On Structural Parameterizations of Continuous Facility Location Problems on Graphs

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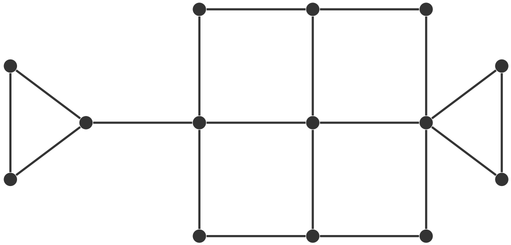
joint work with Tim A. Hartmann

GROW 2022

Continuous Facility Location on Graphs



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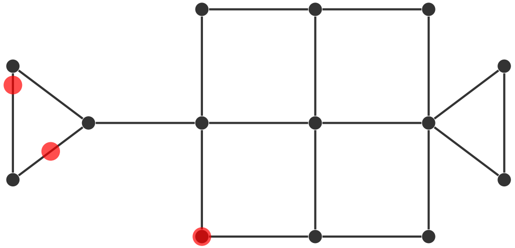

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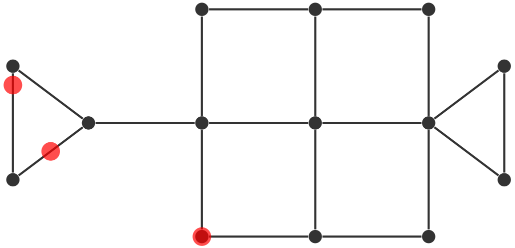
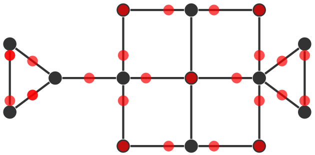


- Graph $G = (V, E)$, connected, unit length
- $P(G)$ continuum set of points on edges and vertices
- $d(p, q)$ shortest distance between p and q

δ-Dispersion and δ-Covering Number



- $S \subset P(G)$ δ-dispersed: $\forall\, p \neq q \in S\colon\ d(p,q) \geq \delta$

$$\delta\text{-disp}(G) = \max\{|S| \colon S \subset P(G), S\ \delta\text{-dispersed}\}$$

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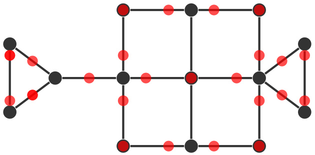

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- $S \subset P(G)$ δ-covering: $\forall\, p \in P(G)\ \exists\, s \in S\colon d(p,s) \leq \delta$

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[Hartmann, L. Woeginger, IPCO 2020] Similar results for δ-covering

Influence of Structural Graph Parameters?

Studied structural parameters:

PART I: Parameters leading to sparse graphs:

- treewidth $\mathrm{tw}(G)$
- pathwidth $\mathrm{pw}(G)$
- size of a feedback vertex set $\mathrm{fvs}(G)$
- treedepth $\mathrm{td}(G)$

PART II: Structural parameterizations of dense graphs:

- neighborhood diversity $\mathrm{nd}(G)$

Overview of results

[Hartmann, L.; MFCS 2022] Main ingredients:

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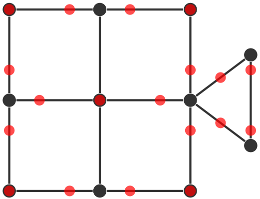
Dispersion and Independent Set

$\alpha_d(G)$ maximum size of a distance-d independent set

Lemma

Consider integers a, b and a $2b$-subdivision G_{2b} of a graph G. Then $\frac{a}{b}$-disp$(G) = \alpha_{2a}(G_{2b})$.



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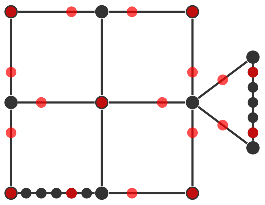

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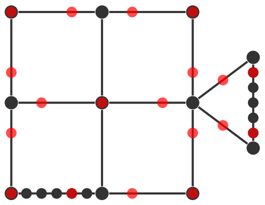

Using [Katsikarelis, Lampis, Paschos; DAM 2022] we get

Theorem

$\frac{a}{b}$-disp(G) can be computed in time $(2a)^{\mathrm{tw}(G)}(bn)^{\mathcal{O}(1)}$.

Translating δ-Dispersion

Lemma

For $\delta \in (0, 3]$ *we have* δ*-disp*$(G) = \frac{\delta}{\delta+1}$*-disp*$(G) + |E(G)|$.

