GROW 2022, Koper

# SEMI-PROPER INTERVAL GRAPHS

## AN IDEA IS COMING BACK TO KOPER. . .

Robert Scheffler

Brandenburg
University of Technology
Cottbus - Senftenberg

## ONCE UPON A TIME IN KOPER…

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is a vertex ordering $(v_1, \ldots, v_n)$ such that

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### Claim (of Koper)

*A graph has a* ***connected PEO*** *if and only if it is a connected* ***proper interval graph****.*

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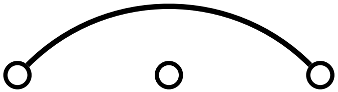

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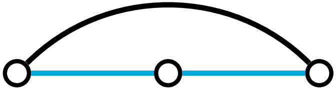

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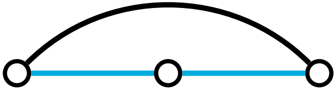

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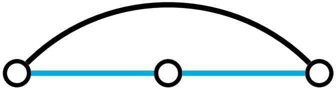

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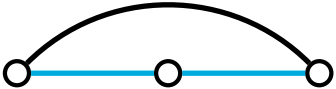

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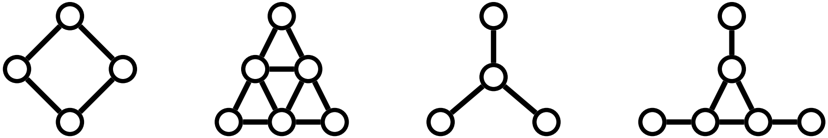

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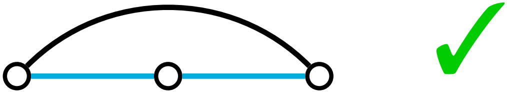

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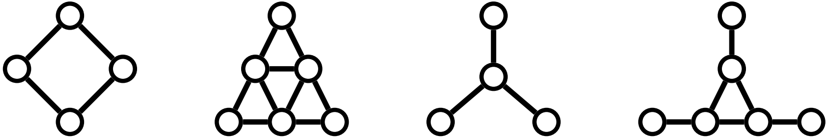

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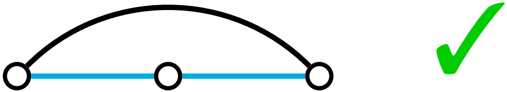

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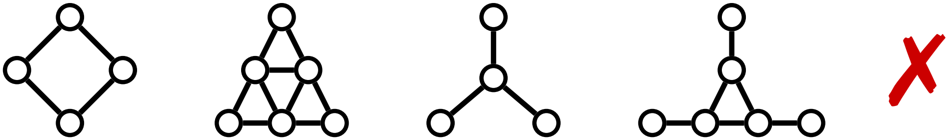

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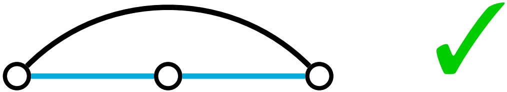

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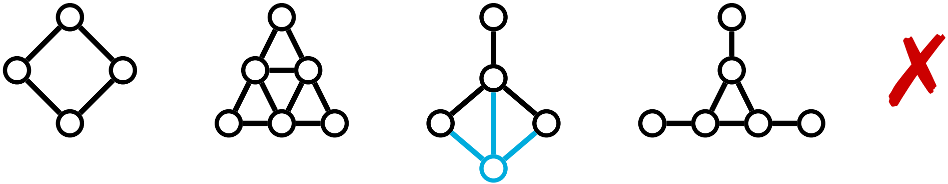

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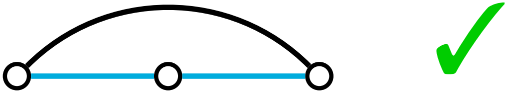

