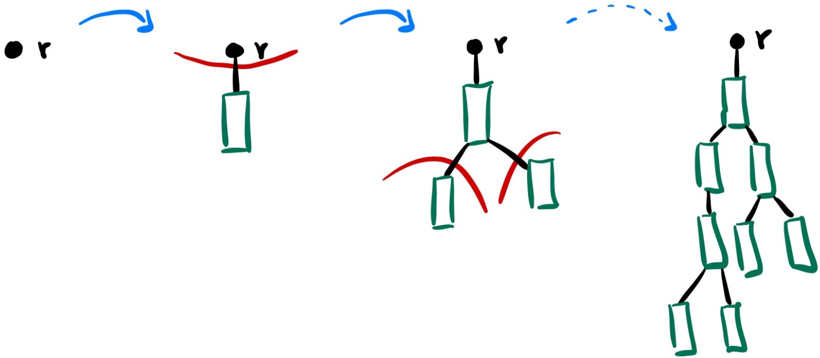

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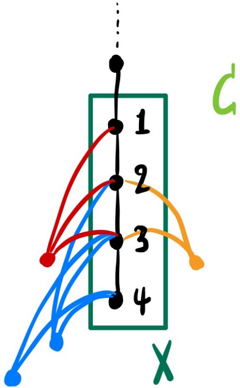
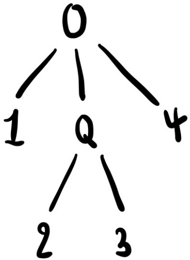
To obtain an indifference tree-layout, it remains to:

1. **Order** vertices within blocks;
2. **Properly attach** children blocks to parent block.

Ordering blocks





1 2 3

2 3 4

3 4

2 before 34

2 3

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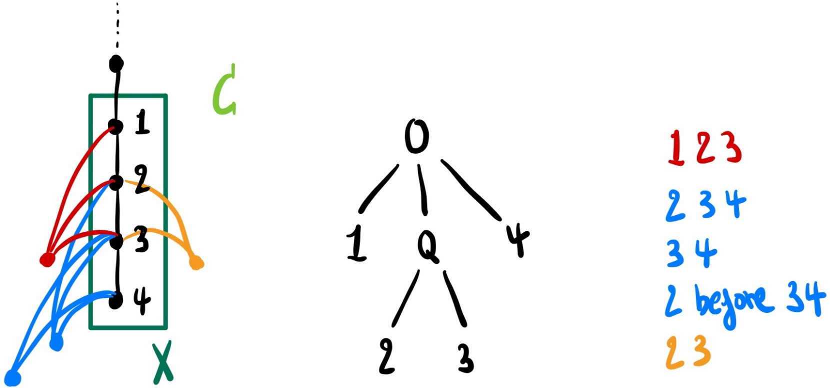

The **order** of X needs to satisfy certain **convexity** conditions:

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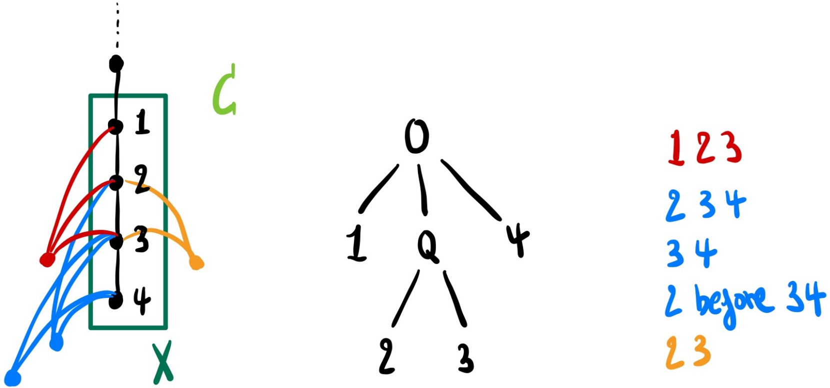

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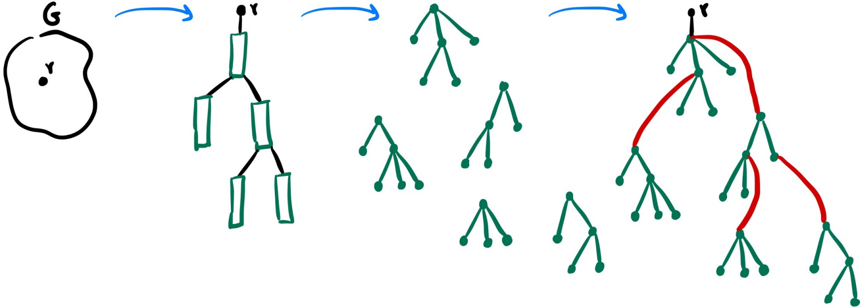
Unique OPQ-tree represents all **possible permutations** of X.

Canonical representation



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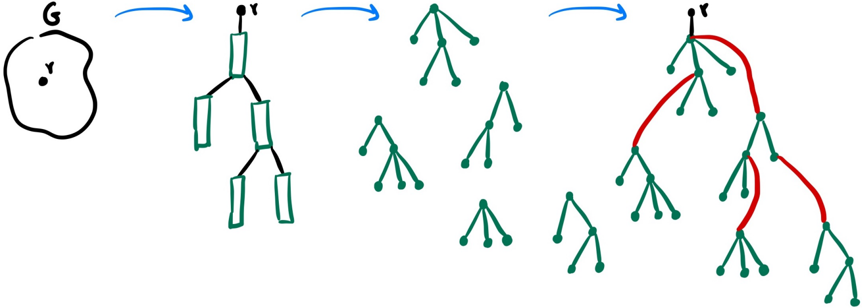

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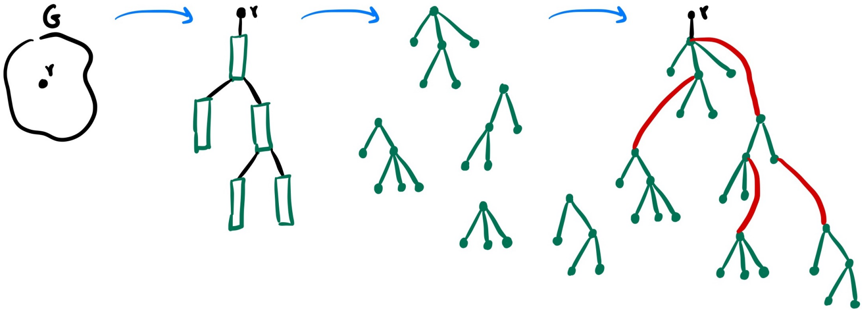

Unique OPQ-hierarchy represents all **indifference tree-layouts** of G **rooted** at $r \in V(G)$.

Remark: We can compute it in **poly-time** and has size **linear** in $|G|$.

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Lemma

Two proper chordal graphs G and H are isomorphic iff there exist $u \in V(G)$ and $v \in V(H)$ such that the OPQ*-hierarchies rooted at u and v are isomorphic.*

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- Other applications of OPQ-hierarchies?

Thank you!