# Evaluating Restricted First-Order Counting Properties on Nowhere Dense Classes and Beyond

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# Overview

FO Model Checking

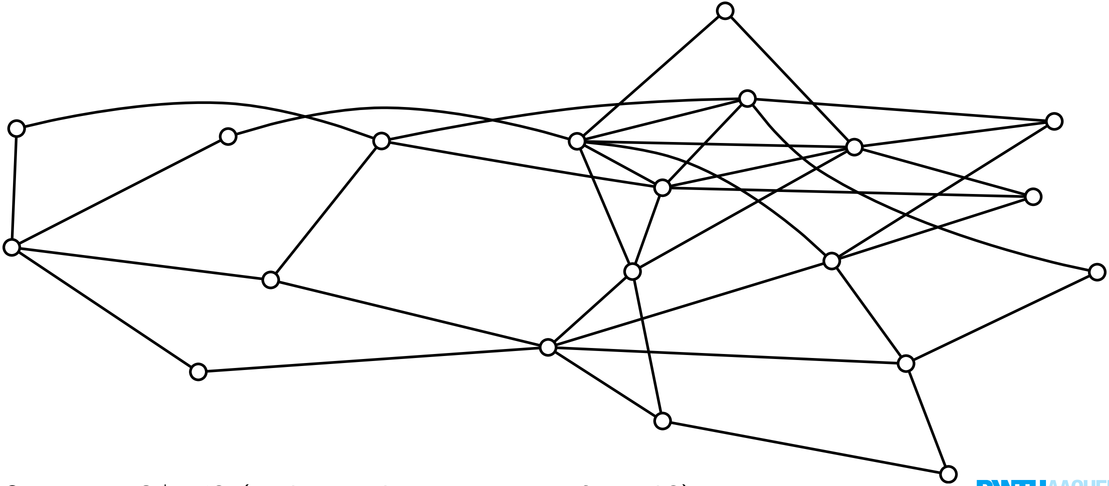
Some results about counting

New results on nowhere dense classes

Beyond nowhere dense

# First Order Model Checking

$$\phi = \exists x_1 \ldots \exists x_k \forall y \bigvee_{i=1}^{k} \Big( y = x_i \vee adj(y, x_i) \Big)$$



Question: $G \models \phi$? (Is there a dominating set of size $k$?)

# First Order Model Checking

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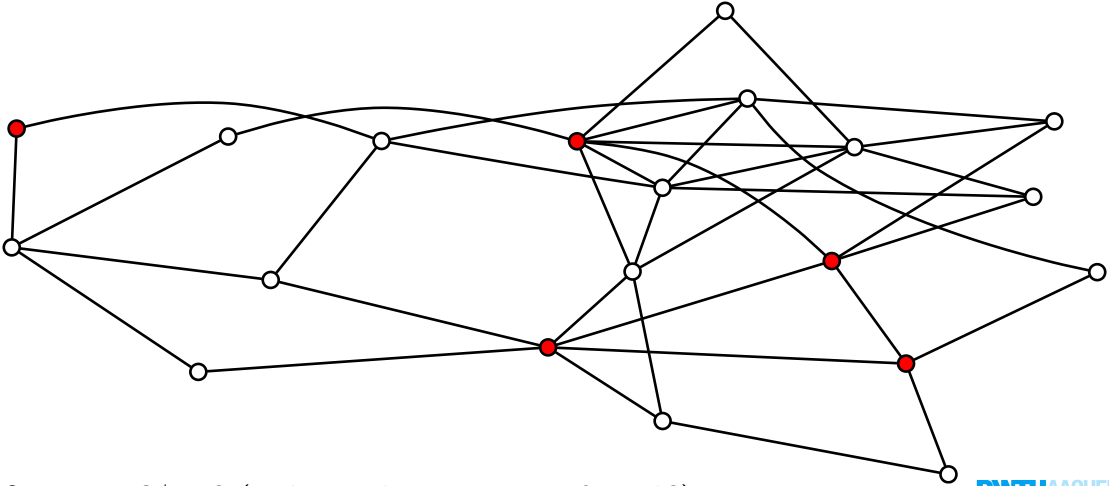


Question: $G \models \phi$? (Is there a dominating set of size $k$?)

# First Order Model Checking

**FO-Model Checking**

*Input:* A graph $G$ and an FO-formula $\phi$
*Parameter:* $|\phi|$
*Problem:* $G \models \phi$

**Query Evaluation**

*Input:* A database $D$ and a query $\phi(\bar{x})$
*Parameter:* $|\phi|$
*Problem:* Enumerate all $\bar{x}$ with $D \models \phi(\bar{x})$

