Model-Checking for First-Order Logic with Disjoint Paths Predicates in Proper Minor-Closed Graph Classes

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Joint work with

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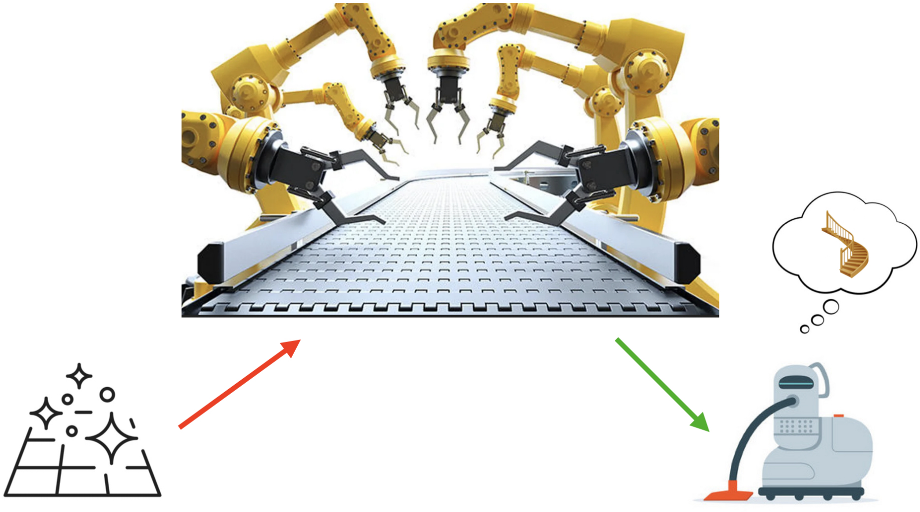
GROW, Koper, Slovenia, 21/09/2022

Algorithmic Meta-Theorems (AMTs):



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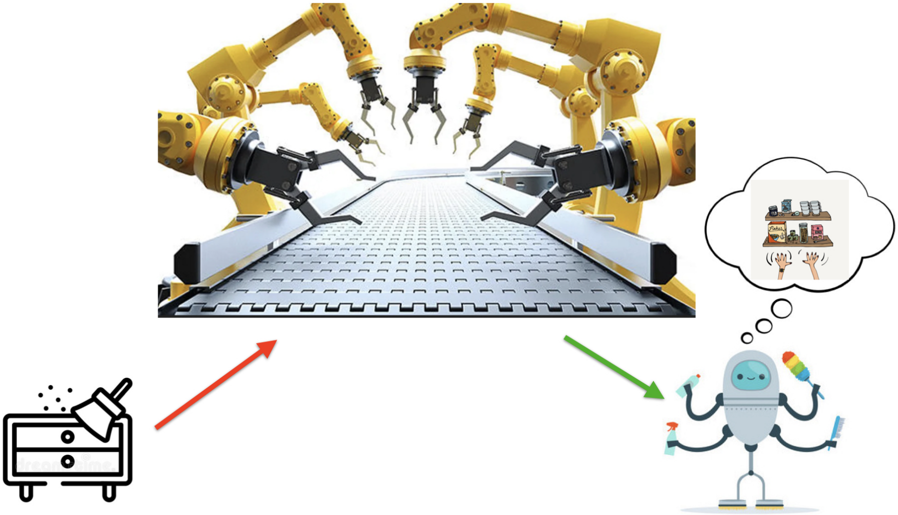

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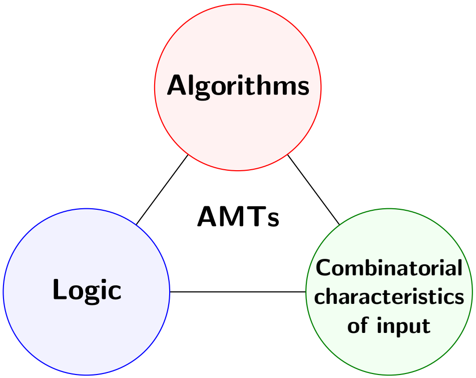

