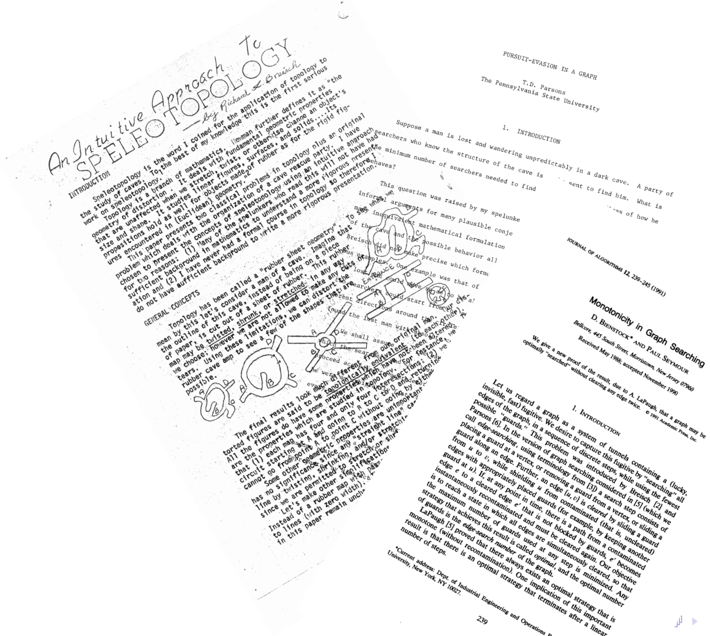
Mixed graph searching games are all monotone

Christophe PAUL
(CNRS – Univ. Montpellier, LIRMM, France)

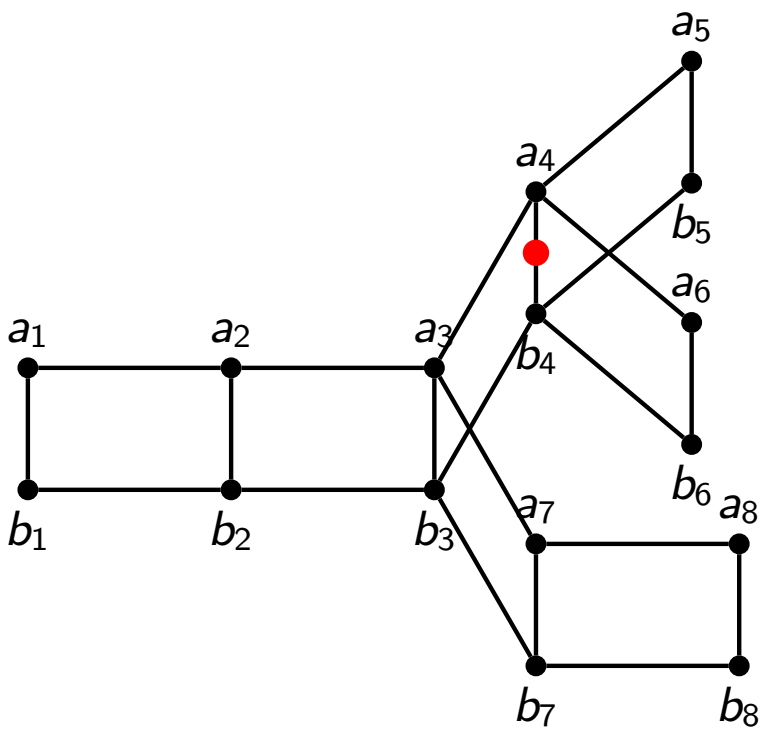
Joint work with
Dimitrios M. Thilikos and **Guillaume Mescoff**

September 22, 2022

Back to the roots - Let us revisit mixed search games



Mixed search game

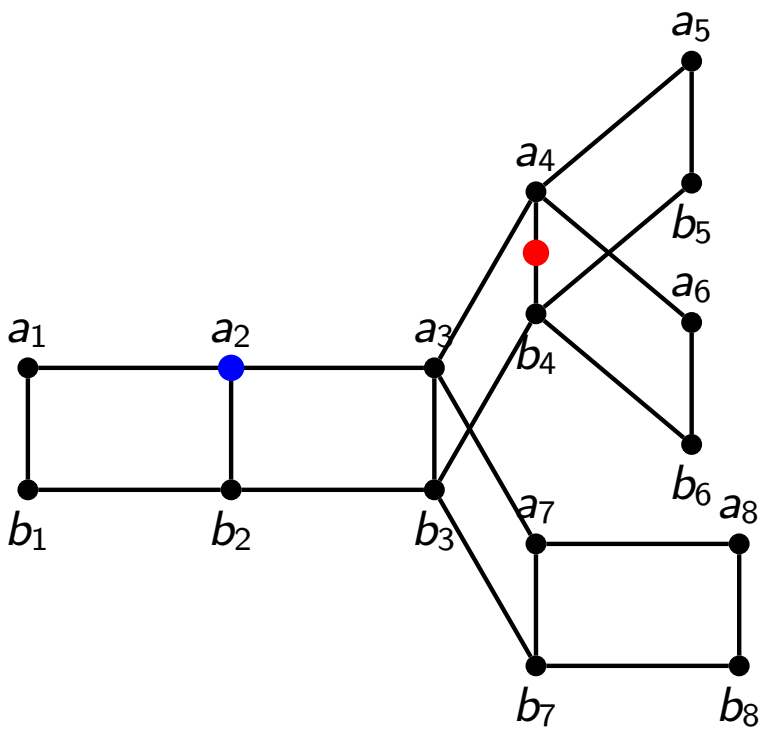


The fugitive is located on an edge

A play of a mixed search game is a sequence

$$\mathcal{P} = \langle \emptyset, a_4\, b_4, \ldots \rangle$$

Mixed search game



The fugitive is located on an edge

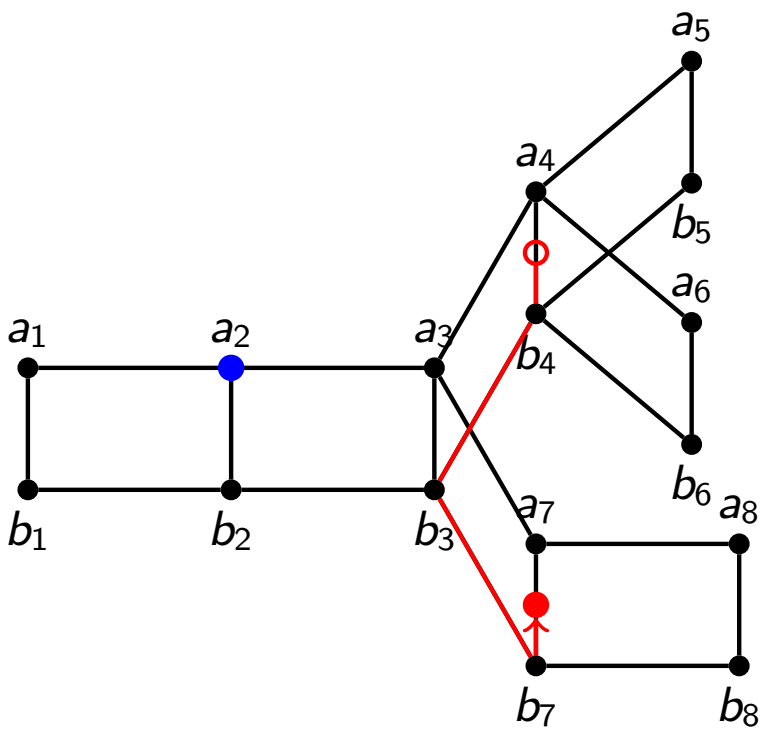
Searchers' move :

$\rightsquigarrow$ placement of a searcher on a vertex

A play of a mixed search game is a sequence

$$\mathcal{P} = \langle \emptyset, a_4 b_4, \{a_2\}, \dots \rangle$$

Mixed search game



The fugitive is located on an edge

$\rightsquigarrow$ the fugitive slides along a pathway

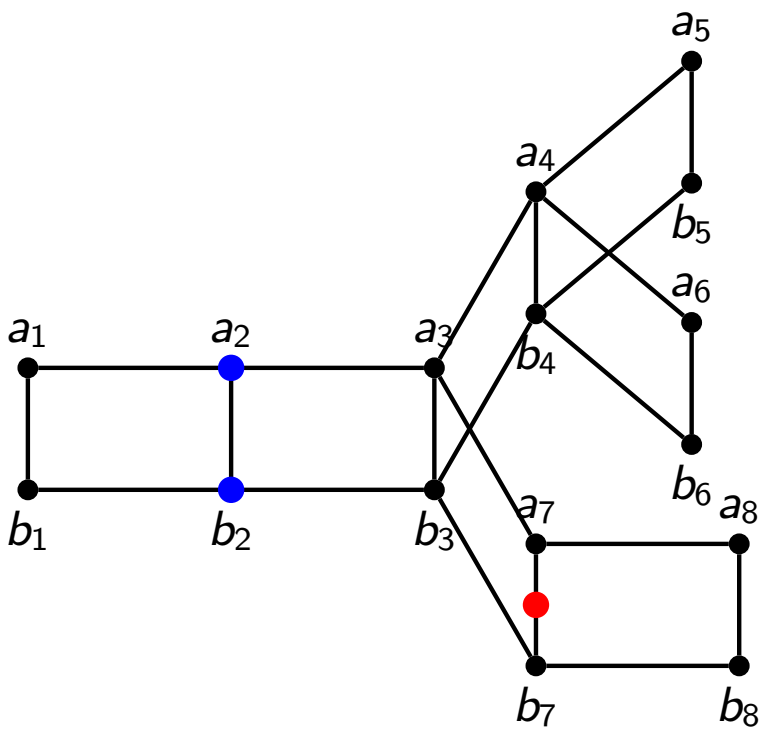
Searchers' move :