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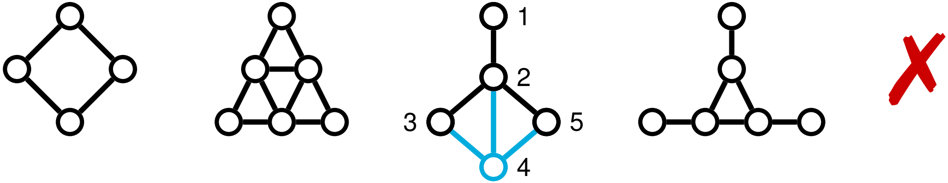

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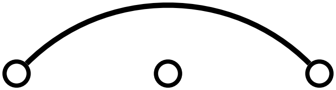

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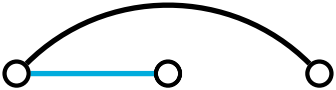

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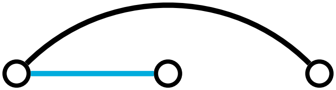

### Corollary

*Graphs having a* ***connected PEO*** *are connected* ***interval graphs****.*

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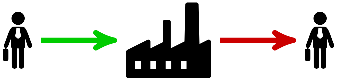

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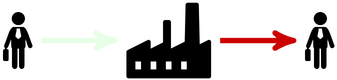

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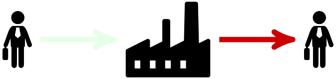

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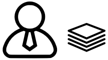

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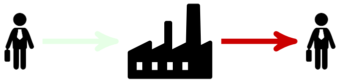

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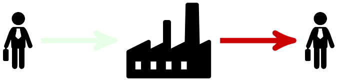

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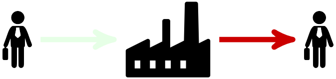

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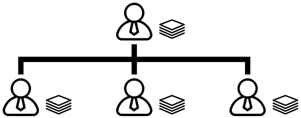

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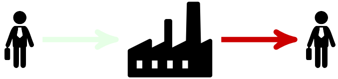

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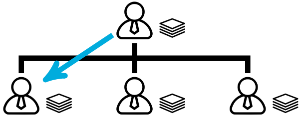

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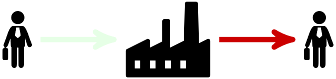

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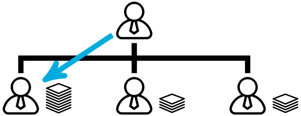

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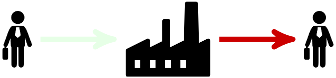

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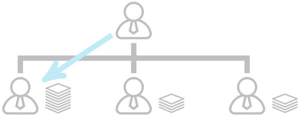

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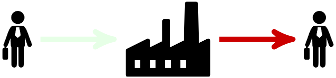

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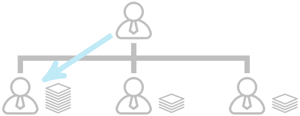

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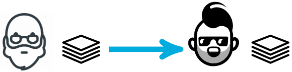

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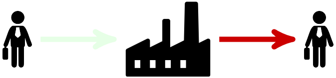

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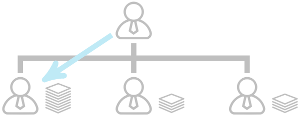

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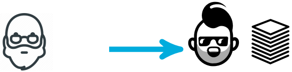

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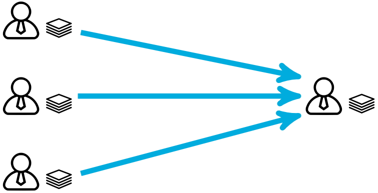

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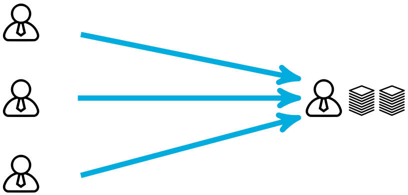

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- 3 sweeps of LBFS give proper interval ordering **[Corneil, 2004]**

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**Examples** are LBFS, LDFS, MCS, and MNS.

