The complexity of computing optimum labelings for temporal connectivity

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The 10th Workshop on Graph Classes, Optimization, and Width Parameters (GROW)

September 2022

Temporal Graphs

Definition

A temporal graph $\mathcal{G}$ is a pair (G, λ) where:

- $G = (V, E)$ is an underlying (di)graph and
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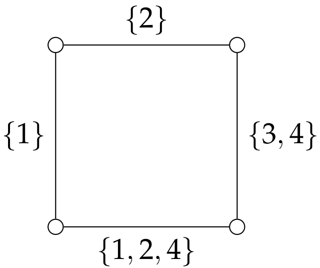

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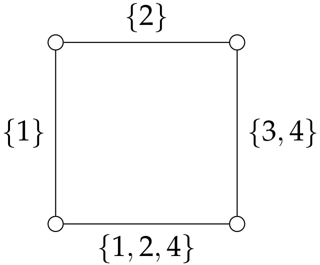
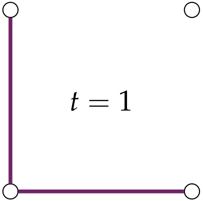


Temporal instances:



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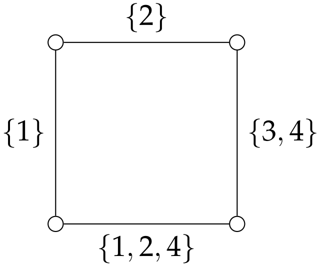

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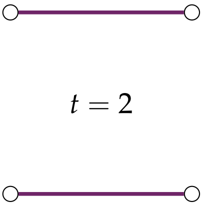

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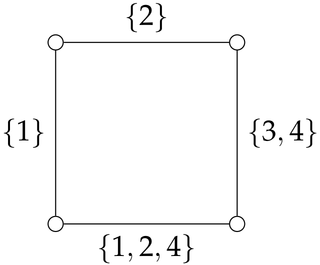

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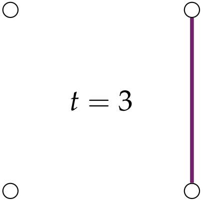

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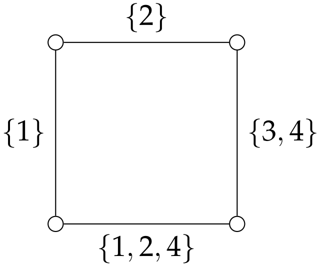

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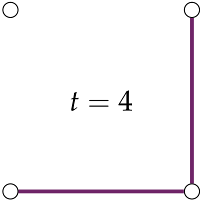

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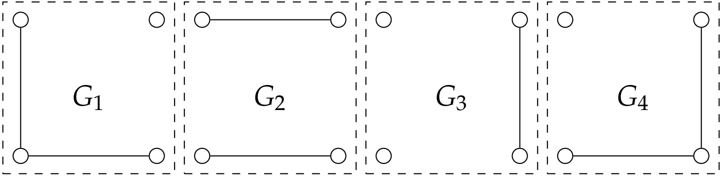
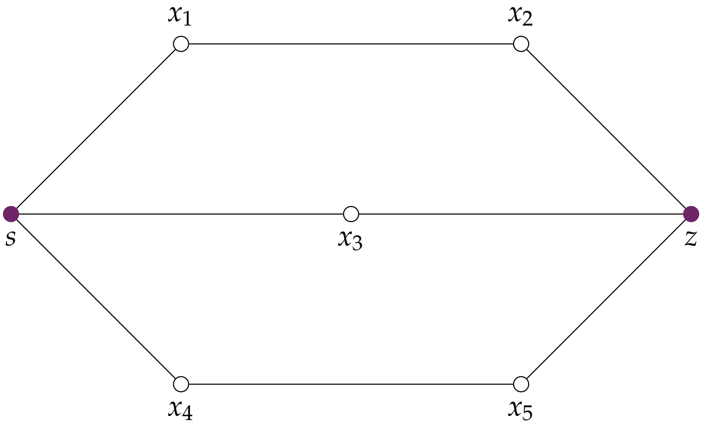
Alternatively we can view it as a sequence of static graphs, called *snapshots*:



From Paths to Temporal Paths



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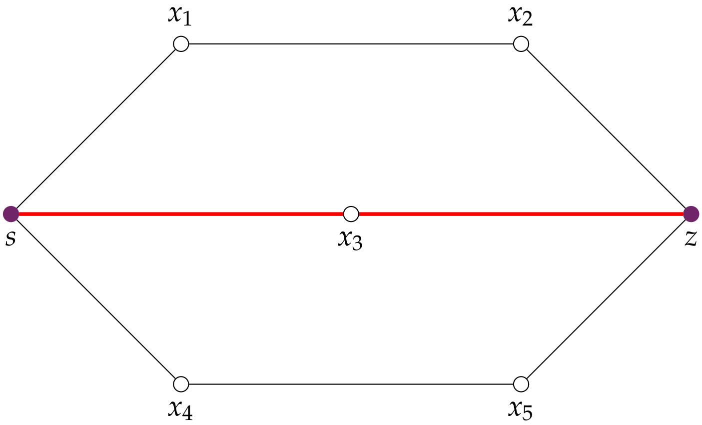

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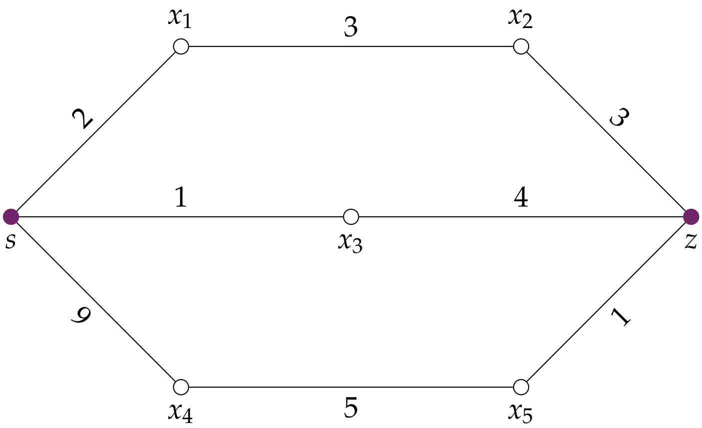

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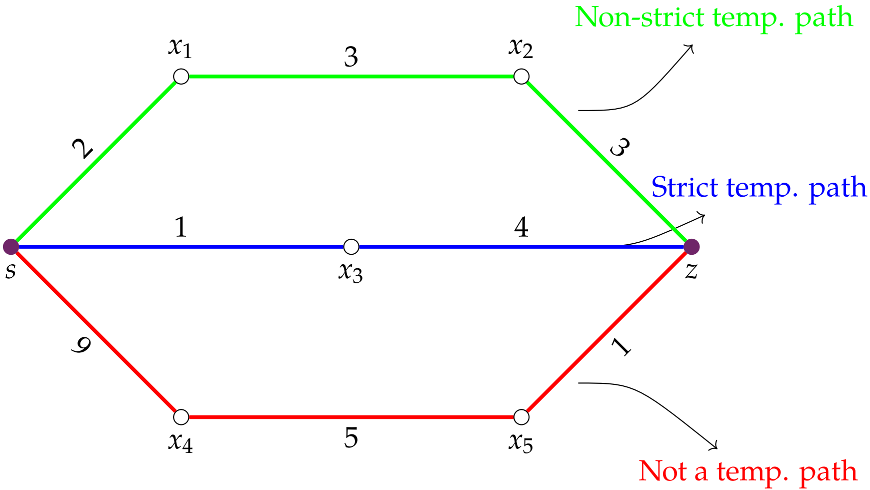

Definitions and Notations

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A temporal graph $\mathcal{G}$ is (temporally) connected iff for all $u, v \in V(G)$ there exists a temporal (u, v)-path[1].