$\rightsquigarrow$ placement of a searcher on a vertex

A play of a mixed search game is a sequence

$$\mathcal{P} = \langle \emptyset, a_4 b_4, \{a_2\}, a_7 b_7, \dots \rangle$$

Mixed search game



The fugitive is located on an edge

$\rightsquigarrow$ the fugitive slides along a pathway

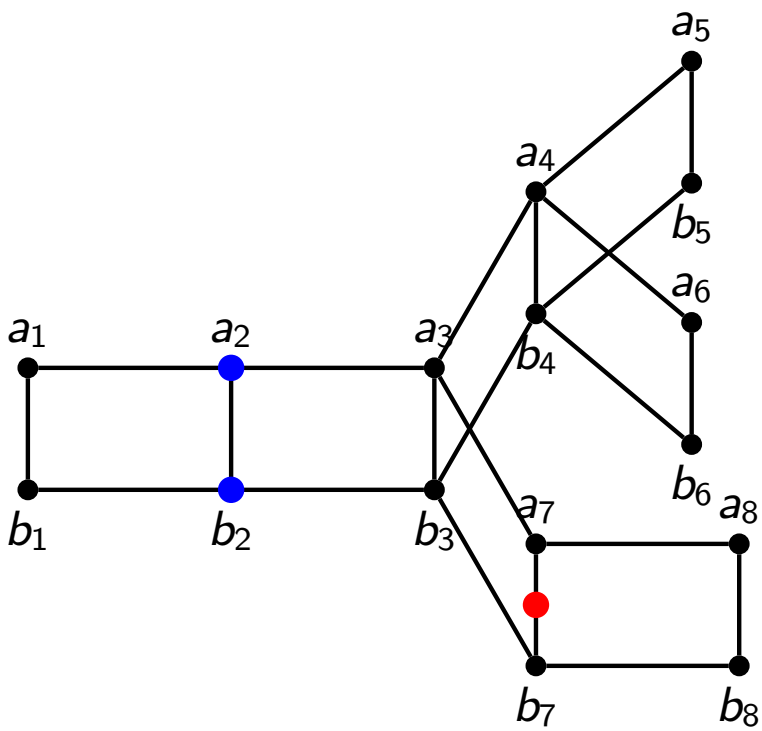
Searchers' move :

$\rightsquigarrow$ placement of a searcher on a vertex

A play of a mixed search game is a sequence

$$\mathcal{P} = \langle \emptyset, a_4 b_4, \{a_2\}, a_7 b_7, \{a_2, b_2\}, \dots \rangle$$

Mixed search game



The fugitive is located on an edge

$\rightsquigarrow$ the fugitive slides along a pathway

$\rightsquigarrow$ the fugitive may stay at its location

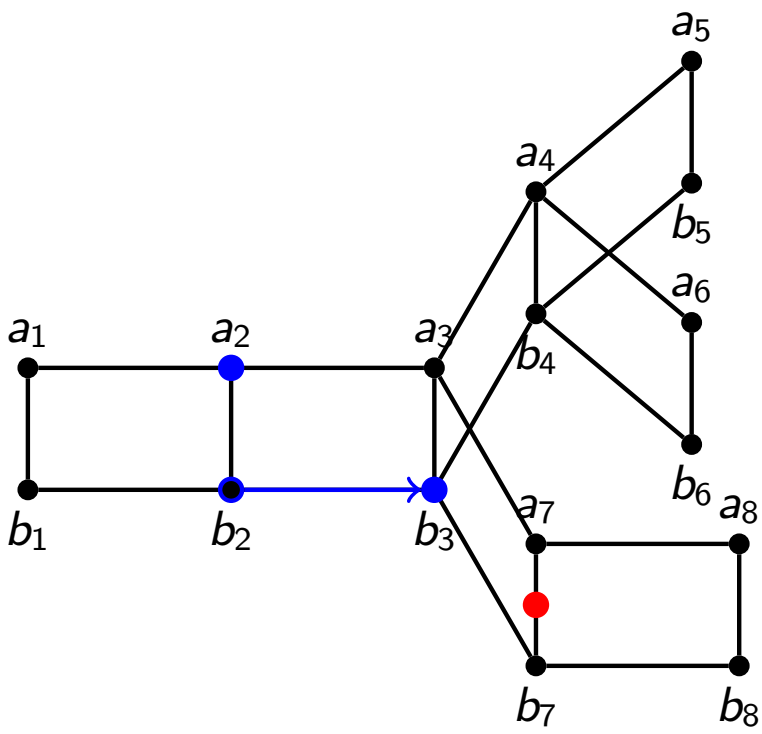
Searchers' move :

$\rightsquigarrow$ placement of a searcher on a vertex

A play of a mixed search game is a sequence

$$\mathcal{P} = \langle \emptyset, a_4 b_4, \{a_2\}, a_7 b_7, \{a_2, b_2\}, a_7 b_7, \ldots \rangle$$

Mixed search game



The fugitive is located on an edge

$\rightsquigarrow$ the fugitive slides along a pathway

$\rightsquigarrow$ the fugitive may stay at its location

Searchers' move :

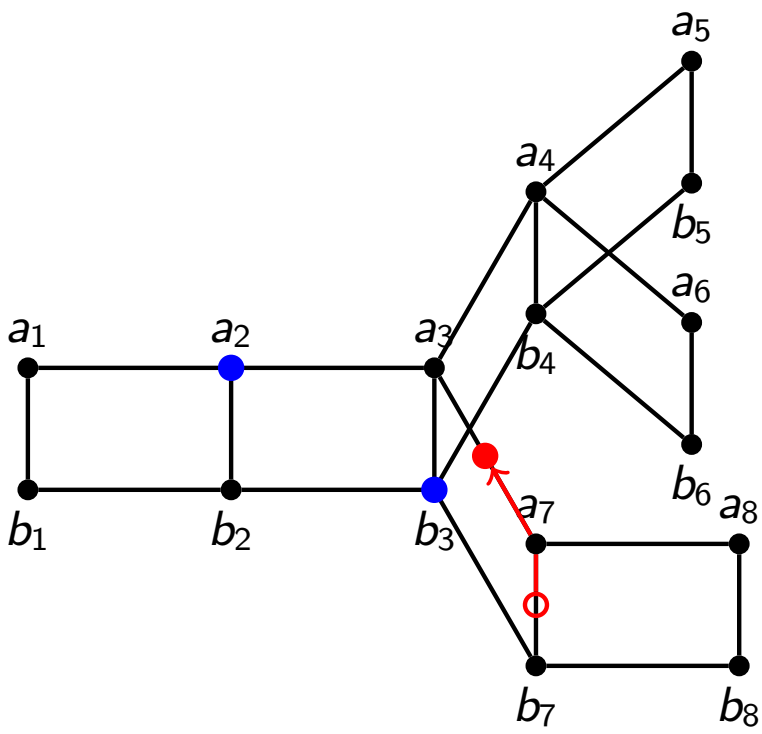
$\rightsquigarrow$ placement of a searcher on a vertex

$\rightsquigarrow$ sliding a searcher along an edge

A play of a mixed search game is a sequence

$$\mathcal{P} = \langle \emptyset, a_4\,b_4, \{a_2\}, a_7\,b_7, \{a_2, b_2\}, a_7\,b_7, \{a_2, b_3\}, \ldots \rangle$$

Mixed search game



The fugitive is located on an edge

$\rightsquigarrow$ the fugitive slides along a pathway

$\rightsquigarrow$ the fugitive may stay at its location

Searchers' move :

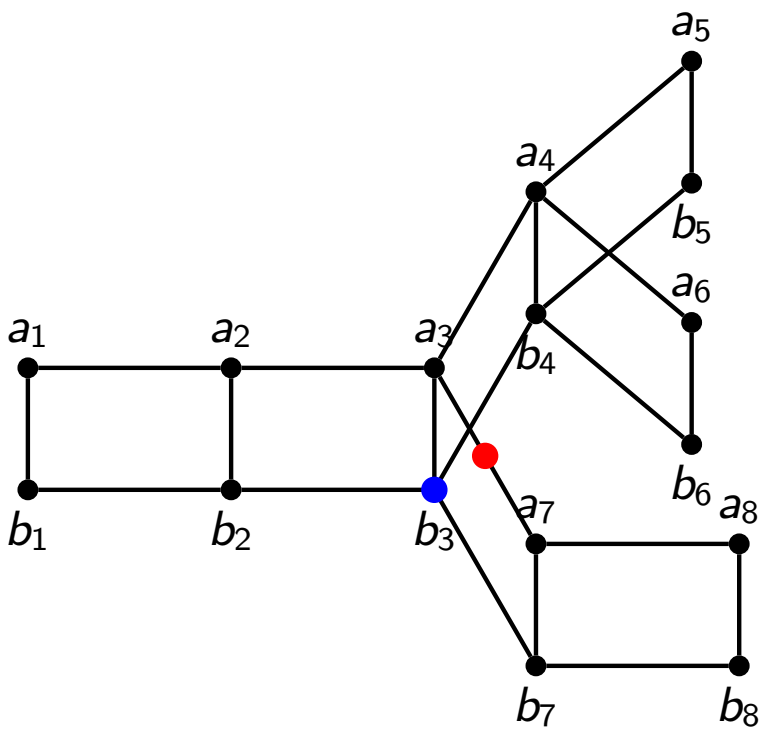
$\rightsquigarrow$ placement of a searcher on a vertex

$\rightsquigarrow$ sliding a searcher along an edge

A play of a mixed search game is a sequence

$$\mathcal{P} = \langle \emptyset, a_4 b_4, \{a_2\}, a_7 b_7, \{a_2, b_2\}, a_7 b_7, \{a_2, b_3\}, a_3 a_7, \dots \rangle$$

Mixed search game



The fugitive is located on an edge

$\rightsquigarrow$ the fugitive slides along a pathway

$\rightsquigarrow$ the fugitive may stay at its location

Searchers' move :

