From even-hole-free graphs to treewidth

10th Workshop on Graph Classes, Optimization, and Width Parameters (GROW), Koper

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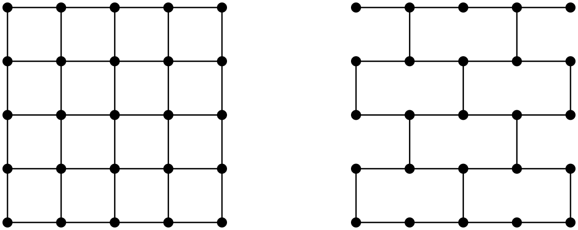
The celebrated grid theorem

Theorem (Robertson and Seymour, 1986)

There exists a function f such that any graph of treewidth at least $f(k)$ contains wall $k \times k$ as a minor



How about replacing "minor" with "induced subgraph"?

A theorem by Lozin and Razgon

Theorem (Lozin and Razgon 2020)

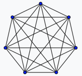
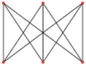
The treewidth of graphs in a hereditary class defined by a finite set $\mathcal{F}$ of forbidden induced subgraphs is bounded if and only if $\mathcal{F}$ includes:

- *a complete graph,*
- *a complete bipartite graph,*
- *a tripod, and*
- *the line graph of a tripod.*

Tripod: a forest in which every connected component has at most 3 leaves

The easy direction of Lozin and Razgon theorem

If the treewidth is bounded, $\mathcal{F}$ must contain one of the listed obstruction [Lozin 2008], because of:

- K_l 
- $K_{l,l}$ 
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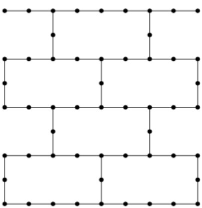

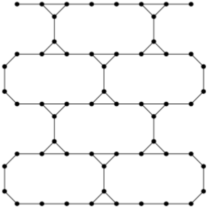

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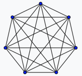
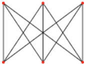
- K_l 
- $K_{l,l}$ 
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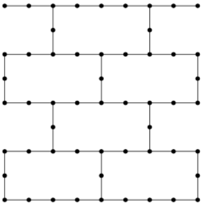

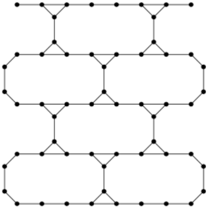

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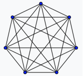
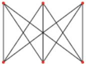
- K_l 
- $K_{l,l}$ 
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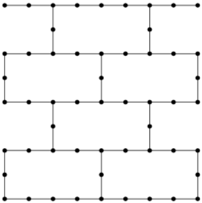

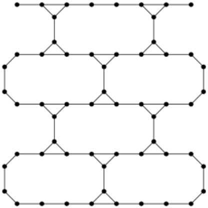

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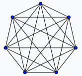
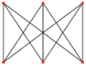
- K_l 
- $K_{l,l}$ 
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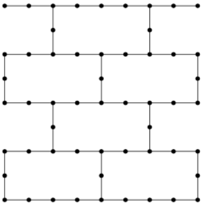

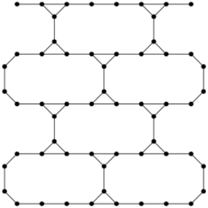

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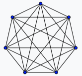
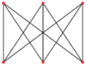
- K_l 
- $K_{l,l}$ 
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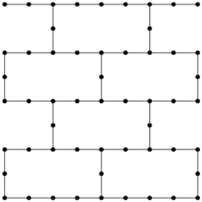

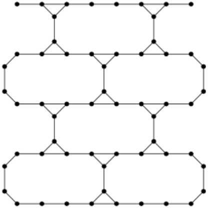

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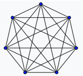
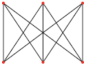
- K_l 
- $K_{l,l}$ 
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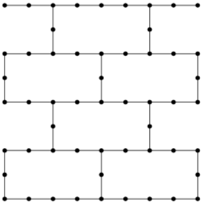