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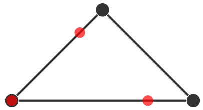
Problem for $\delta > 3$: locally-injective p to p walk.

$$\delta = 3 + \varepsilon \Rightarrow \frac{\delta}{\delta+1} > \frac{3}{4}$$



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Illustrative example: P_6

$$\delta = \frac{15}{11}$$

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$\delta = \frac{3}{4}$

δ^* depends on L, the length of the longest (non-induced) path in G

Rounding the Distance

Theorem

Let $\delta \in \mathbb{R}^+$. Let L be an upper bound on the length of paths in G. Let $\delta^\star = \frac{a^\star}{b^\star} \geq \delta$ minimal with $a^\star \leq 2L$ and $b^\star \in \mathbb{N}$. Then δ-disp$(G) = \delta^\star$-disp(G).

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(Event 1) A δ-uncritical pair of points $\{p, q\}$ becomes $(\delta + \varepsilon)$-critical.

Coordination of Movement

Consider a sequence of point $p_0, p_1, p_2, \ldots$ with $\{p_i, p_{i+1}\}$ critical.

Move p_0 by 0, p_1 by ε, p_2 by 2ε, $\ldots$.

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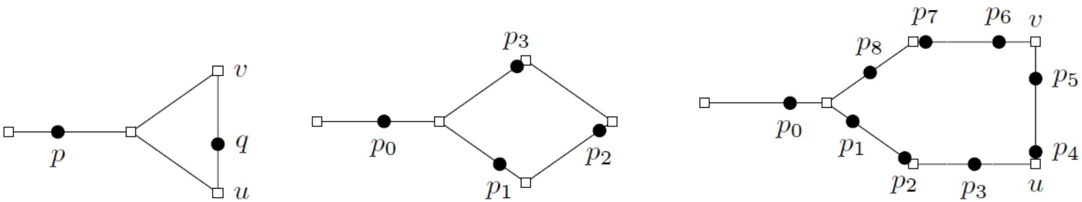
Problems:



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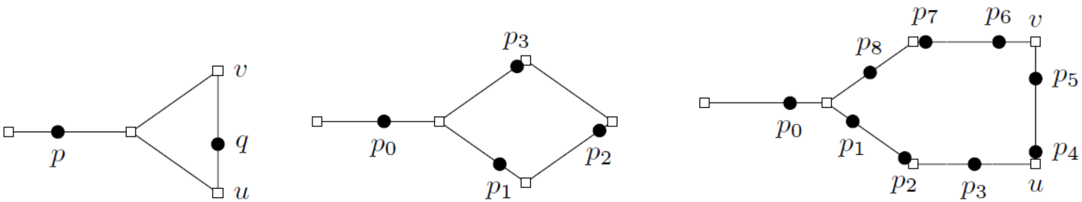

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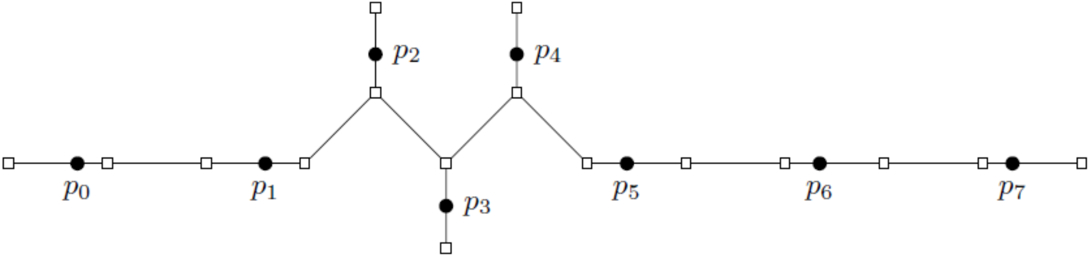
Spines (pushed sequences of points) start with a root (half-integral point if possible).

Velocities

Another problem appears within our simple pushing idea:



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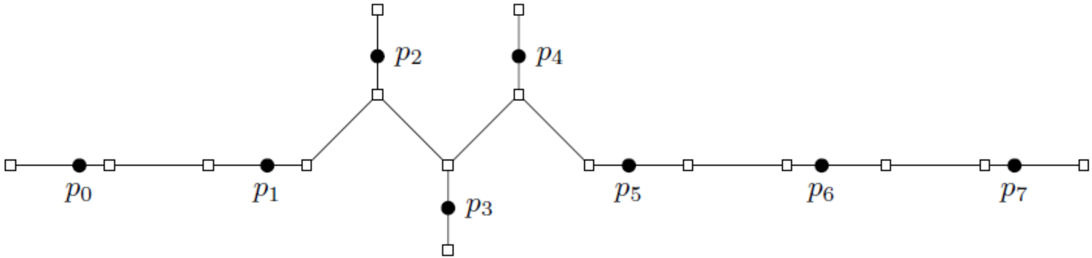

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- well-defined directions between points,
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- spines only starting in a defined set of roots.