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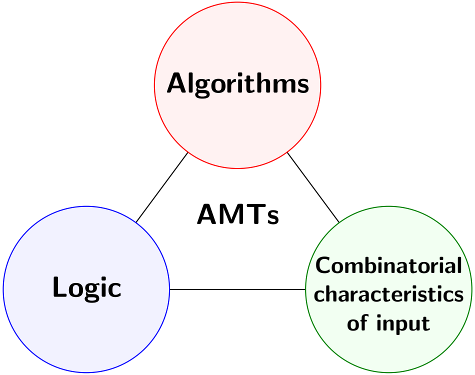

AMTs: General conditions that imply the automatic derivation of efficient algorithms.

"Algorithms that output algorithms"

Form of an **AMT**:

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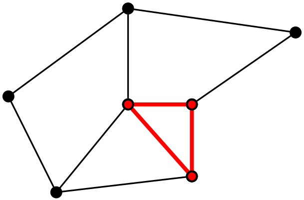
- • Structure → Structural Graph Theory
  - width parameters
  - containment in graph classes
  - sparsity measures
  - ...

Logic on Graphs

	First Order Logic (FOL)	**Monadic Second Order Logic (MSOL)**
Variables	vertices	vertices/edges **sets** of vertices/edges
Predicates	$=$, $\sim$ (adjacency)	$=$, $\sim$, $\in$
Quantifiers over	vertices	**sets** of vertices/edges

Subgraph Isomorphism

Does G contain H as a subgraph?



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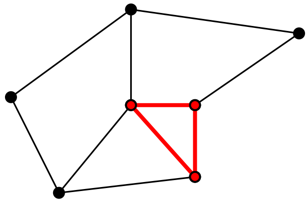
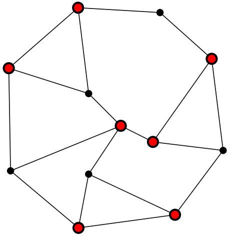


FOL-expressible:

$$\exists x \exists y \exists z \left((x \sim y) \wedge (y \sim z) \wedge (x \sim z) \right)$$

Vertex Cover

Does G contain a set S of k vertices that intersects all the edges of G?



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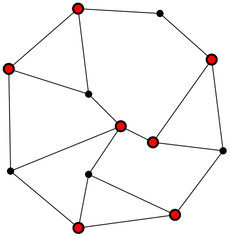
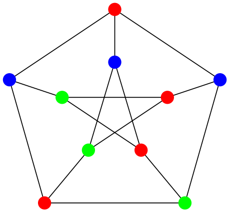


FOL-expressible:

$$\exists v_1, \ldots, v_k \quad \forall x \; \forall y \left(x \sim y \;\rightarrow \bigvee_{i \in \{1,\ldots,k\}} (v_i = x \vee v_i = y) \right)$$

3-colorability

Is there a 3-coloring of G?



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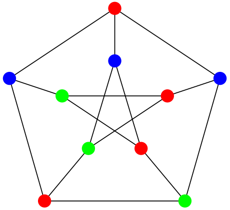

MSOL-expressible:

$$\exists V_1 \exists V_2 \exists V_3 \Bigg(\Big(\forall x \; (x \in V_1 \lor x \in V_2 \lor x \in V_3) \Big) \land$$

$$\Big(\forall x, y \in V_1 \neg (x \sim y) \Big) \land \Big(\forall x, y \in V_2 \neg (x \sim y) \Big) \land \Big(\forall x, y \in V_3 \neg (x \sim y) \Big) \Bigg)$$

Connectivity

Is the graph G connected?