$\rightsquigarrow$ placement of a searcher on a vertex

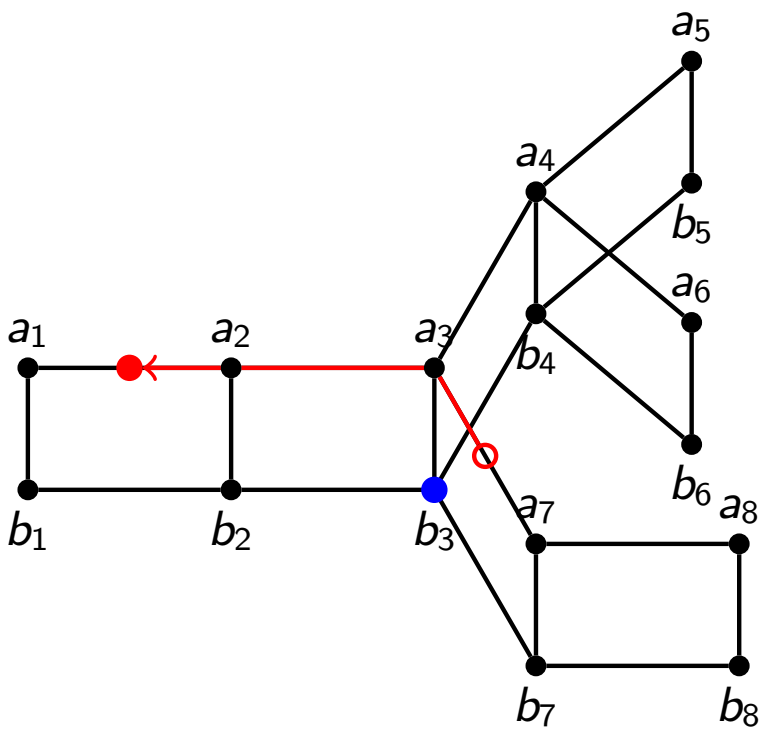
$\rightsquigarrow$ sliding a searcher along an edge

$\rightsquigarrow$ removal of a searcher

A play of a mixed search game is a sequence

$$\mathcal{P} = \langle \emptyset, a_4 b_4, \{a_2\}, a_7 b_7, \{a_2, b_2\}, a_7 b_7, \{a_2, b_3\}, a_3 a_7, \{b_3\}, \dots \rangle$$

Mixed search game



The fugitive is located on an edge

$\rightsquigarrow$ the fugitive slides along a pathway

$\rightsquigarrow$ the fugitive may stay at its location

Searchers' move :

$\rightsquigarrow$ placement of a searcher on a vertex

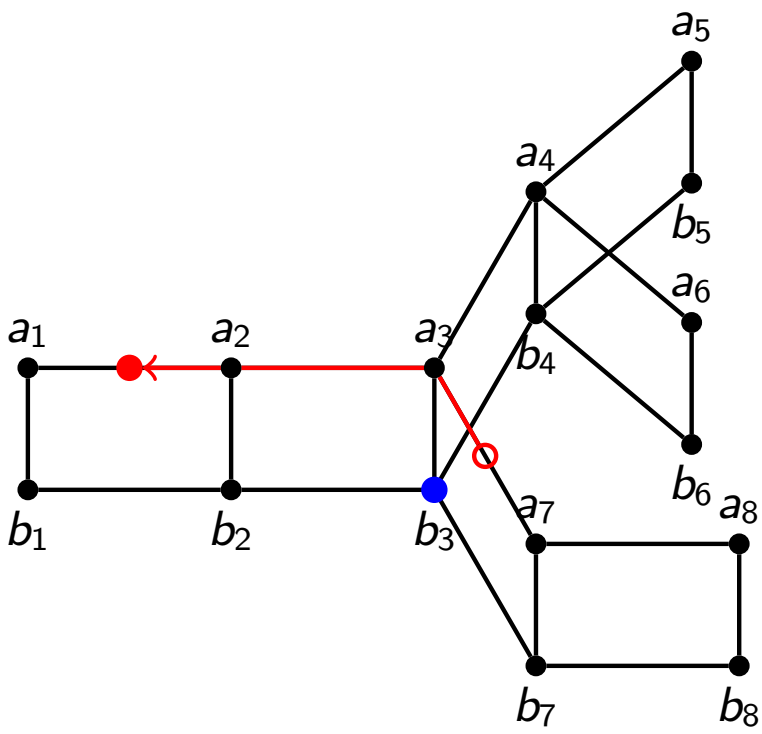
$\rightsquigarrow$ sliding a searcher along an edge

$\rightsquigarrow$ removal of a searcher

A play of a mixed search game is a sequence

$$\mathcal{P} = \langle \emptyset, a_4 b_4, \{a_2\}, a_7 b_7, \{a_2, b_2\}, a_7 b_7, \{a_2, b_3\}, a_3 a_7, \{b_3\}, a_1 a_2, \dots \rangle$$

Mixed search game



The fugitive is located on an edge

$\rightsquigarrow$ the fugitive slides along a pathway

$\rightsquigarrow$ the fugitive may stay at its location

Searchers' move :

$\rightsquigarrow$ placement of a searcher on a vertex

$\rightsquigarrow$ sliding a searcher along an edge

$\rightsquigarrow$ removal of a searcher

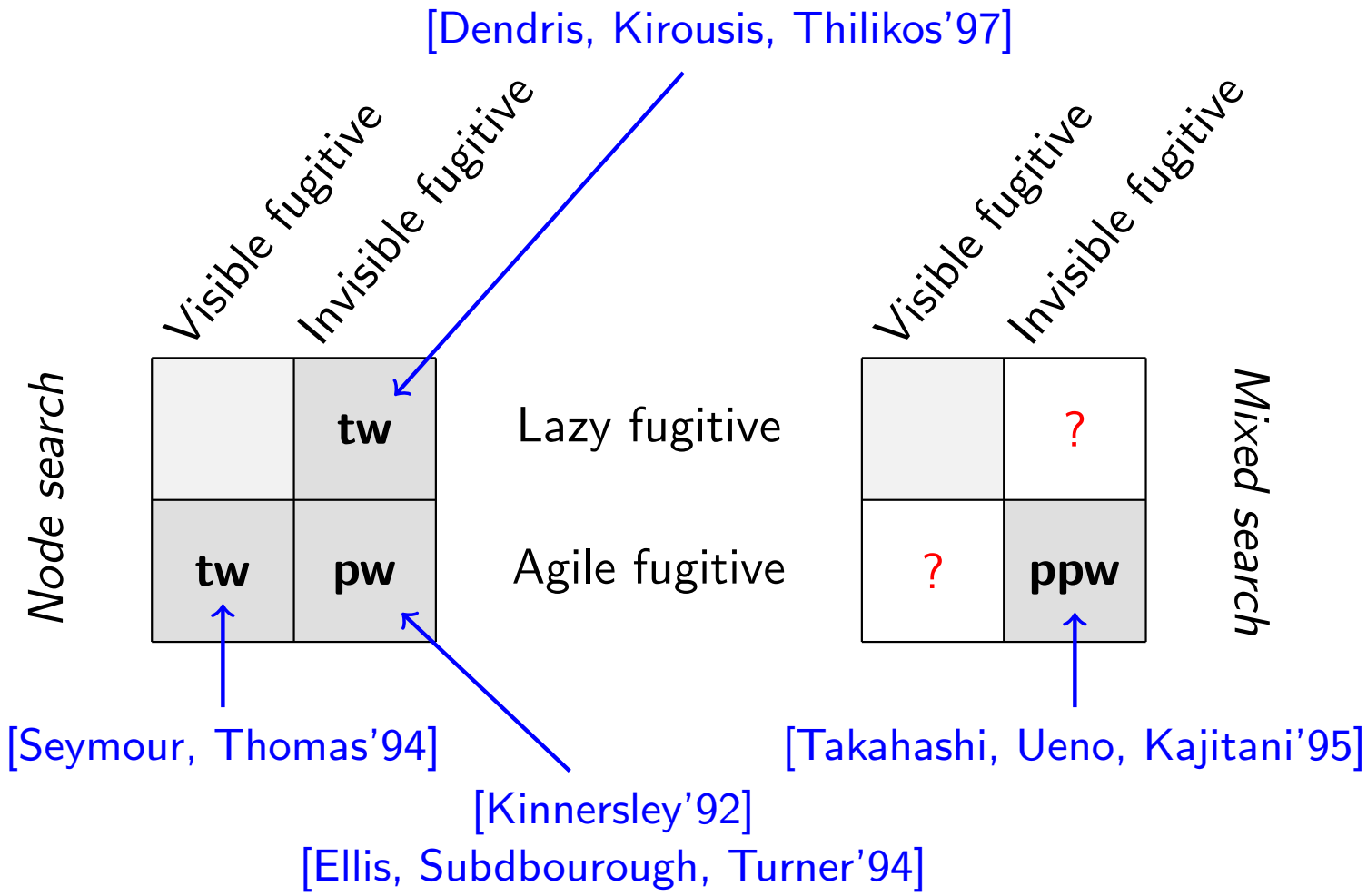
A play of a mixed search game is a sequence

$$\mathcal{P} = \langle \emptyset, a_4 b_4, \{a_2\}, a_7 b_7, \{a_2, b_2\}, a_7 b_7, \{a_2, b_3\}, a_3 a_7, \{b_3\}, a_1 a_2, \dots \rangle$$

$\rightsquigarrow$ The fugitive is captured if it cannot escape its location: the two vertices incident to its location are occupied by searchers.

