Bounds for the twin-width of graphs

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Twin-width is defined by contraction sequences

Definition

A trigraph is a triple $G = (V(G), B(G), R(G))$ where $B(G)$ and $R(G)$ are disjoint edge sets. The edges in $B(G)$ are **black** edges and the edges in $R(G)$ are **red** edges.

For a vertex v of a trigraph G,

1. $N^B(v)$ is the set of vertices adjacent to v by a **black** edge,
2. $N^R(v)$ is the set of vertices adjacent to v by a **red** edge, and
3. $N_G(v) := N^B(v) \cup N^R(v)$.

Twin-width is defined by contraction sequences

Contracting two vertices u and v, not necessarily adjacent, of a (tri)graph G is defined as follows.

We remove u and v and make a new vertex w. For $x \in V(G) \setminus \{u, v\}$,

1. if $x \in N^B(u) \cap N^B(v)$, then add a **black** edge wx,
2. if $x \notin N_G(u) \cup N_G(v)$, then add no edge between w and x, and
3. otherwise add a **red** edge wx.

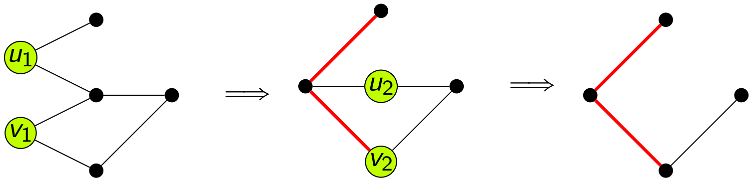

Figure: Contracting u and v.

Twin-width is defined by contraction sequences

A contraction sequence of a graph G on n vertices is a sequence $G_1 := G, \ldots, G_n := K_1$ such that for each $i \in [n-1]$, G_{i+1} is obtained from G_i by contracting two distinct vertices.
A width of a contraction sequence is the maximum red-degree of G_i.

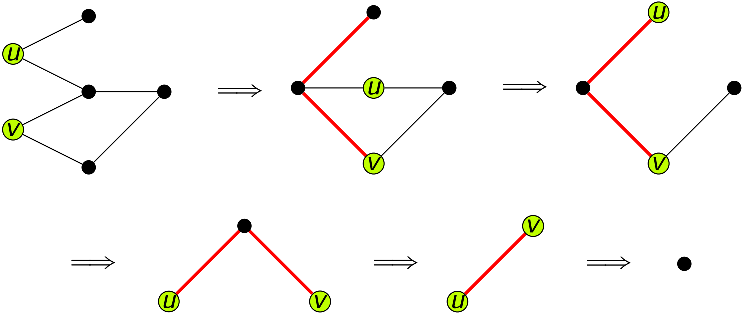

Figure: A contraction sequence of width 2.

Twin-width is defined by contraction sequences

The twin-width of a graph G is the minimum possible width of a contraction sequence of G.

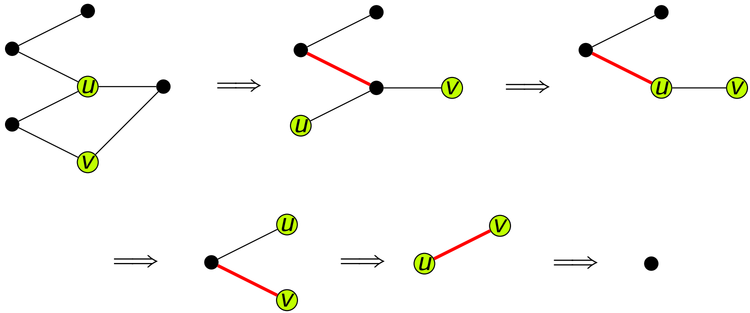

Figure: The twin-width of the graph is 1.

Bounded twin-width

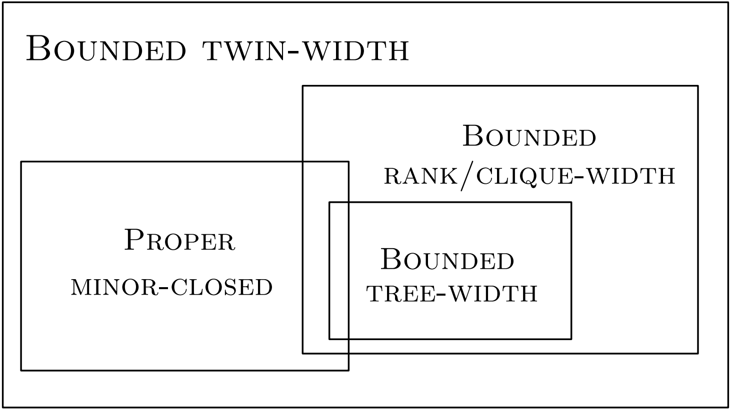

Figure: Bounded twin-width graph classes generalize several graph classes.

