# Locally checkable problems parameterized by treewidth, clique-width and mim-width

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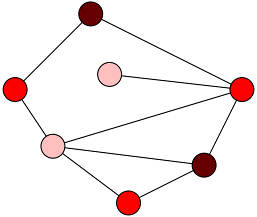
Joint work with:

- Flavia Bonomo-Braberman[1] (treewidth)
- Narmina Baghirova[2], Bernard Ries[2] and David Schindl[2] (clique-width)
- Felix Mann[2] (mim-width)

# Introductory problems

**$k$-coloring**



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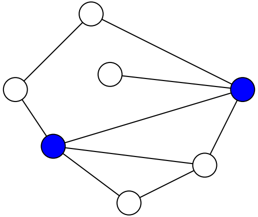

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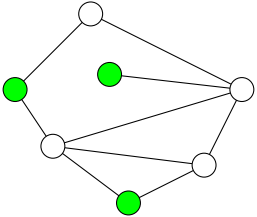

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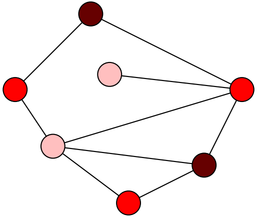

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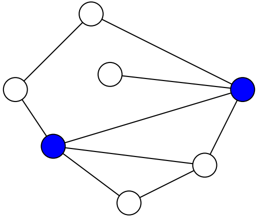

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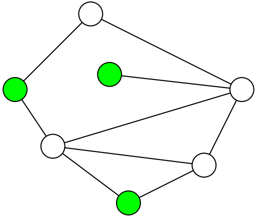

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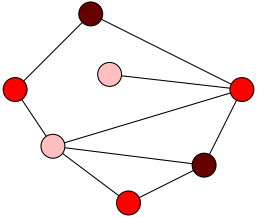

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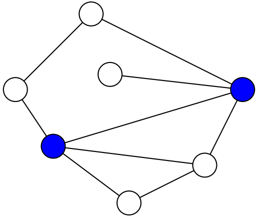

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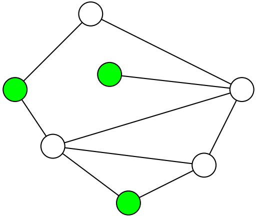

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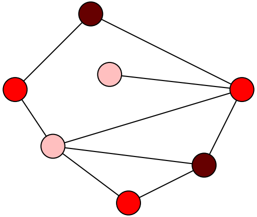

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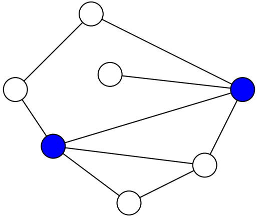

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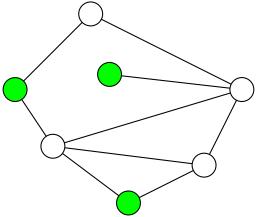

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$\rightarrow$ **Locally checkable problems**

# Related work

- Monadic Second Order Logic (Courcelle's theorem)
  [B. Courcelle 1990]

- DN-logic
  [B. Bergougnoux, J. Dreier and L. Jaffke 2022]

- Locally Checkable Vertex Partitioning (LCVP) problems
  [J.A. Telle 1994]

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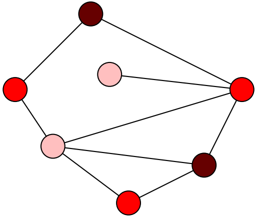

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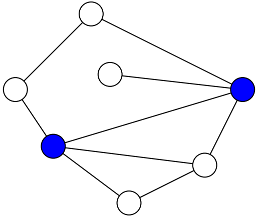

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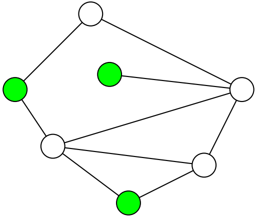

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Minimum Dominating Set:

$\textsc{Colors} = \{\mathbf{s}, \overline{\mathbf{s}}\}$

$\mathrm{W}(v,c) = 1$ if $c(v) = \mathbf{s}$, and $0$ otherwise

Order of weights: $\leq$

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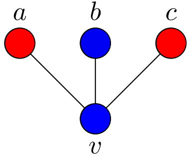

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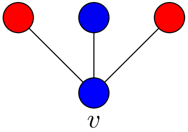


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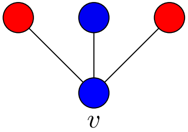


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We say $f$ is ***d*-stable** if it is color-counting and

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We can reduce Minimum Dominating Set in general graphs to a locally checkable problem (with 2 colors) in complete graphs (clique-width $\leq 2$).

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Standard dynamic programming algorithm using clique-width expressions.

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**Max-Cut** is color-counting with 2 colors, and **W[1]-hard** parameterized by clique-width.

[F.V. Fomin, P.A. Golovach, D. Lokshtanov and S. Saurabh 2014]

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Dynamic programming algorithm based on [B.M. Bui-Xuan, J.A. Telle and M. Vatshelle 2013].

Alternative: modeling with DN-logic.

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**Minimum Dominating Set** is 1-stable with 2 colors, and **W[1]-hard** parameterized by mim-width.
[F.V. Fomin, P.A. Golovach, J.F. Raymond 2018]

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**Max-Cut** is color-counting with 2 colors, and **NP-complete** on interval graphs (mim-width $\leq 1$).

[R. Adhikary, K. Bose, S. Mukherjee, B. Roy 2020]

# Adding global properties

- Treewidth: size, connectivity, acyclicity
- Clique-width: size, connectivity
- Mim-width: size, connectivity and any other expressible in DN-logic

# Applications

Using this framework we proved that:

| **Problem** | **clique-width** | **mim-width** |
|---|---|---|
| $[k]$-Roman domination | (linear$^*$) FPT | XP |
| Conflict-free $k$-coloring<br>Bhyravarapu, Hartmann, Kalyanasundaram and Vinod Reddy, 2021: similar results for clique-width | (linear$^*$) FPT | XP |
| b-coloring with fixed number of colors<br>Jaffke, Lima and Lokshtanov, 2020: XP parameterized by clique-width with unfixed number of colors | (linear$^*$) FPT | XP |
| $k$-community | XP | — |

and similar results for some variations of these problems.

(*) if a clique-width expression is given as input.

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Given a graph $G$, compute the minimum weight of a function $f\colon V \to \{0, \ldots, k+1\}$ such that

$$f(v) + \sum_{\substack{u \in N(v) \\ f(u) \geq 1}} (f(u) - 1) \geq k \quad \forall\, v \in V.$$

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→ **FPT clique-width**
→ **XP mim-width**

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# Our framework (complete formulation)

## $r$-locally checkable problems

Given a graph $G$ and

- $\textsc{Colors}$: a set of colors,
- $L_v$: for every vertex $v$, a subset of $\textsc{Colors}$ of allowed colors,
- $\ell_e$: for every edge $e$, a label,
- $(\textsc{Weights}, \preceq, \oplus)$: a weight set,
- $\mathrm{w}(v, c)$: a weight function (input: vertex $v$ and coloring $c$ of $N^r[v]$, output: a weight),
- $check(v, c)$: a check function (input: vertex $v$ and coloring $c$ of $N^r[v]$, output: true or false)

find the minimum weight of a coloring $c$ such that $check(v, c|_{N^r[v]}) = \textsc{True}\ \forall\, v \in V(G)$.

| Width | $\lvert\textsc{Colors}\rvert$ | $\ell_e$ | $check, \mathrm{w}$ | Global properties |
|---|---|---|---|---|
| tw | Polynomial | Yes | Polynomial partial neighborhood system | Size, connectivity, acyclicity |
| cw | Constant or log | No | Color-counting | Size, connectivity |
| mimw | Constant | No | $d$-stable | Size, connectivity and any other expressible in DN-logic |