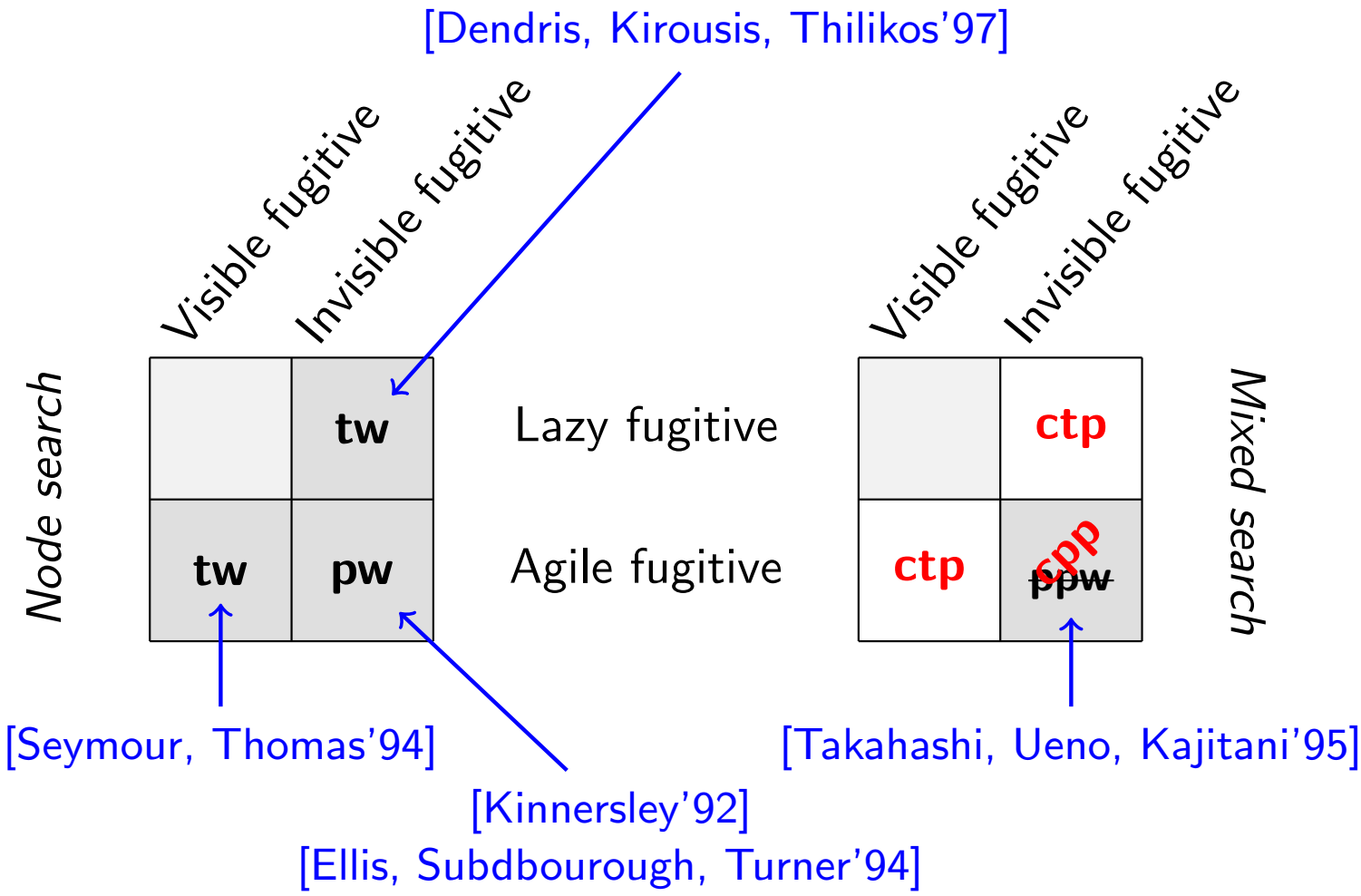
Known results on (node/mixed) search games (1/3)

$\rightsquigarrow$ width parameters



Known results on (node/mixed) search games (1/3)

$\rightsquigarrow$ width parameters

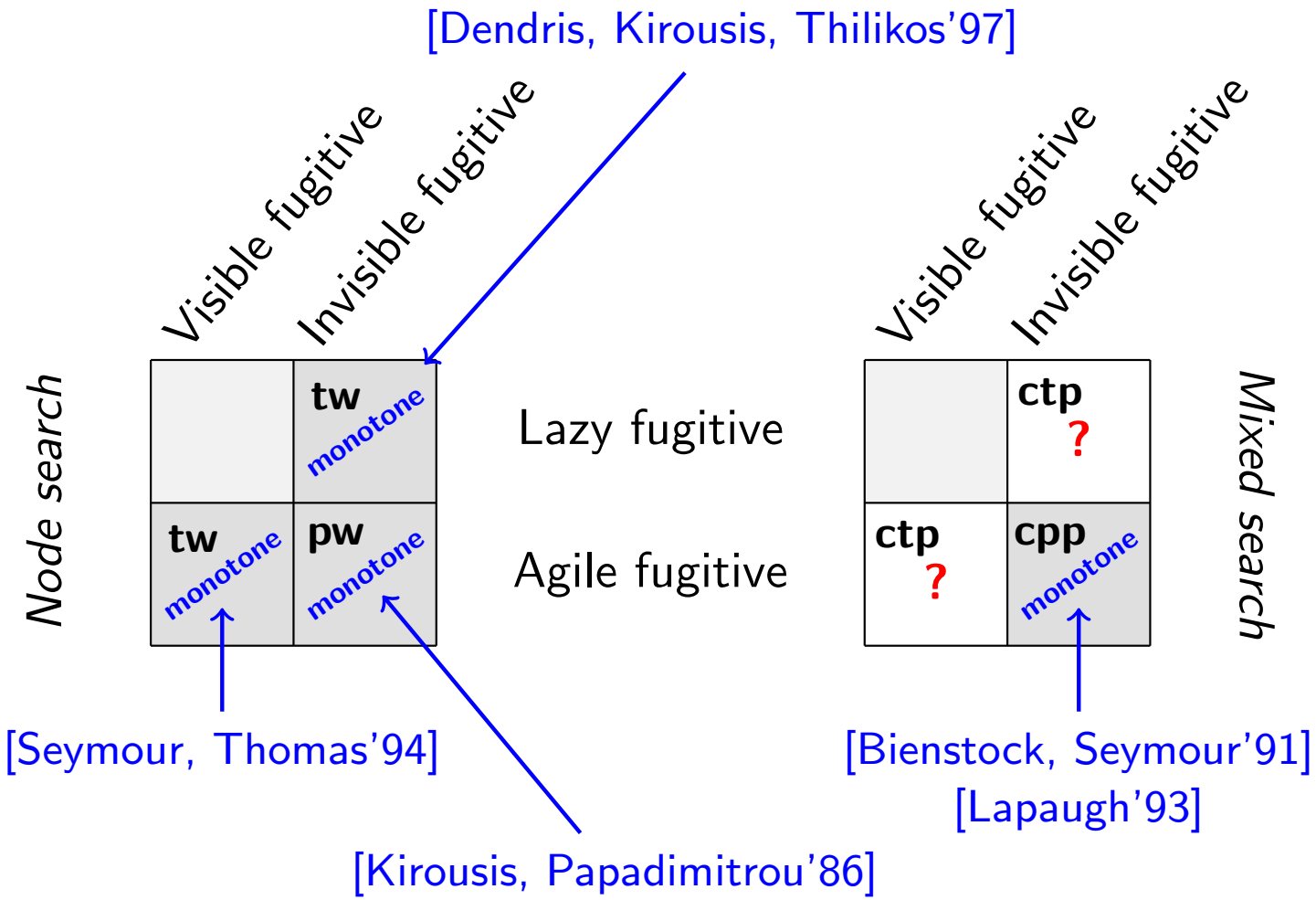


ctp - Cartesian Tree Product number [Harvey'14]; also known as

la - Largeur arborescente [Colin de Verdière'98]

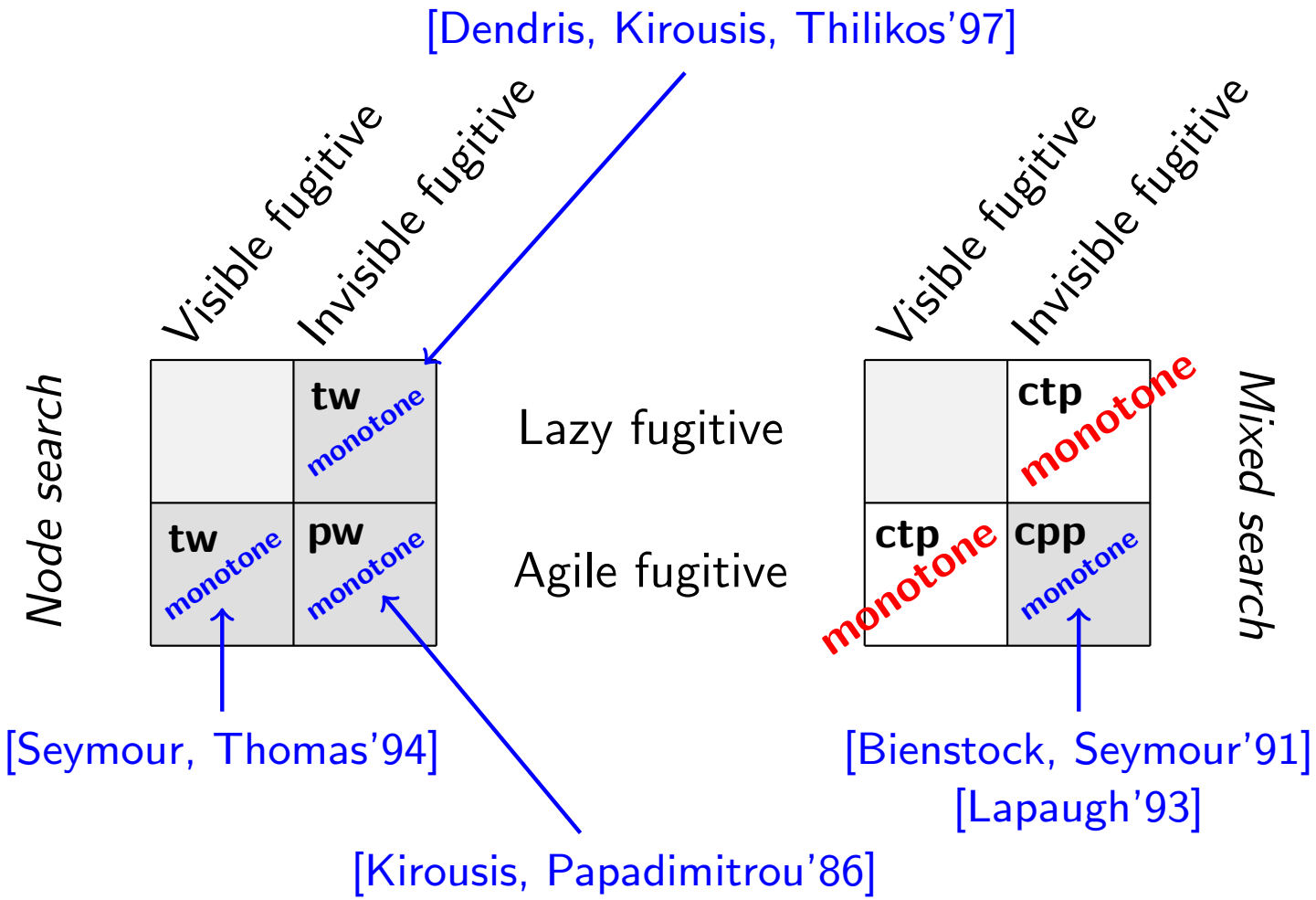
Known results on (node/mixed) search games (2/3)