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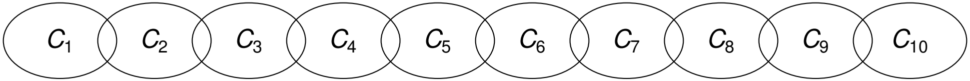

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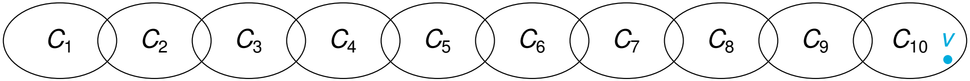

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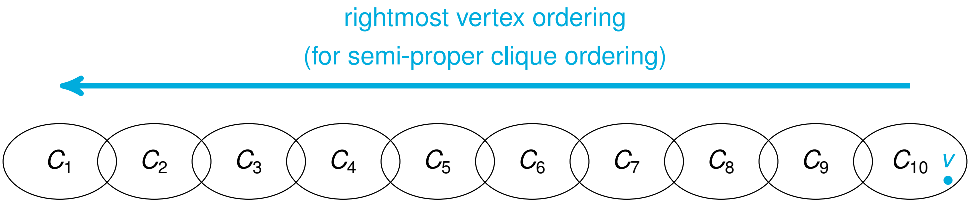
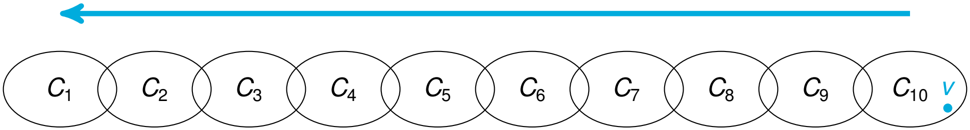


## THE ALGORITHM

if $r_\Phi(x) < r_\Phi(y)$, then $y \prec_\sigma x$

rightmost vertex ordering

(for semi-proper clique ordering)



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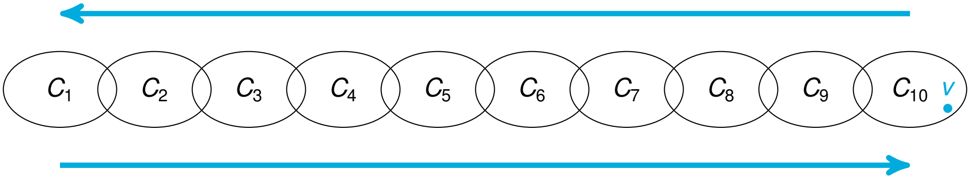


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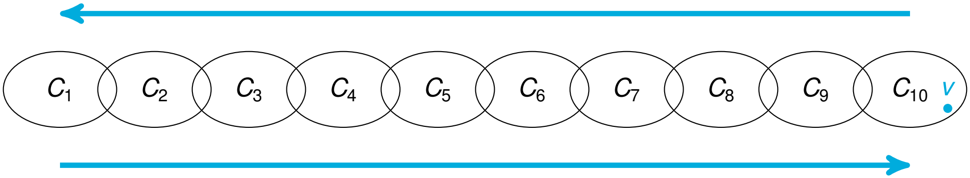


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Lemma

*A **lexicographic vertex ordering** of a semi-proper clique ordering is a **connected PEO**.*

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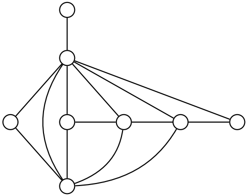
# HOW TO FIND THE END VERTEX?



