## Computing Tree Decompositions with Small Independence Number

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## Tree Decompositions



Graph G


A tree decomposition of $G$

## Dynamic programming for maximum independent set



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## All applications need the decomposition as an input!

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- For every constant $k \geq 4$, NP-hard to decide if $\operatorname{tree}-\alpha(G) \leq k$
- (For $k=1$ linear time, $k=2,3$ remain open)


## The Algorithm

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1. Container with bounded $\alpha$
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3. Linear programming

## Balanced separators

Input: Graph $G$, integer $k$, and a vertex set $X$ with $\alpha(X)=9 k$
Task: Find a separation $\left(C_{1}, S, C_{2}\right)$ with $\alpha(S) \leq 2 k, \alpha\left(X \cap C_{1}\right) \leq 7 k$, and $\alpha\left(X \cap C_{2}\right) \leq 7 k$ or conclude tree- $\alpha(G)>k$

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Algorithm: Guess independent set $t_{1} \subseteq X \cap C_{1}$ with $\left|I_{1}\right|=2 k$ and $I_{2} \subseteq X \cap C_{2}$ with $\left|I_{2}\right|=2 k$, and then find an $I_{1}-I_{2}$ separator $S$ with $\alpha(S) \leq 2 k$

## 2-Approximation Algorithm for separators

Input: Graph $G$, integer $k$, and two sets of vertices $V_{1}, V_{2}$
Task: Find an $\left(V_{1}, V_{2}\right)$-separator $S$ with $\alpha(S) \leq 2 k$, or conclude that no ( $V_{1}, V_{2}$ )-separators with $\alpha(S) \leq k$ exist

## Container with bounded $\alpha$

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Proof: By induction on $\alpha$ ( $\boldsymbol{S}$ )

$\Rightarrow R$ can be guessed by guessing $\mathcal{O}(k)$ bags of TD

## Branching

Have: A vertex set $R$ with $\alpha(R) \leq \mathcal{O}\left(k^{2}\right)$ so that $S \subseteq R$
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## Observation

Branches (2) and (3) decrease $\alpha\left(R \backslash N\left(V_{1} \cup V_{2}\right)\right)$.

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Branches (2) and (3) decrease $\alpha\left(R \backslash N\left(V_{1} \cup V_{2}\right)\right)$.
$\Rightarrow$ Branching tree of size $n^{2 \alpha(R)}$

## Linear Programming

Input: Graph $G$, integer $k$, three disjoint sets of vertices $V_{1}, V_{2}, R$ with $R=N\left(V_{1} \cup V_{2}\right)$
Task: Find an $\left(V_{1}, V_{2}\right)$-separator $S \subseteq R$ with $\alpha(S) \leq 2 k$, or conclude that no such separators with $\alpha(S) \leq k$ exist

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$x_{v}+x_{u} \geq 1$ for all $v \in N\left(V_{1}\right), u \in N\left(V_{2}\right)$ with $v-u$ path with internal vertices in $G \backslash R$

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Independence number inequalities:
$\sum_{v \in I} x_{v} \leq k$ for all independent sets $I \subseteq R$ with $|I|=2 k+1$

## Linear Programming

Input: Graph $G$, integer $k$, three disjoint sets of vertices $V_{1}, V_{2}, R$ with $R=N\left(V_{1} \cup V_{2}\right)$
Task: Find an $\left(V_{1}, V_{2}\right)$-separator $S \subseteq R$ with $\alpha(S) \leq 2 k$, or conclude that no such separators with $\alpha(S) \leq k$ exist

Variables: $x_{v}$ for all vertices $v \in R$
Separator inequalities:
$x_{v}+x_{u} \geq 1$ for all $v \in N\left(V_{1}\right), u \in N\left(V_{2}\right)$ with $v-u$ path with internal vertices in $G \backslash R$
Independence number inequalities:
$\sum_{v \in I} x_{v} \leq k$ for all independent sets $I \subseteq R$ with $|I|=2 k+1$

## Lemma

Rounding a fractional solution gives a solution with independence number at most $2 k$

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- Testing tree- $\alpha(G) \leq k$ is NP-hard for every $k \geq 4$
- Open problem: Complexity of testing tree $-\alpha(G) \leq k$ for $k=2,3$ ?


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