$\rightsquigarrow$ monotonicity (recontamination does not help the capture)



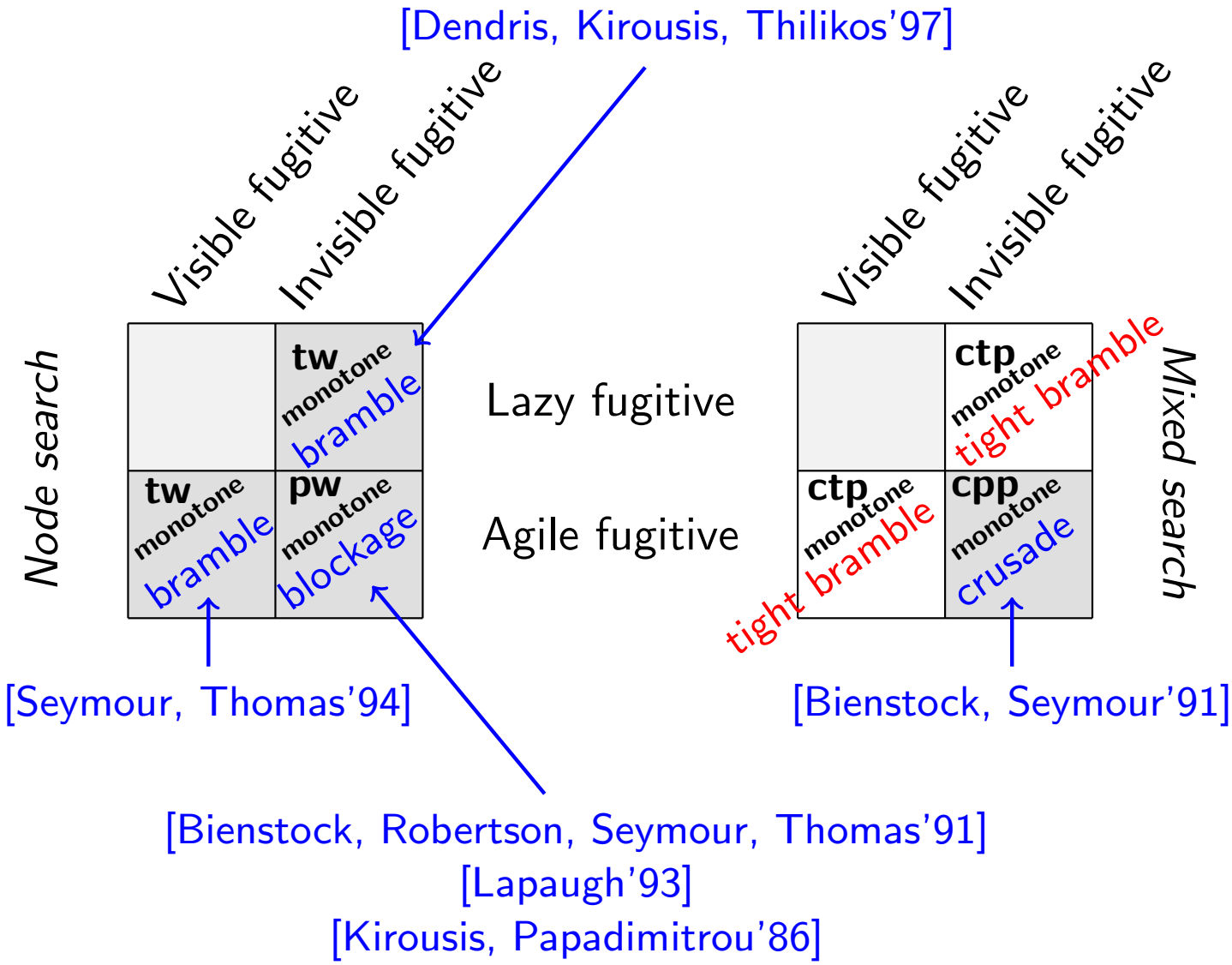
Known results on (node/mixed) search games (2/3)

$\rightsquigarrow$ monotonicity (recontamination does not help the capture)



Known results on (node/mixed) search games (3/3)

$\rightsquigarrow$ obstacle / certificate



Our result

Theorem:
Let G be a graph and $k \in \mathbb{N}$. Then the following conditions are equivalent:

1. G has a loose tree-decomposition of width k;
2. $\mathbf{ctp}(G) \leq k \rightsquigarrow G$ is a minor of $T^{(k)} = T \square K_k$;
3. every tight bramble of G has order at most k;
4. $\mathrm{avms}(G) \leq k \rightsquigarrow$ the mixed search number against an agile and visible fugitive is at most k;
5. $\mathrm{mavms}(G) \leq k \rightsquigarrow$ the monotone mixed search number against an agile and visible fugitive is at most k.

Mixed search strategy (against a visible fugitive)

A mixed search strategy is a function

$$\mathbf{s}_G : 2^{V(G)} \times E(G) \to 2^{V(G)}$$

st. $\forall (S, e) \in 2^{V(G)} \times E(G)$, $\big(S, \mathbf{s}_G(S, e)\big)$ is a legitimate searchers' move:

$\rightsquigarrow$ [Placement of a searcher]: $\mathbf{s}_G(S, e) = S \cup \{x\}$;

$\rightsquigarrow$ [Removal of a searcher]: $\mathbf{s}_G(S, e) = S \setminus \{x\}$;

$\rightsquigarrow$ [Sliding on an edge]: $\mathbf{s}_G(S, e) \ominus S = \{x, y\}$ and $xy \in E(G)$.

Mixed search strategy (against a visible fugitive)

A mixed search strategy is a function

$$\mathbf{s}_G : 2^{V(G)} \times E(G) \to 2^{V(G)}$$

st. $\forall (S, e) \in 2^{V(G)} \times E(G)$, $\big(S, \mathbf{s}_G(S, e)\big)$ is a legitimate searchers' move:

$\rightsquigarrow$ [Placement of a searcher]: $\mathbf{s}_G(S, e) = S \cup \{x\}$;

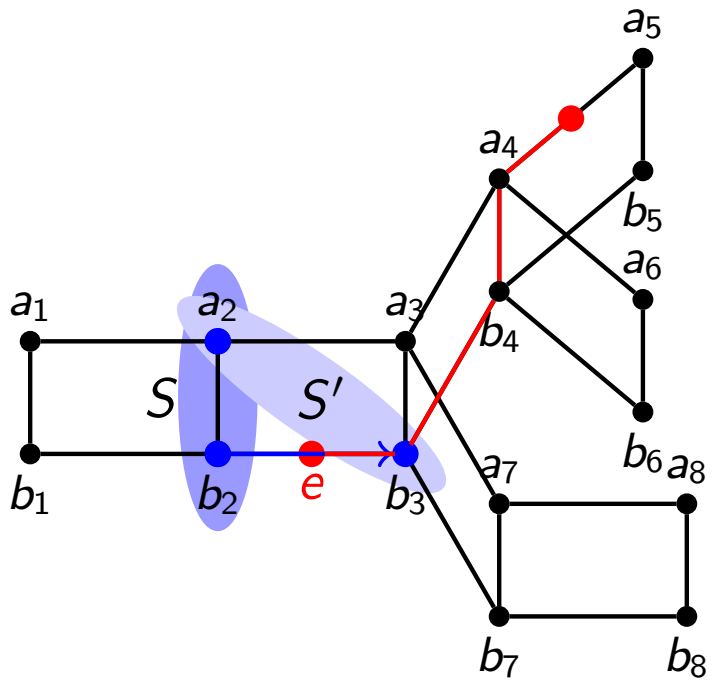
$\rightsquigarrow$ [Removal of a searcher]: $\mathbf{s}_G(S, e) = S \setminus \{x\}$;

$\rightsquigarrow$ [Sliding on an edge]: $\mathbf{s}_G(S, e) \ominus S = \{x, y\}$ and $xy \in E(G)$.