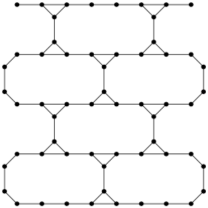

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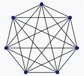
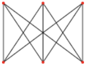
- K_l 
- $K_{l,l}$ 
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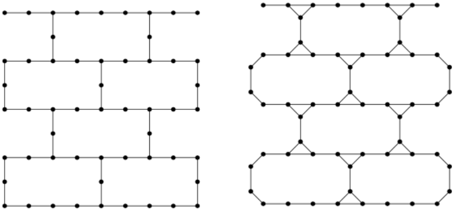
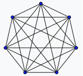
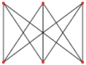


No !

Does a large treewidth must come from one of the reasons below (call them the l-obstructions):

- K_l 
- $K_{l,l}$ 
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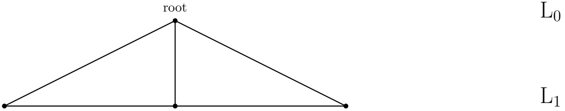
No: construction of Sintiari and T.

TTF layered wheel construction

$G(\ell, k)$, with $\ell = 2$ and $k = 4$

TTF layered wheel construction



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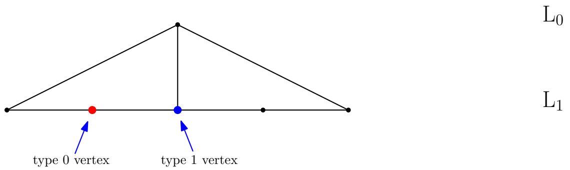

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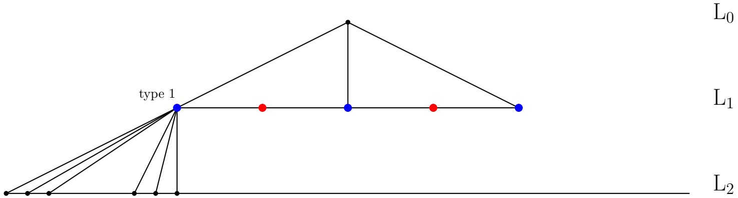

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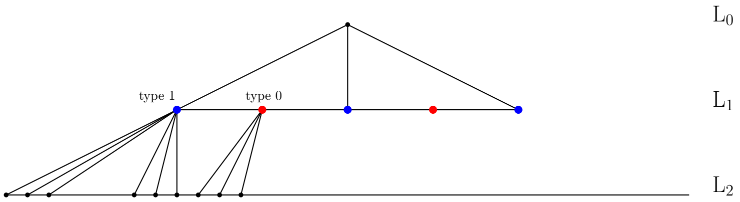

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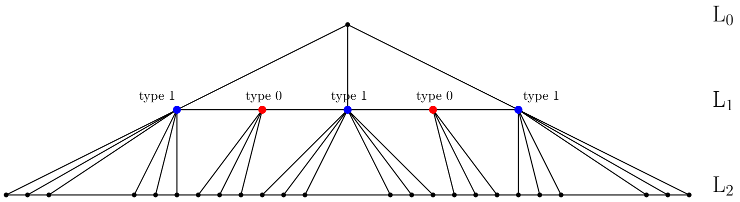

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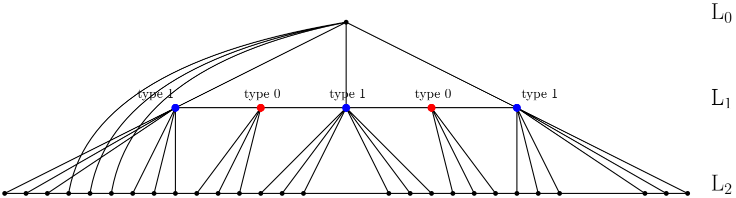