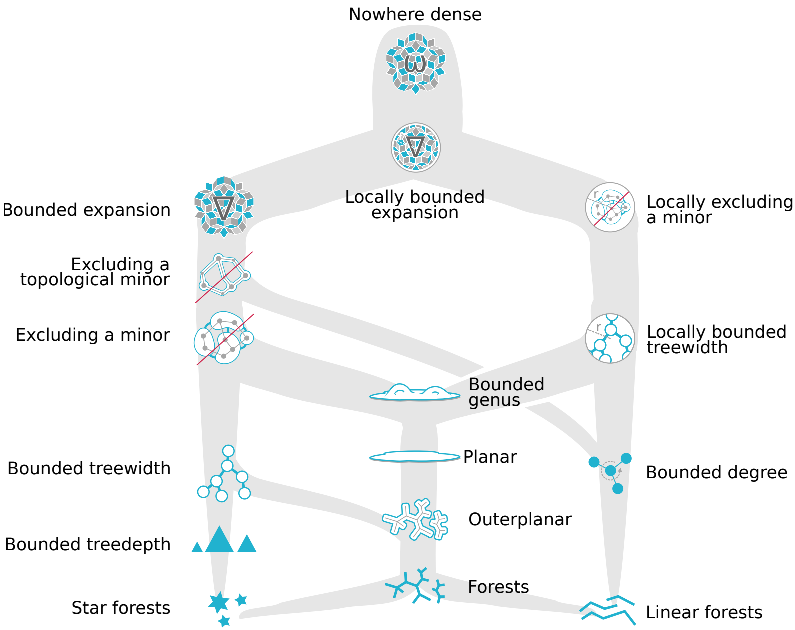
**Query Counting**

*Input:* A database $D$ and a query $\phi(\bar{x})$
*Parameter:* $|\phi|$
*Problem:* Compute $|\{\, \bar{x} \mid D \models \phi(\bar{x}) \,\}|$

# Some older results

- FO-model checking is PSPACE-complete as a classical problem.
  [Stockmeyer 1994]
- FO-model checking is AW[$*$]-complete (if parameterized by $|\phi|$).
  [Downey, Fellows, Taylor 1996]
- If $\mathcal{G}$ has is $H$-minor free, then we can decide $G \models \varphi$ in linear time if $G \in \mathcal{G}$ and $\varphi$ is an FO-formula.
  [Flum, Grohe 2001]
- If $\mathcal{G}$ has bounded expansion, then we can decide $G \models \varphi$ in linear time if $G \in \mathcal{G}$ and $\varphi$ is an FO-formula.
  [Dvořák, Kráľ, Thomas 2010] and [Kazana, Segoufin 2011]
- If $\mathcal{G}$ is nowhere dense, then we can decide $G \models \varphi$ in time $n^{1+\epsilon}$ if $G \in \mathcal{G}$ and $\varphi$ is an FO-formula.
  [Kreutzer, Grohe, 2011]

Moreover: If $\mathcal{G}$ is monotone: FO-model checking in FPT iff $\mathcal{G}$ is nowhere-dense.


Nowhere dense
Locally bounded expansion
Bounded expansion
Locally excluding a minor
Excluding a topological minor
Excluding a minor
Locally bounded treewidth
Bounded genus
Planar
Bounded degree
Bounded treewidth
Outerplanar
Bounded treedepth
Forests
Star forests
Linear forests
r
r

# Approximate and exact counting on bounded expansion

Assume that the counting subformulas have the form

$$\#y\,\phi(\bar{x}, y) > m,$$

where $m$ is an arbitrary fixed number.

## Theorem (Dreier and R. 2021)

- *Approximate model checking of such formulas is in linear FPT on graph classes of bounded expansion.*
- *For formulas of the form* $\exists x_1 \ldots \exists x_k \# y\,\phi(x_1, \ldots, x_k, y) > m$, *the result is exact if* $\phi$ *is an FO-formula.*
- *For modulo-counting (*$\equiv m \bmod q$ *instead of* $> m$*) the result is exact for constant* $q$*.*

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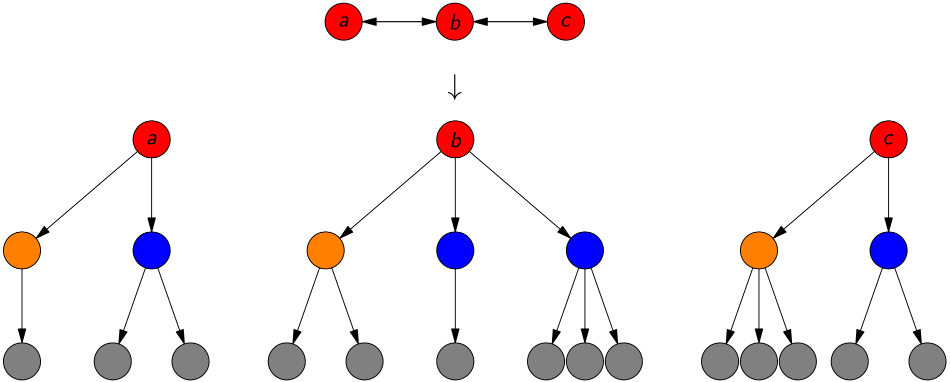
# Counting is hard




$$adj(a,b) \to \phi_E(a,b) = \exists x \exists y \Big( adj(a,x) \wedge adj(b,y) \wedge \#z\ adj(x,z) = \#z\ adj(y,z) \Big)$$

$$k\text{-clique iff } G' \models \exists x_1 \ldots \exists x_k \bigwedge_{1 \leq i,j \leq k} \phi_E(x_i, x_j)$$

| Partial Dominating Set | |
|---|---|
| *Input:* | A graph $G$, $k \in \mathbf{N}$, $m \in \mathbf{N}$. |
| *Parameter:* | $k$ |
| *Problem:* | Are there $k$ vertices dominating $m$ vertices? |

$$\exists x_1 \ldots \exists x_k \, \# y \bigvee_{1 \le i \le k} adj(x_i, y) \ge m$$