A legitimate searchers' move (S, S') clears the following set of edges:

$$\mathrm{Clear}_G(S, S') = \begin{cases} \{xy \mid y \in S\} \cap E(G), & \text{if } S' \setminus S = \{x\} \\ \emptyset, & \text{if } S' \setminus S = \emptyset. \end{cases}$$

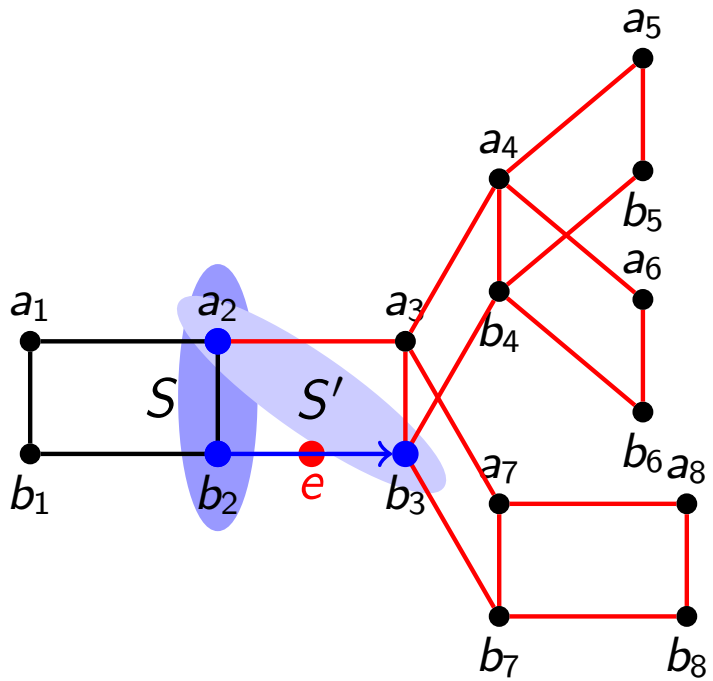
The (agile) fugitive strategy (1/2)



The set of *accessible* edges of G from e is:

$$\mathrm{Acc}_G(S, e, S') = \left\{ e' \in E(G) \setminus \binom{S'}{2} \mid \exists \text{ an } (S, S')\text{-avoiding } (e, e')\text{-pathway} \right\}.$$

The (agile) fugitive strategy (1/2)



The set of *accessible* edges of G from e is:

$$\mathrm{Acc}_G(S, e, S') = \left\{ e' \in E(G) \setminus \binom{S'}{2} \mid \exists \text{ an } (S, S')\text{-avoiding } (e, e')\text{-pathway} \right\}.$$

The fugitive space is:

$$\mathrm{freeSp}_G(S, e, S') = (\{e\} \setminus \mathrm{Clear}_G(S, S')) \cup \mathrm{Acc}_G(S, e, S').$$

Search program and mixed search number

A fugitive strategy on G is a pair $(e_1, \mathbf{f}_G)$ with $e_1 \in E(G)$ and

$$\mathbf{f}_G : 2^{V(G)} \times E(G) \times 2^{V(G)} \to E \cup \{\star\}.$$

and such that

- if $\text{freeSp}_G(S, e, S') \neq \emptyset$, then $\mathbf{f}_G(S, e, S') \in \text{freeSp}_G(S, e, S')$
- otherwise $\mathbf{f}_G(S, e, S') = \star$ (the fugitive is captured).

Search program and mixed search number

A fugitive strategy on G is a pair $(e_1, \mathbf{f}_G)$ with $e_1 \in E(G)$ and

$$\mathbf{f}_G : 2^{V(G)} \times E(G) \times 2^{V(G)} \to E \cup \{\star\}.$$

A search program on G is a pair $\left(\mathbf{s}_G, (e_1, \mathbf{f}_G)\right)$ *generating* a play:

$$\mathcal{P}(\mathbf{s}_G, e_1, \mathbf{f}_G) = \langle S_0, e_1, S_1, \ldots, S_{i-1}, e_i, S_i, e_{i+1}, \ldots \rangle$$

where for each $i \geq 1$,

- $S_i = \mathbf{s}_G(S_{i-1}, e_{i-1})$ and
- $e_{i+1} = \mathbf{f}_G(S_{i-1}, e_i, S_i)$.

Search program and mixed search number

A fugitive strategy on G is a pair $(e_1, \mathbf{f}_G)$ with $e_1 \in E(G)$ and

$$\mathbf{f}_G : 2^{V(G)} \times E(G) \times 2^{V(G)} \to E \cup \{\star\}.$$

A search program on G is a pair $\big(\mathbf{s}_G, (e_1, \mathbf{f}_G)\big)$ *generating* a play:

$$\mathcal{P}(\mathbf{s}_G, e_1, \mathbf{f}_G) = \langle S_0, e_1, S_1, \ldots, S_{i-1}, e_i, S_i, e_{i+1}, \ldots \rangle$$

The cost of a search program is:

$$\mathbf{cost}\big(\mathcal{P}(\mathbf{s}_G, e_1, \mathbf{f}_G)\big) = \max_{i \geq 1} |S_i|$$

Search program and mixed search number

A fugitive strategy on G is a pair $(e_1, \mathbf{f}_G)$ with $e_1 \in E(G)$ and

$$\mathbf{f}_G : 2^{V(G)} \times E(G) \times 2^{V(G)} \to E \cup \{\star\}.$$

A search program on G is a pair $\big(\mathbf{s}_G, (e_1, \mathbf{f}_G)\big)$ *generating* a play:

$$\mathcal{P}(\mathbf{s}_G, e_1, \mathbf{f}_G) = \langle S_0, e_1, S_1, \ldots, S_{i-1}, e_i, S_i, e_{i+1}, \ldots \rangle$$

The cost of a search program is:

$$\mathbf{cost}\big(\mathcal{P}(\mathbf{s}_G, e_1, \mathbf{f}_G)\big) = \max_{i \geq 1} |S_i|$$

$\rightsquigarrow$ The mixed search number (against an agile and visible fugitive) is:

$$\mathbf{avms}(G) = \min_{\mathbf{s}_G \, winning} \max \Big\{ \mathbf{cost}\big(\mathcal{P}(\mathbf{s}_G, e_1, \mathbf{f}_G)\big) \mid (e_1, \mathbf{f}_G) \text{ is a fugitive strategy} \Big\}.$$

Monotone search program

$\rightsquigarrow$ The search program $(\mathbf{s}_G, e_1, \mathbf{f}_G)$ is monotone if, in $\mathcal{P}(\mathbf{s}_G, e_1, \mathbf{f}_G)$, for every $i \geq 1$, the edge e_{i+1} has not been cleared at any step prior to i, that is:

$$\forall j \leq i, e_i \notin \mathrm{Clear}_G(S_{j-1}, S_j).$$

Monotone search program

$\rightsquigarrow$ The search program $(\mathbf{s}_G, e_1, \mathbf{f}_G)$ is monotone if, in $\mathcal{P}(\mathbf{s}_G, e_1, \mathbf{f}_G)$, for every $i \geq 1$, the edge e_{i+1} has not been cleared at any step prior to i, that is:

$$\forall j \leq i, e_i \notin \mathrm{Clear}_G(S_{j-1}, S_j).$$

$\rightsquigarrow$ A search strategy $\mathbf{s}_G$ is monotone if for every fugitive strategy $(e_1, \mathbf{f}_G)$, the program $\left(\mathbf{s}_G, (e_1, \mathbf{f}_G)\right)$ is monotone.

Monotone search program

$\rightsquigarrow$ The search program $(\mathbf{s}_G, e_1, \mathbf{f}_G)$ is monotone if, in $\mathcal{P}(\mathbf{s}_G, e_1, \mathbf{f}_G)$, for every $i \geq 1$, the edge e_{i+1} has not been cleared at any step prior to i, that is:

$$\forall j \leq i, e_i \notin \mathrm{Clear}_G(S_{j-1}, S_j).$$

$\rightsquigarrow$ A search strategy $\mathbf{s}_G$ is monotone if for every fugitive strategy $(e_1, \mathbf{f}_G)$, the program $\big(\mathbf{s}_G, (e_1, \mathbf{f}_G)\big)$ is monotone.

$\rightsquigarrow$ The monotone mixed search number is:

$$\mathbf{mavms}(G) = \min\big\{\mathbf{cost}(\mathbf{s}_G) \mid \mathbf{s}_G \text{ is a monotone winning search strategy}\big\}.$$

Our result

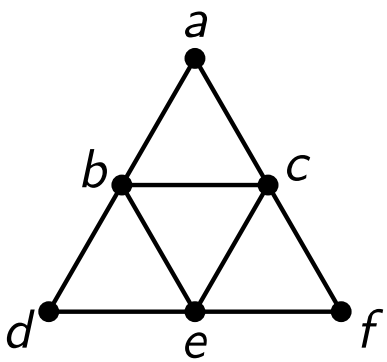
Theorem:
Let G be a graph and $k \in \mathbb{N}$. Then the following conditions are equivalent:

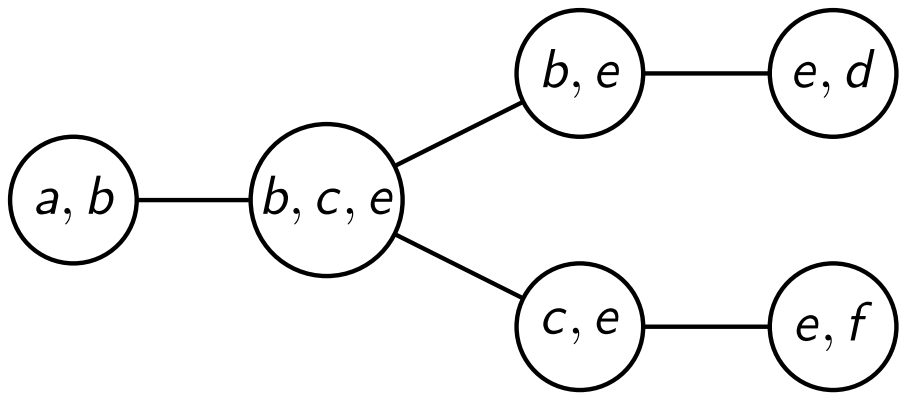
1. G has a loose tree-decomposition of width k;
2. $\mathbf{ctp}(G) \leq k \rightsquigarrow G$ is a minor of $T^{(k)} = T \square K_k$;
3. every tight bramble of G has order at most k;
4. $\mathrm{avms}(G) \leq k \rightsquigarrow$ the mixed search number against an agile and visible fugitive is at most k;
5. $\mathrm{mavms}(G) \leq k \rightsquigarrow$ the monotone mixed search number against an agile and visible fugitive is at most k.

Loose tree-decomposition

A loose tree-decomposition is a pair $\mathcal{D} = (T, \chi)$ such that T is a tree and $\chi : V(T) \to 2^{V(G)}$ satisfying the following properties:

(L1) $\forall x \in V(G)$, $T_x = \{t \in V(T) \mid x \in \chi(t)\}$ is non-empty and connected in T.