Bounded twin-width

Theorem (Bonnet, Kim, Thomassé, and Watrigant, 2021)

For any FO ϕ and a graph G with a contraction sequence of width d, checking whether G satisfies ϕ can be done in $f(d, |\phi|) \cdot |V(G)|$ time.

Theorem (Bonnet, Geniet, Kim, Thomassé, and Watrigant, 2021)

For a graph with a contraction sequence of constant width, k-Independent Set, k-Clique, k-Dominating Set *can be solved in* $2^{O(k)}n$ *time.*

Weighted k-Independent Set, Subgraph Isomorphism, Induced Subgraph Isomorphism *can be solved in* $2^{O(k \log k)}n$ *time.*

Main results

Theorem (A., Hendrey, Kim, and Oum, 2022)

For a graph G with n vertices and m edges,

$$\mathrm{tww}(G) < \frac{1}{2}(n + \sqrt{n \ln n} + \sqrt{n} + 2 \ln n),$$

$$\mathrm{tww}(G) < \sqrt{3m} + \frac{m^{1/4}\sqrt{\ln m}}{4 \cdot 3^{1/4}} + \frac{3m^{1/4}}{2}.$$

Proof idea

Lemma

Let M be a maximal set of disjoint pairs $\{u, v\}$ of vertices in G such that

$$|(N_G(u) \triangle N_G(v)) \setminus \{u, v\}| \leqslant \frac{n + \sqrt{n} - 1}{2}.$$

Then $|M| > (n - \sqrt{n})/2$.

Note that $(n + \sqrt{n} - 1)/2$ is slightly smaller than our bound.

As contracting a pair in M induces a small red-degree, one possible strategy is to contract the pairs in M sequentially.

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Proof idea

But there is a problem...

If we contract $\{u_1, v_1\}$, $\{u_2, v_2\}$ in sequel, contracting the second pair may affect the red-degree of the new vertex obtained by contracting $\{u_1, v_1\}$.

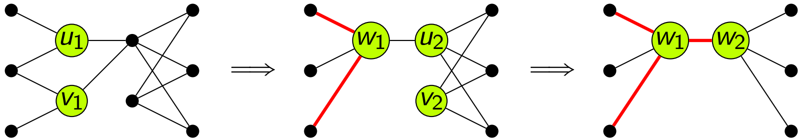

Figure: Contracting u_2 and v_2 increases the red-degree of w_1 by 1.

Proof idea

By probabilistic method, we can find an order of M so that contracting the elements in M in this order does not increase much the red-degree of new vertices.

After contracting $> (n + \sqrt{n})/2$ pairs in M, the remaining graph has small number of vertices, so we can take any contraction sequence of the remaining graph.

$\Rightarrow$ As the result, we obtain $\mathrm{tww}(G) < \frac{1}{2}(n + \sqrt{n \ln n} + \sqrt{n} + 2 \ln n)$. $\square$

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$n/2 + o(n)$ is tight

In the upper bound by n, the coefficient of n is tight.

Definition (Conference graph)

A conference graph is a graph G on n vertices with $n \equiv 1 \pmod 4$ such that G is $((n-1)/2)$-regular and for each distinct vertices v and w,

1. if v and w are adjacent, then $|N_G(v) \cap N_G(w)| = (n-5)/4$, and
2. otherwise $|N_G(v) \cap N_G(w)| = (n-1)/4$.

$\Rightarrow$ After contracting any pair of vertices, the new vertex has red-degree $(n-1)/2$, so its twin-width is at least $(n-1)/2$.

$n/2 + o(n)$ is tight

Is there a graph on n vertices having twin-width more than $(n-1)/2$?

Theorem (Schidler and Szeider, 2022)

Suggest a SAT-encodings for computing twin-width, and compute the twin-width of several small graphs.