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- Maximum label $\alpha(\lambda)$ is called the age of $\mathcal{G}$.
- The total cost of (G, λ) is $|\lambda| = \sum_{e \in E} |\lambda_e|$.

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Problem(s) Definitions I

Min. Labeling (ML)

Input: A static graph $G = (V, E)$ and an integer $k \in \mathbb{N}$.
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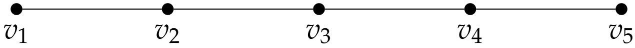

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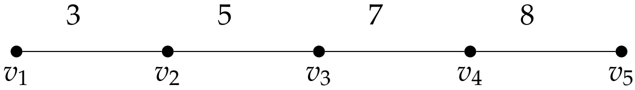

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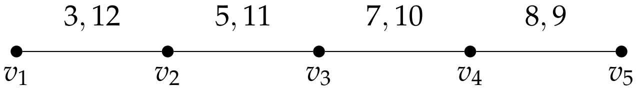

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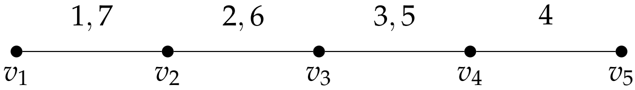

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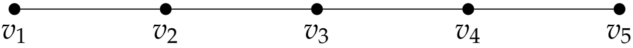

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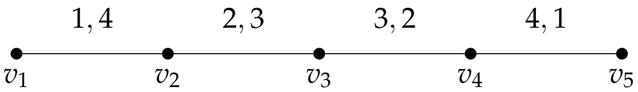

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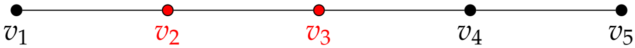

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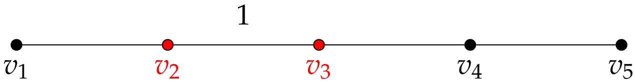

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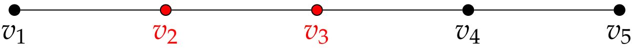

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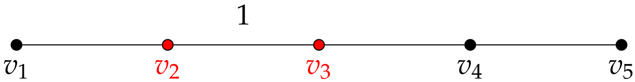

Results

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Temp. connected		
R-connected		

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Reduction from Monotone Max XOR(3):

- conjunction of XOR clauses,
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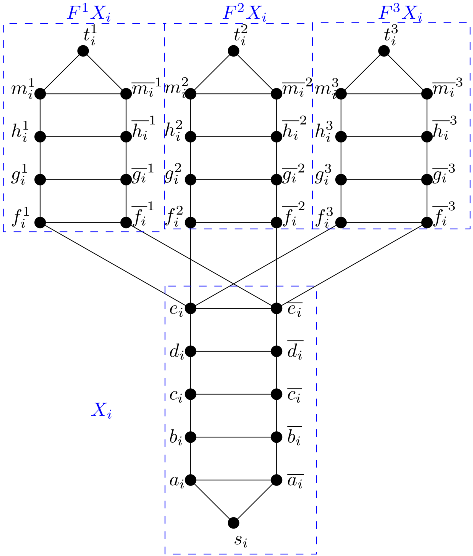
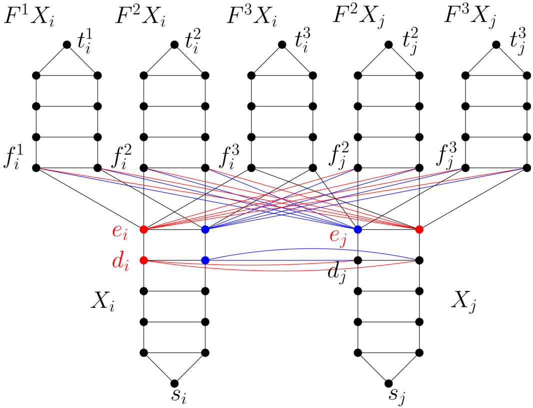
- variables appear exactly 3 times.