Is $W[1]$-hard for 2-degenerate graphs.
[Golovach, Villanger 2008]

Can be solved on $H$-minor free graphs in time $(g(H)k)^k n^{O(1)}$.
[Amini, Fomin, Saurabh, 2008]

Can be solved on apex-minor-free graphs in time $2^{\sqrt{k}} n^{O(1)}$.
[Fomin, Lokshtanov, Raman, Saurabh, 2011]

Can be solved in $f(k)n$ time on graphs of bounded expansion.
[Dreier, R. 2021]

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# New result on nowhere dense classes

**Theorem**

*Model checking of formulas of the form*

$$\exists x_1 \ldots \exists x_k \# y\, \phi(x_1, \ldots, x_k, y) > N,$$

*can be done in time $f(k)n^{1+\epsilon}$ on nowhere-dense graph classes. Here $\phi$ has to be quantifier-free.*

**Corollary**

*Partial domination set can be solved in $f(k)n^{1+\epsilon}$ time on nowhere-dense graph classes.*

# Proof sketch (1. Preprocessing of the formula)

Consider a quantifier-free FO-formula $\phi(y\bar{x})$ with signature $\sigma$. In time $f(|\phi|)$ one can construct a set $\Omega$ with the following properties:

1. The set $\Omega$ contains pairs of the form $(\mu, \omega(y\bar{x}))$ where $\mu \in \mathbf{Z}$ and $\omega(y\bar{x})$ is a conjunctive clause containing only positive literals,
2. $|\Omega| \leq 4^{|\phi|}$,
3. $|\omega| \leq |\phi|$ for each $(\mu, \omega) \in \Omega$,
4. $|\mu| \leq 4^{|\phi|}$ for every $(\mu, \omega) \in \Omega$,
5. for every graph $G$ and every $\bar{u} \in V(G)^{|\bar{x}|}$,

$$[\![\#y\ \phi(y\bar{u})]\!]^G = \sum_{(\mu,\omega)\in\Omega} \mu[\![\#y\ \omega(y\bar{u})]\!]^G.$$

# Proof sketch (2. Solving an optimization problem)

For $\varepsilon > 0$ and $G \in \mathcal{C}$ and every quantifier-free first-order formula $\phi(y\bar{x})$ we can compute a vertex tuple $\bar{u}^*$ that maximizes $[\![\#y\ \phi(y\bar{u}^*)]\!]^G$ in time $O(n^{1+\varepsilon})$.

- Use last slide for $[\![\#y\ \phi(y\bar{u})]\!]^G = \sum_{(\mu,\omega)\in\Omega} \mu[\![\#y\ \omega(y\bar{u})]\!]^G$.
- Guess the structure of $G[\bar{u}^*]$, i.e., a quantifier-free complete $\psi$ such that $G \models \psi(\bar{u}^*)$.
- Let $\Omega'$ be the formulas in $\Omega$ that fulfill $\psi$.
- Now find $\bar{u}^*$ with $G \models \psi(\bar{u}^*)$ that maximizes $\sum_{(\mu,\omega)\in\Omega'} \mu[\![\#y\ \omega(y\bar{u})]\!]^G$.

Last step most complicated one: DP using the game-tree of a splitter game on $G$. Use sparse neighborhood covers to keep the complexity low.

# Lower bound

**Theorem**
*Model checking of formulas of the form*

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*can be done in time $f(k)n^{1+\epsilon}$ on nowhere-dense graph classes. Here $\phi$ has to be quantifier-free.*

Can this theorem be improved beyond nowhere-dense?

*No:* Dominating set becomes $\mathrm{W}[1]$-hard on bipartite graphs if one side has polylogarithmic size.

Such graphs have subpolynomial weak coloring numbers.

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# An approximate solution beyond nowhere dense

**Theorem**

*For every graph $G$ and every quantifier-free first-order formula $\phi(y\bar{x})$ we can compute a vertex tuple $\bar{u}^*$ that maximizes $[[\#y\ \phi(y\bar{u})]]^G$ with an additive error of $\pm 4^{|\phi|}\mathrm{wcol}_2(G)^{O(|\phi|)}$ in time $\mathrm{wcol}_{f(|\phi|)}(G)^{f(|\phi|)}n$.*

This result is useful if the weak coloring numbers are subpolynomial.

Graph class: Almost nowhere dense.

# Conclusion and some open questions

- Evaluating restricted counting properties is fpt in nowhere dense graph classes.
- Question: Generalization to first-order $\phi$?
- Evaluating restricted counting properties approximately is fpt in almost nowhere dense graph classes.
- Evaluation of FO with modulo counting is fpt in bounded expansion.
- Question: Can we generalize this result to nowhere dense?