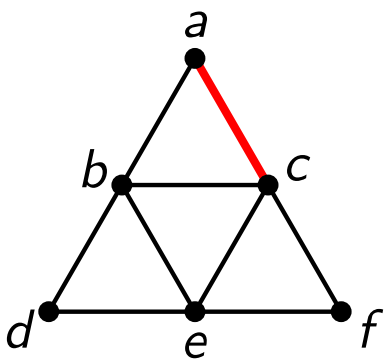


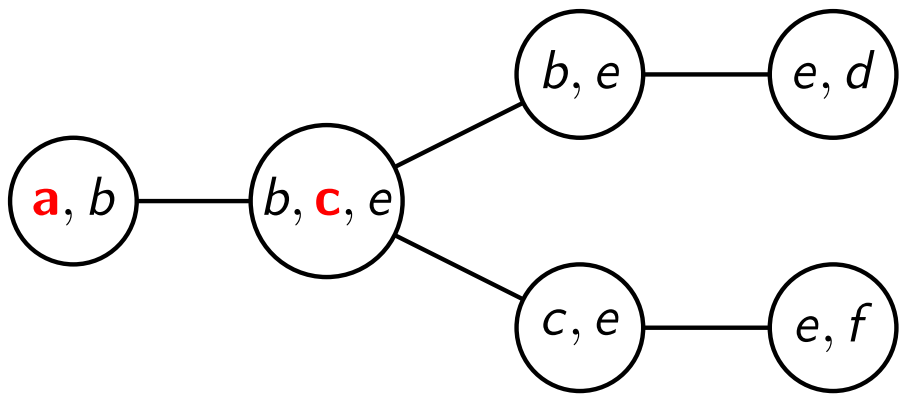
Loose tree-decomposition

A loose tree-decomposition is a pair $\mathcal{D} = (T, \chi)$ such that T is a tree and $\chi : V(T) \to 2^{V(G)}$ satisfying the following properties:

(L1) $\forall x \in V(G)$, $T_x = \{t \in V(T) \mid x \in \chi(t)\}$ is non-empty and connected in T.

(L2) $\forall e = xy \in E(G)$, $\exists \{t_1, t_2\} \in E(T)$ st. $e \in E(G[\chi(t_1) \cup \chi(t_2)])$;





Loose tree-decomposition

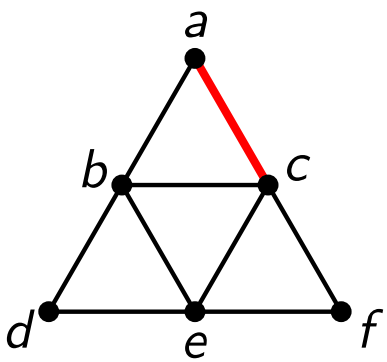
A loose tree-decomposition is a pair $\mathcal{D} = (T, \chi)$ such that T is a tree and $\chi : V(T) \to 2^{V(G)}$ satisfying the following properties:

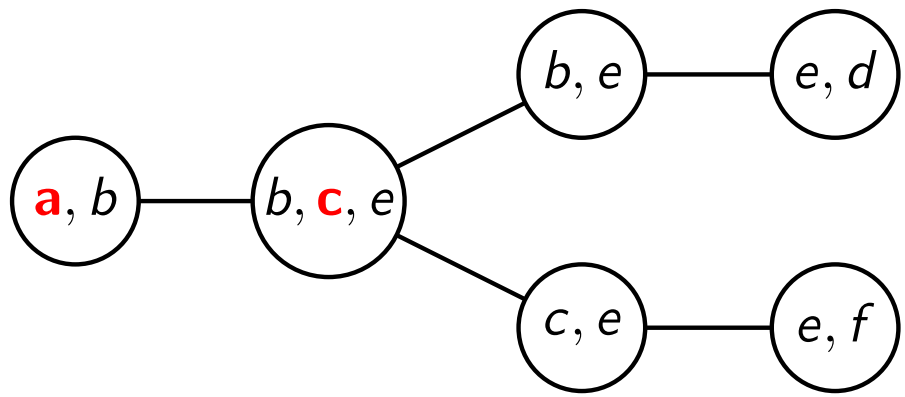
(L1) $\forall x \in V(G)$, $T_x = \{t \in V(T) \mid x \in \chi(t)\}$ is non-empty and connected in T.

(L2) $\forall e = xy \in E(G)$, $\exists \{t_1, t_2\} \in E(T)$ st. $e \in E(G[\chi(t_1) \cup \chi(t_2)])$;

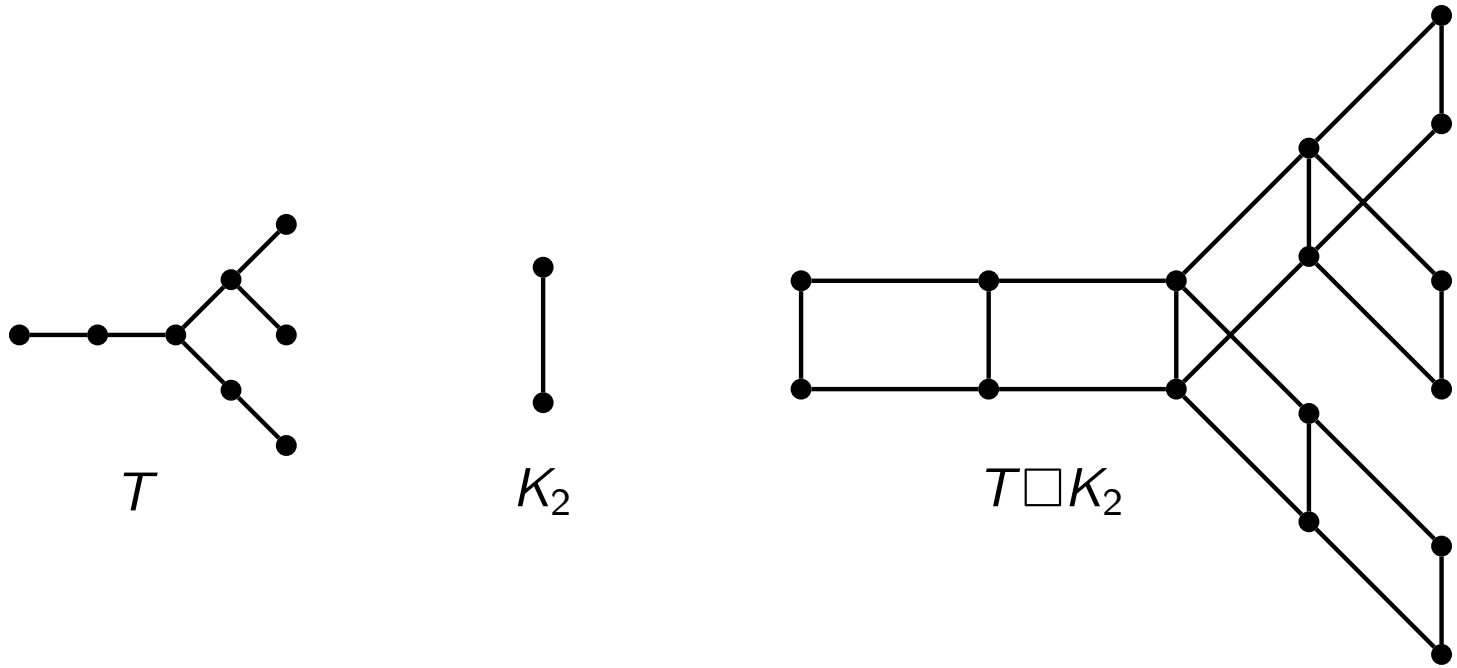
(L3) $\forall \{t_1, t_2\} \in E(T)$,

$$\left|E(G[\chi(t_1) \cup \chi(t_2)]) \setminus \left(E(G[\chi(t_1)]) \cup E(G[\chi(t_2)])\right)\right| \leq 1.$$





Cartesian tree product number



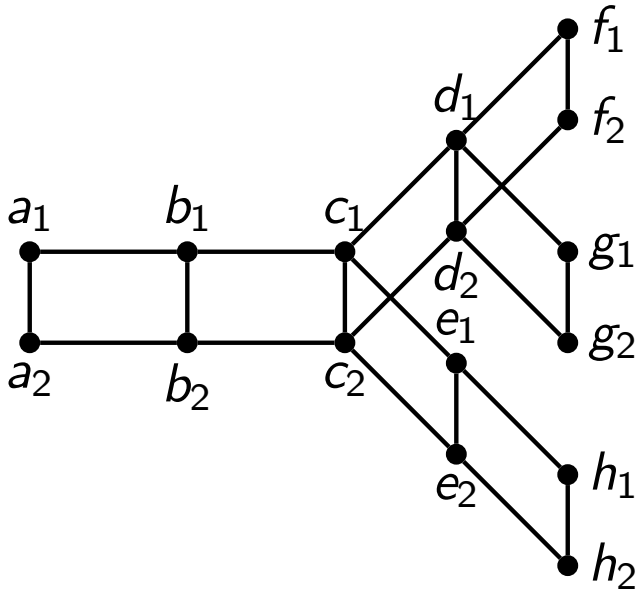
Definition [Harvey'14, Colin De Verdière'98]
The *cartesian tree product number* of a graph G is

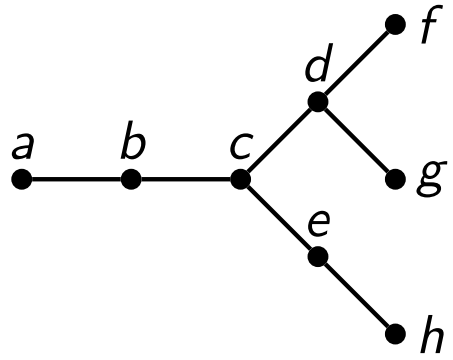
$$\mathbf{ctp}(G) = \min\{k \in \mathbb{N} \mid G \text{ is a minor of } T^{(k)}\}.$$