Variable gadgets



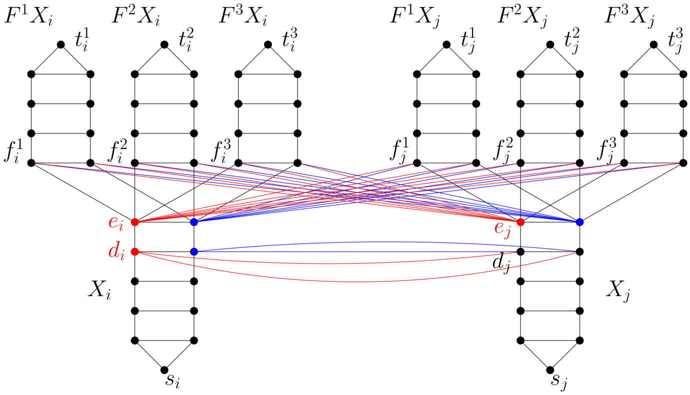
Connecting variable gadgets I

Clause $(x_i \oplus x_j)$ with 3rd and 1st appearance of x_i, x_j, respectively.



Connecting variable gadgets II

No clause with x_i and x_j.



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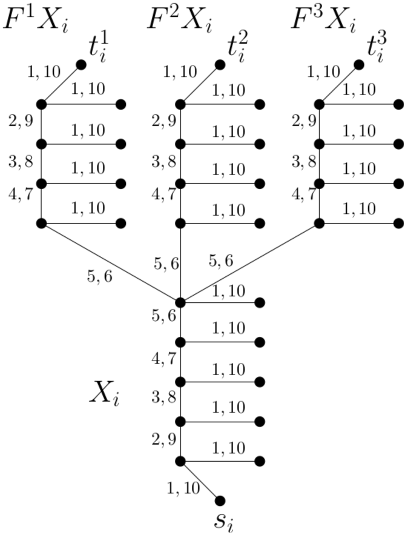

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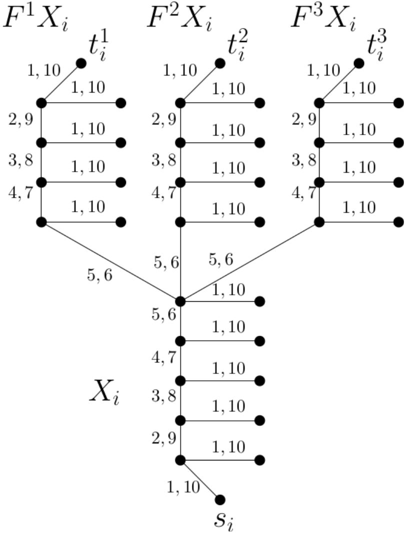

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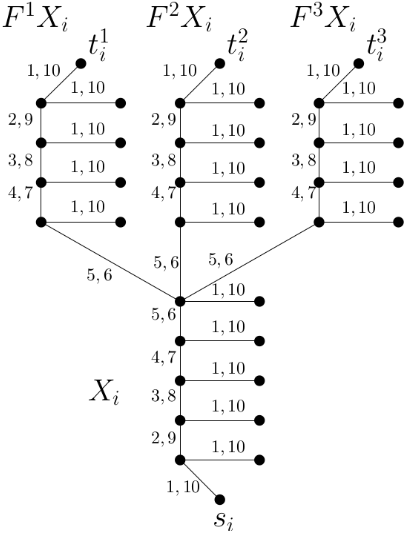
$$OPT(G_\phi, d_\phi) \leq \text{poly}(n, k)$$
$$\Leftrightarrow$$
$$OPT(\phi) \geq k.$$



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Idea of the algorithm:

- Use an FPT algorithm for Steiner Tree.
- Iterate over all C_4s in G, check if one can be labeled in an optimum solution.

Thank you!

Questions?

- MAL is NP-hard for $\alpha = d$ and poly-time solvable for $\alpha = 2d$, what happens for $d < \alpha < 2d$?

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- Is there an XP algorithm form MASL?
- What if we require two temporally disjoint paths among vertices?