$n/2 + o(n)$ is tight

Graph	$\|V\|$	$\|E\|$	lb_1	tww	ub_{greedy}	Variables	Clauses
Brinkmann	21	42	6	6	6	34526	150770
Chvátal	12	24	2	3	5	5611	18288
Clebsch	16	40	6	6	8	15510	64517
Desargues	20	30	4	4	5	28383	132636
Dodecahedron	20	30	4	4	4	26863	126244
Dürer	12	18	2	3	4	5347	18602
Errera	17	45	4	5	6	17720	75895
FlowerSnark	20	30	4	4	4	28383	119176
Folkman	20	40	2	3	3	10311	35761
Franklin	12	18	2	2	4	5347	16354
Frucht	12	18	2	3	3	5083	17573
Goldner	11	27	2	2	4	4067	11813
Grid 6×8*	48	82	2	3	4	396751	3493676
Grötzsch	11	20	2	3	5	4287	13910
Herschel	11	18	2	2	4	4067	13590
Hoffman	16	32	2	4	5	14070	58051
Holt	27	54	6	6	7	79513	405925
Kittell	23	63	4	5	6	46161	171811
McGee	24	36	4	4	5	50087	238494
Moser	7	11	2	2	2	252	502
Nauru	24	36	4	4	5	50087	239051
Paley-73*	73	1314	36	36	64	2530300	21107035
Pappus	18	27	4	4	5	20399	89670
Peterson	10	15	4	4	4	3009	9388
Poussin	15	39	3	4	5	11571	31049
Robertson	19	38	6	6	6	25369	114592
Rook 6×6*	36	180	10	10	12	216499	1236368
Shrikhande	16	48	6	6	8	15510	64431
Sousselier	16	27	4	4	5	14070	51414
Tietze	12	18	2	4	4	5347	18628
Wagner	8	12	2	2	2	1418	3909

$n/2 + o(n)$ is tight

Definition (Paley graph)

For each prime power q with $q \equiv 1 \pmod 4$, the Paley graph $P(q)$ is a graph on $\mathbb{F}_q$ with $|\mathbb{F}_q| = q$ such that vertices a and b are adjacent in $P(q)$ if and only if $a - b = c^2$ for some $c \in \mathbb{F}_q \setminus \{0\}$.

Theorem (A., Hendrey, Kim, and Oum, 2022)

Every n-vertex Paley graph has twin-width exactly $(n-1)/2$.

$n/2 + o(n)$ is tight

Definition (Paley graph)

For each prime power q with $q \equiv 1 \pmod 4$, the Paley graph $P(q)$ is a graph on $\mathbb{F}_q$ with $|\mathbb{F}_q| = q$ such that vertices a and b are adjacent in $P(q)$ if and only if $a - b = c^2$ for some $c \in \mathbb{F}_q \backslash \{0\}$.

Question

Is there an n-vertex graph having twin-width larger than $(n-1)/2$?

On random graphs, we can do better

Theorem (A., Chakraborti, Hendrey, Kim, and Oum, 2022+)

There exists $p^* \in (0.401, 0.402)$ such that the following hold for a real p and $q := 1 - p$.

(1) *If $p^* < p \leqslant 1/2$, then with high probability,*

$$\mathrm{tww}(G(n,p)) = 2pqn - \sqrt{6pq(1-2pq)n\ln n} + o(\sqrt{n \ln n}).$$

(2) *If $0 < p < p^*$, then there exists $\delta := \delta(p) > 0$ such that with high probability,*

$$\mathrm{tww}(G(n,p)) > (2pq + \delta)n.$$

On random graphs, we can do better

Corollary (A., Chakraborti, Hendrey, Kim, and Oum, 2022+)

With high probability,

$$\operatorname{tww}(G(n,1/2)) = \frac{n}{2} - \frac{\sqrt{3n\ln n}}{2} + o(\sqrt{n\ln n}).$$

Limit Ratio

Theorem (A., Hendrey, Kim, and Oum, 2022)

For a graph G with n vertices and m edges,

$$\mathrm{tww}(G) < \frac{1}{2}(n + \sqrt{n \ln n} + \sqrt{n} + 2 \ln n),$$

$$\mathrm{tww}(G) < \sqrt{3m} + \frac{m^{1/4}\sqrt{\ln m}}{4 \cdot 3^{1/4}} + \frac{3m^{1/4}}{2}.$$

Limit Ratio

Theorem (A., Hendrey, Kim, and Oum, 2022)

$$\lim_{n\to\infty} \max\left\{\frac{\mathrm{tww}(G)}{|V(G)|} : |V(G)| = n\right\} = \frac{1}{2},$$

$$1.088 < \frac{4\sqrt{6}}{9} \leqslant \limsup_{m\to\infty} \max\left\{\frac{\mathrm{tww}(G)}{\sqrt{|E(G)|}} : |E(G)| = m\right\} \leqslant \sqrt{3} < 1.733.$$

Question

What's the correct value of $\limsup \max\{\mathrm{tww}(G)/\sqrt{E(G)} : |E(G)| = m\}$? Can we reduce the upper bound $\sqrt{3}$ to a smaller number?

Thank you for your attention!