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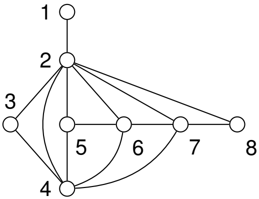

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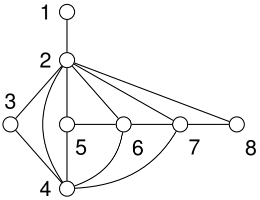


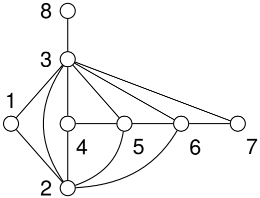

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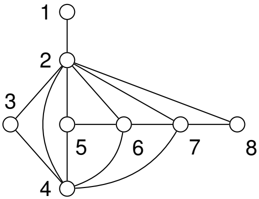


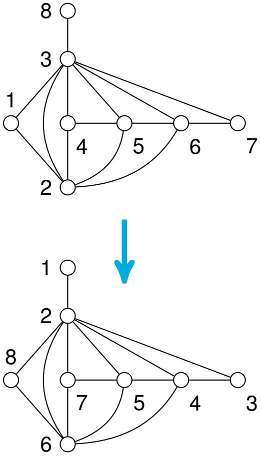

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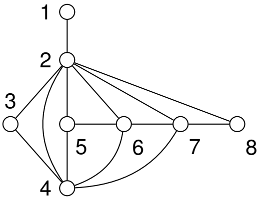


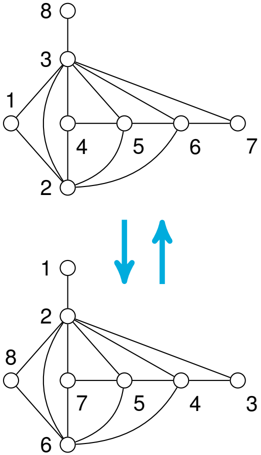

# PQ-TREES

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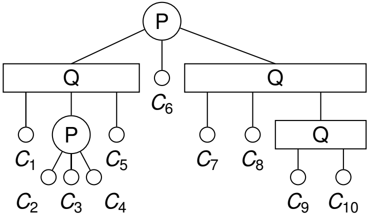

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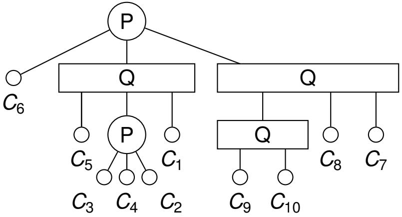

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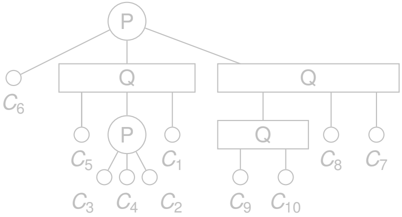


**Semi-proper clique ordering**

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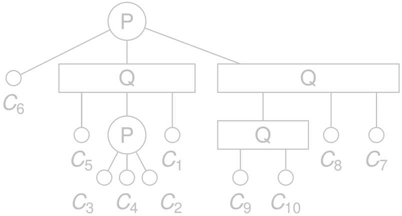
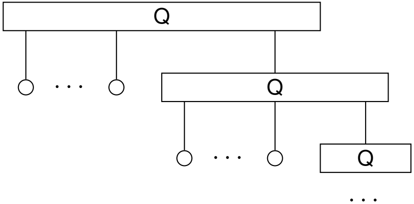



## Semi-proper clique ordering



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*Every semi-proper interval graph has **at most two** semi-proper clique orderings.*

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# HAMILTONIAN PATHS AND CYCLES

### Theorem (Bertossi, 1983; Chen et al, 1997; Broersma et al, 2015)

*Let $G$ be a proper interval graph.*

1. *$G$ has a Hamiltonian path if and only if $G$ is connected.*
2. *$G$ has a Hamiltonian cycle if and only if $G$ is 2-connected.*
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## Theorem (S, 2022)

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### Corollary

*The* ***longest path*** *and the* ***longest cycle*** *of an semi-proper interval graph can be found in* ***linear time****.*

## FURTHER RESEARCH

- complexity of problems that are hard or open on interval graphs

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**Thanks. . . !**