Computing Tree Decompositions with Small Independence Number

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Tree Decompositions



Graph G

A tree decomposition of G

Dynamic programming for maximum independent set



For every node *t* and subset $S \subseteq B_t$

 $dp[t][S] = maximum independent set I below t with <math>I \cap B_t = S$

Dynamic programming for maximum independent set



For every node *t* and subset $S \subseteq B_t$

dp[t][S] = maximum independent set *I* below *t* with $I \cap B_t = S$ $2^{|B_t|}$ states per node

Dynamic programming for maximum independent set



For every node *t* and **independent** subset $S \subseteq B_t$

dp[t][S] = maximum independent set *I* below *t* with $I \cap B_t = S$ #*IS*(*B*_t) states per node

What kind of tree decompositions have bounded $\#IS(B_t)$?

• Clique-trees of chordal graphs: $\#IS(B_t) \le n$

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- Most general parameter over tree decompositions that gives XP algorithms for maximum independent set (with some assumptions)

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All applications need the decomposition as an input!

Theorem

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 - (For k = 1 linear time, k = 2, 3 remain open)

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 - 2. Branching
 - 3. Linear programming

Input: Graph *G*, integer *k*, and a vertex set *X* with $\alpha(X) = 9k$

Task: Find a separation (C_1, S, C_2) with $\alpha(S) \leq 2k$, $\alpha(X \cap C_1) \leq 7k$, and $\alpha(X \cap C_2) \leq 7k$ or conclude tree- $\alpha(G) > k$

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Why balanced separators exists:



Sufficient to ensure that $\alpha(X \cap C_1) \ge 2k$ and $\alpha(X \cap C_2) \ge 2k$

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Algorithm: Guess independent set $l_1 \subseteq X \cap C_1$ with $|l_1| = 2k$ and $l_2 \subseteq X \cap C_2$ with $|l_2| = 2k$, and then find an $l_1 - l_2$ separator *S* with $\alpha(S) \leq 2k$

2-Approximation Algorithm for separators

Input: Graph G, integer k, and two sets of vertices V_1 , V_2

Task: Find an (V_1, V_2) -separator S with $\alpha(S) \leq 2k$, or conclude that no (V_1, V_2) -separators with $\alpha(S) \leq k$ exist

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Lemma

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 \Rightarrow *R* can be guessed by guessing $\mathcal{O}(k)$ bags of TD

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- 3. v goes to V₂

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Observation

Branches (2) and (3) decrease $\alpha(R \setminus N(V_1 \cup V_2))$.

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 \Rightarrow Branching tree of size $n^{2\alpha(R)}$

Input: Graph G, integer k, three disjoint sets of vertices V_1 , V_2 , R with $R = N(V_1 \cup V_2)$

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 $x_v + x_u \ge 1$ for all $v \in N(V_1)$, $u \in N(V_2)$ with v - u path with internal vertices in $G \setminus R$

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Lemma

Rounding a fractional solution gives a solution with independence number at most 2k

Conclusion

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- First XP approximation algorithm for tree-independence number
- Testing tree- $\alpha(G) \leq k$ is NP-hard for every $k \geq 4$
- Open problem: Complexity of testing tree- $\alpha(G) \leq k$ for k = 2, 3?

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