Brandenburgische Technische Universität Cottbus - Senftenberg

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CHARACTERISING GRAPHS WITH CONVEXITY

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Brandenburg University of Technology

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ABSTRACT CONVEXITY

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The convex hull of $X \subset V$, namely $\mathrm{conv}(X)$, is the smallest convex set containing X.

An element $p \in X$ is called an extreme point of X if $p \notin \mathrm{conv}(X - p)$ and the set of all of these points is $\mathrm{ex}(X)$

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This property is also called the anti-exchange property.

Theorem (Edelman and Jamison, 1985)

Let $(V, \mathcal{C})$ be a convexity space. Then the following statements are equivalent:

1. $(V, \mathcal{C})$ *is a convex geometry.*
2. *For every convex set X there exists a point $p \in V \setminus X$ such that $X + p$ is convex.*
3. *We have $\operatorname{conv}(\operatorname{ex}(X)) = X$ for every convex set X.*

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Definition (Chvátal Property)

For all $a, b, c \in V$ and $y \in I[b,c]$ and $z \in I[a,y]$ it holds that $z \in I[a,b]$ or $z \in I[a,c]$ or $z \in I[b,c]$.

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An interval operator that fulfils the Chvátal Property generates a convex geometry.

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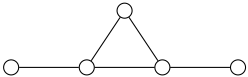

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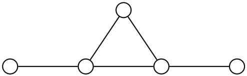

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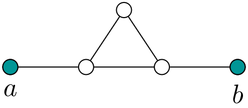

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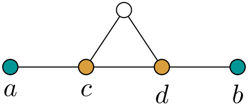

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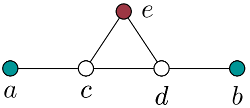

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A graph G is contained in $\mathcal{G}$ if and only if $\mathcal{C}_{\mathcal{G}}$ is a convex geometry.

INTERVAL GRAPHS

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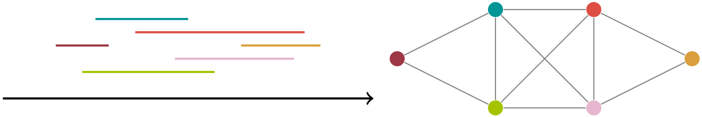

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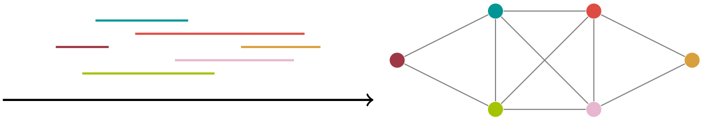

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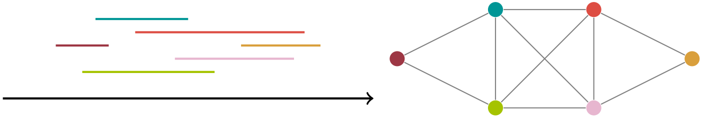

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The interval model implies a strong *convex structure* to be found in the graph.

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Definition

An asteroidal triple of a graph is a set of three independent vertices such that there is a path between each pair of these vertices that does not contain any vertex of the neighbourhood of the third. A graph is called asteroidal triple free (AT-free) if it does not contain such an asteroidal triple.

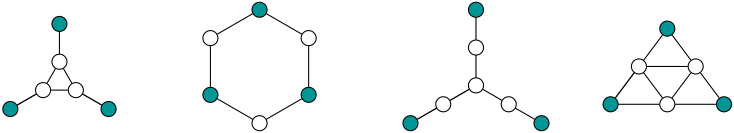

Figure: Examples of graphs containing asteroidal triples.

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Combining domination and monophonic intervals, we get line intervals:

$$I_{\text{line}}[a,b] := I_{\text{mon}}[a,b] \cup I_{\text{dom}}[a,b].$$

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For any graph $G = (V, E)$ the following properties are equivalent:

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1. $\Rightarrow$ 2. is the main part of the proof.

CARATHÉODORY NUMBER

We say that $(V, \mathcal{C})$ has Carathéodory number d if d is the smallest positive integer, such that for every $X \subseteq V$ and every $p \in \operatorname{conv}(X)$ there is a subset $X' \subseteq X$ with $p \in \operatorname{conv}(X')$ and $|X'| \leq d$.

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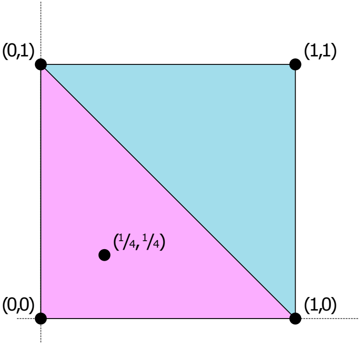

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For many classes such as interval graphs this question is still open.

Thank you for your attention!