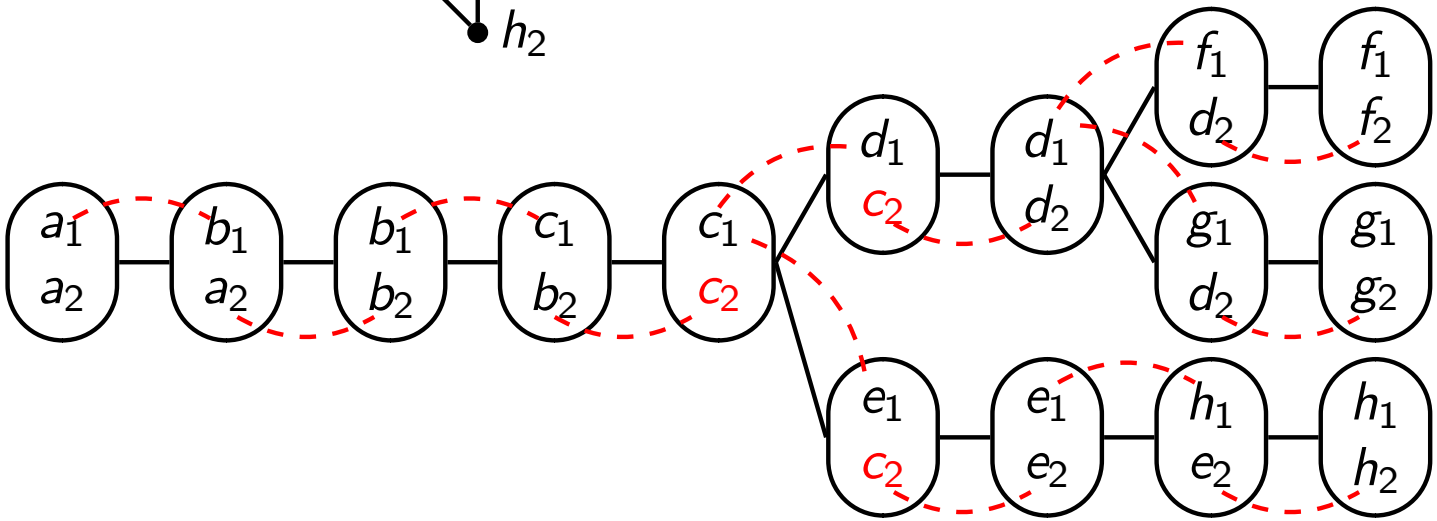
Cartesian tree product number

Theorem

$$\mathbf{ctp}(G) = \min\left\{\text{width}(\mathcal{D}, G) \mid \mathcal{D} \text{ is a loose tree-decomposition of } G\right\}.$$

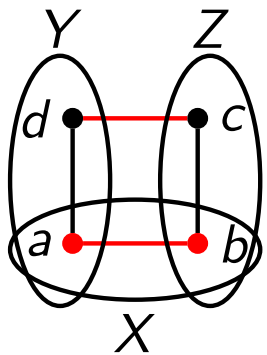






Tight bramble

Two subsets S_1 and S_2 of $V(G)$ are tightly touching if
$\rightsquigarrow$ either $S_1 \cap S_2 \neq \emptyset$
$\rightsquigarrow$ or $E(G)$ contains two distinct edges x_1x_2 and y_1y_2 such that $x_1, y_1 \in S_1$ and $x_2, y_2 \in S_2$.

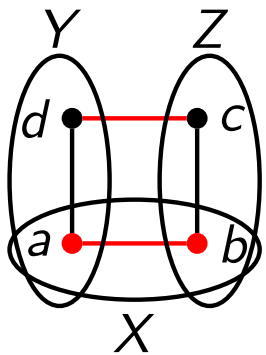


$\{X, Y, Z\}$ is a tight bramble

Definition
A set $\mathcal{B} \subseteq 2^{V(G)}$ of pairwise tightly touching connected subsets of $V(G)$ is a tight bramble of G.

Tight bramble

Two subsets S_1 and S_2 of $V(G)$ are tightly touching if
$\rightsquigarrow$ either $S_1 \cap S_2 \neq \emptyset$
$\rightsquigarrow$ or $E(G)$ contains two distinct edges x_1x_2 and y_1y_2 such that $x_1, y_1 \in S_1$ and $x_2, y_2 \in S_2$.



$\{X, Y, Z\}$ is a tight bramble

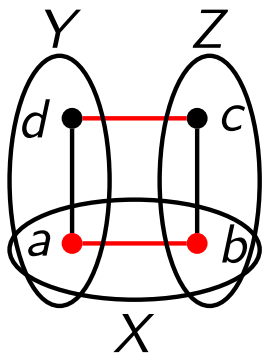
$\{a, b\}$ is a cover of $\{X, Y, Z\}$

Definition
A set $\mathcal{B} \subseteq 2^{V(G)}$ of pairwise tightly touching connected subsets of $V(G)$ is a tight bramble of G.

A set $S \subseteq V(G)$ is a cover of $\mathcal{B}$ if for every set $B \in \mathcal{B}$, $S \cap B \neq \emptyset$.
The order of the bramble $\mathcal{B}$ is the smallest size of a cover of $\mathcal{B}$.

Tight bramble

Two subsets S_1 and S_2 of $V(G)$ are tightly touching if
$\rightsquigarrow$ either $S_1 \cap S_2 \neq \emptyset$
$\rightsquigarrow$ or $E(G)$ contains two distinct edges x_1x_2 and y_1y_2 such that $x_1, y_1 \in S_1$ and $x_2, y_2 \in S_2$.



$\{X, Y, Z\}$ is a tight bramble

$\{a, b\}$ is a cover of $\{X, Y, Z\}$

Theorem

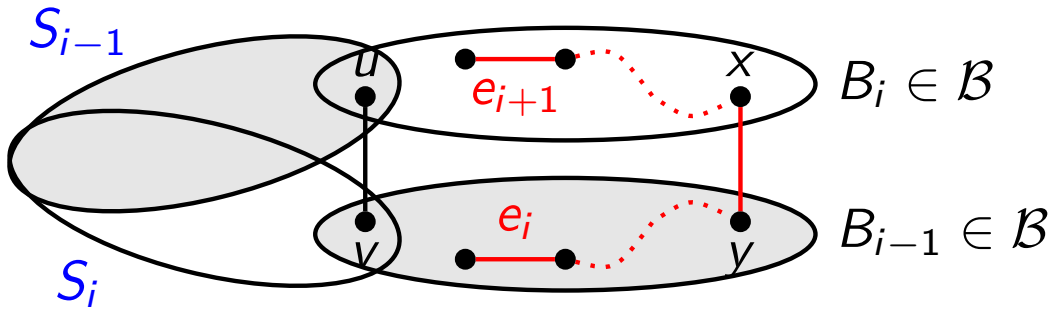
ctp$(G) \leq k$ if and only if every tight bramble of G has order at most k.

Escape strategy derived from a tight bramble

Theorem: If G has a tight bramble $\mathcal{B}$ of order k, then $\mathbf{avms}(G) \geq k$.

Suppose that a searcher slides on the edge uv and that

$$\mathcal{P}(\mathbf{s}_G, e_1, \mathbf{f}_G) = \langle \emptyset, e_1, \ldots S_{i-1}, e_i, S_i, e_{i+1} \ldots \rangle$$



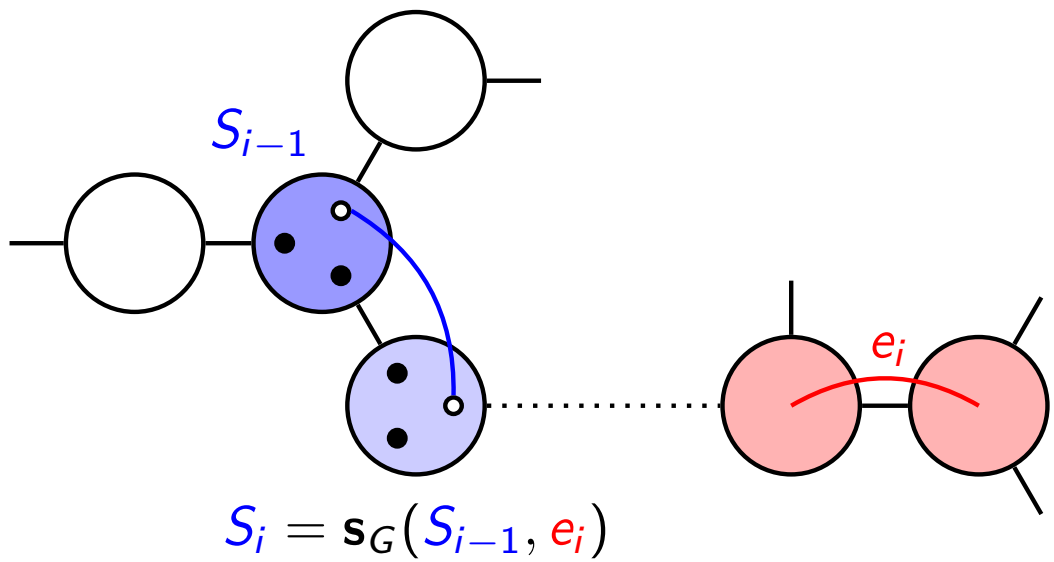
$$e_{i+1} = \mathbf{f}_G(S_{i-1}, e_i, S_i).$$

$\rightsquigarrow$ there exists a pathway from e_i to e_{i+1} going through the edge xy that avoids the edge uv.

Monotone search strategy derived from a loose tree-decomposition

Theorem: If $\mathbf{ctp}(G) \leq k$, then $\mathbf{mavms}(G) \leq k$.

$$\mathcal{P}(\mathbf{s}_G, e_1, \mathbf{f}_G) = \langle \emptyset, e_1, \ldots S_{i-1}, e_i, S_i, \ldots \rangle$$



Theorem:
Let G be a graph and $k \in \mathbb{N}$. Then the following conditions are equivalent:

1. G has a loose tree-decomposition of width k;
2. $\mathbf{ctp}(G) \leq k \rightsquigarrow G$ is a minor of $T^{(k)} = T \square K_k$;
3. every tight bramble of G has order at most k;
4. $\mathrm{avms}(G) \leq k \rightsquigarrow$ the mixed search number against an agile and visible fugitive is at most k;
5. $\mathrm{mavms}(G) \leq k \rightsquigarrow$ the monotone mixed search number against an agile and visible fugitive is at most k.

Thank you !