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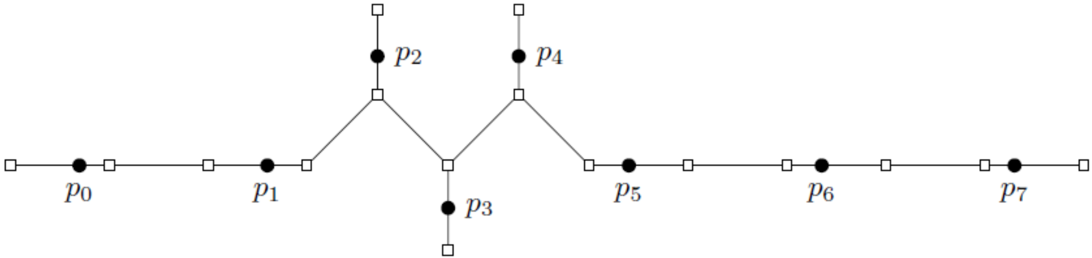

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Lemma

The choice of such a spine does not influence the movement of a point.

Algorithmic Implications

treewidth $\mathrm{tw}(G)$

- **XP** with running time $(2L)^{\mathrm{tw}(G)} n^{\mathcal{O}(1)}$
- no $n^{o(\mathrm{tw}(G)+\sqrt{k})}$, assuming ETH

pathwidth $\mathrm{pw}(G)$, size of a feedback vertex set $\mathrm{fvs}(G)$

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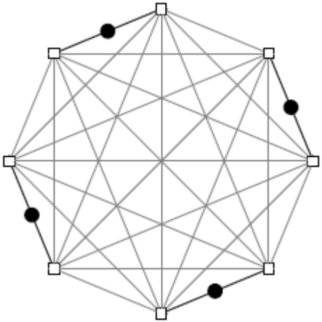
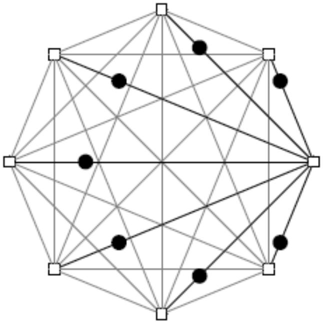
natural parameter k:

- **FPT** if $\delta \leq 2$
- $\mathrm{W}[1]$-hard if $\delta > 2$

Dense Graphs – Cliques



(a) $\delta \in (\frac{3}{2}, 2]$: $F(S)$ is a matching.



(b) $\delta \in (1, \frac{3}{2}]$: $F(S)$ is a star.

Neighborhood Diversity

[Hartmann, L.; 2022+] Structural parameterization of dense graphs

Parameter including large cliques:

Neighborhood diversity nd(G)

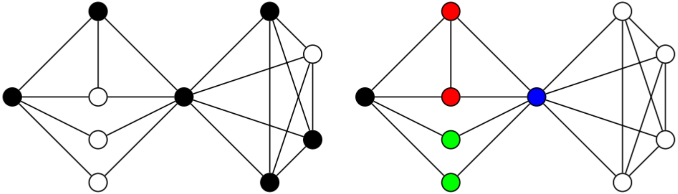
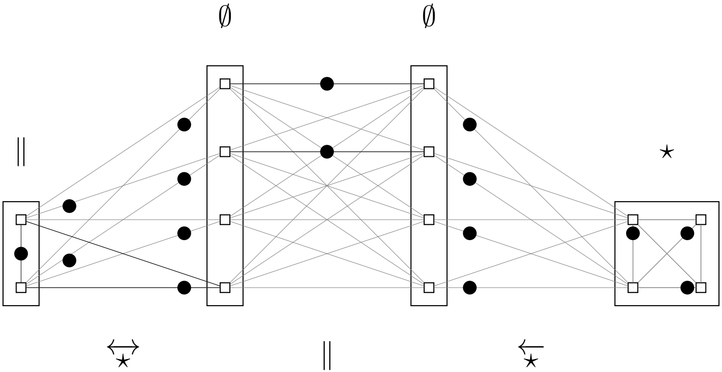


Illustration from [Ganian; SOFSEM 2012]

Canonical Form



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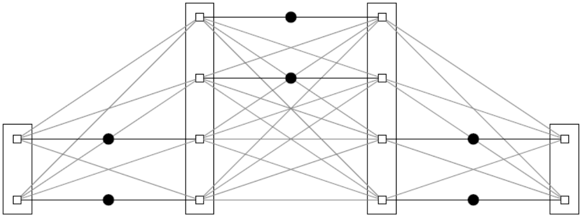

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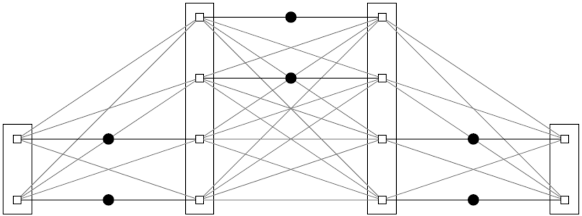

Theorem

DISPERSION *can be solved in time* $2^{\mathcal{O}(\mathrm{nd}(G)^2)} n^{\mathcal{O}(1)}$.

Thank you!