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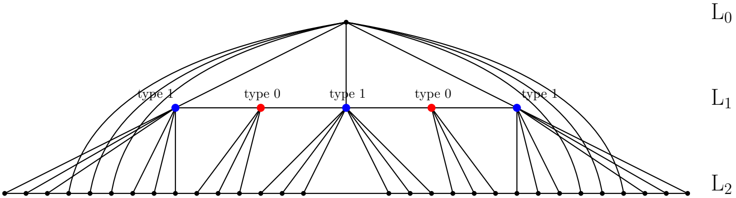

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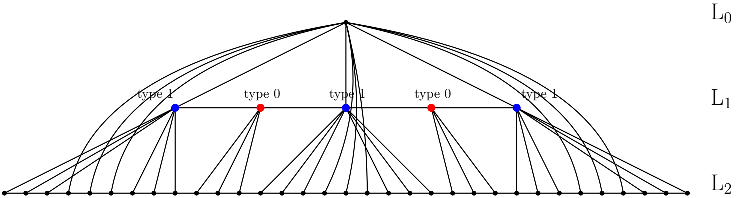

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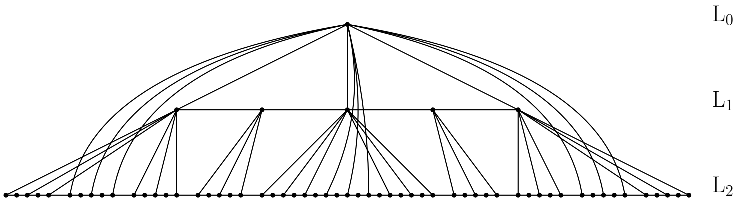

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Analysing the counter-example

To have large treewidth and no I-obstruction, the counter-example needs:

- a large clique minor
- large maximum degree
- a huge number of vertices
- a wheel

Are all these necessary ?

Four conjectures

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Large clique minor

Theorem (Aboulker, Adler, Kim, Sintiari and T., 2020)

A graph with no l-obstruction and no K_l as a minor a treewidth at most $f(l)$

- seemingly known from the community as a consequence of the so-called flat wall theorem
- Our proof relies on the contraction obstructions for treewidth (Fomin, Golovach and Thilikos).

Logarithmic conjecture

Conjecture (Sintiari et T., 2018)

If G contains no I-obstruction, then treewidth$(G) \leq \log(|V(G)|)$.

Interesting pour algorithm in time $O(2^{\text{treewidth}})$

Proved in some particular cases:

- no prism, theta and pyramid (Abrishami, Chudnovsky, Hajebi et Sprikl, 2021)
- no mutually induced disjoint cycles (Bonamy, Bonnet, Déprés, Esperet, Geniet, Hilaire, Thomassé and Wesolek, 2022)

But...

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Max degree

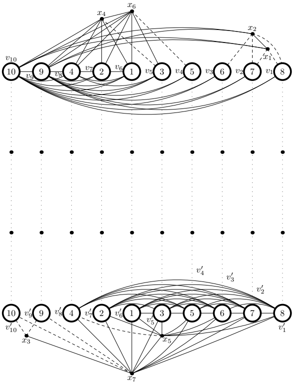
Theorem (Korhonen 2022))

There is a function $f(k,d) = O(k^{10} + 2^{d^5})$ so that if a graph has treewidth at least $f(k,d)$ and maximum degree at most d, then it contains a $k \times k$-grid as an induced minor.

Similar theorem by Hickingbotham (2022), with pathwidth instead of treewidth and complete binary tree instead of walls/grid.

When all holes have the same length

Structure of graphs where all holes have the same length (Cook, Horsfield, Preissmann, Robin, Seymour, Sintiari, T. and Vušković, 2021)



Thanks

Thanks for your attention
Thanks for the nice event !