MSOL-expressible:

$$\forall x\, \forall y\, \neg \left(\exists V \;\; \left(x \in V \;\wedge\; y \notin V \;\wedge \left(\forall u\, \forall v\, (u \sim v) \implies (u \in V \Leftrightarrow v \in V) \right) \right) \right)$$

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Given a graph class $\mathcal{C}$:

$$\boxed{\text{Deciding } \textbf{FOL}/\textbf{MSOL}\text{-expressible properties is FPT}^{1} \text{ on } \mathcal{C}.}$$

[1] there is an $f(|\varphi|) \cdot n^{\mathcal{O}(1)}$-time algorithm, where $|\varphi|$ is the size of the given FOL/MSOL-formula and n is the size of the input graph.

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- Courcelle's Theorem: Deciding **MSOL** properties on graphs of bounded treewidth.
  [Courcelle, 1990]

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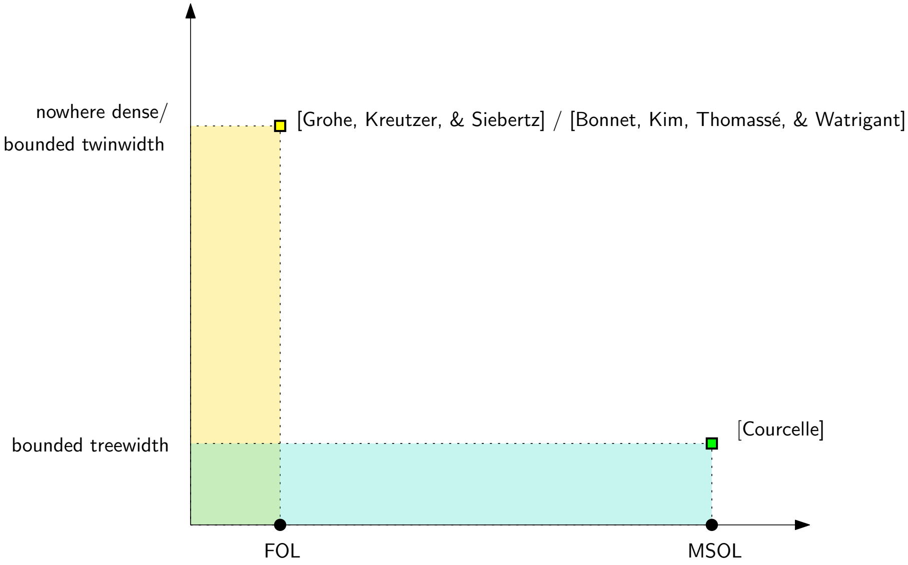
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bounded treewidth
FOL
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What to do next?

- *Meta-algorithmics* of FOL:
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- *Meta-algorithmics* of FOL:
  counting predicates, *transitive-closure operators*, *fixed-point operators*, *successor-invariant formulas*, *FOL-interpretability*,...
- Consider logics between FOL and MSOL.

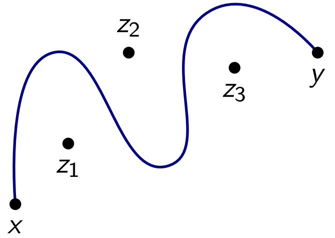
Separator logic (FOL+conn):

[Schirrmacher, Siebertz, & Vigny, 2021]

[Bojańczyk, 2021]

Additional predicate $\mathrm{conn}_k(x, y, z_1, \ldots, z_k)$:

There is an (x, y)*-path that avoids* $z_1, \ldots, z_k$.



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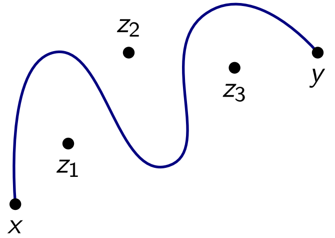

Can express:

k-connectivity.

Elimination distance to ϕ, for $\phi \in$ FOL+conn.

Feedback Vertex Set.

Cannot express:

Planarity, (Topological) Minor Containment.

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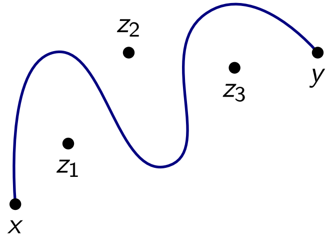

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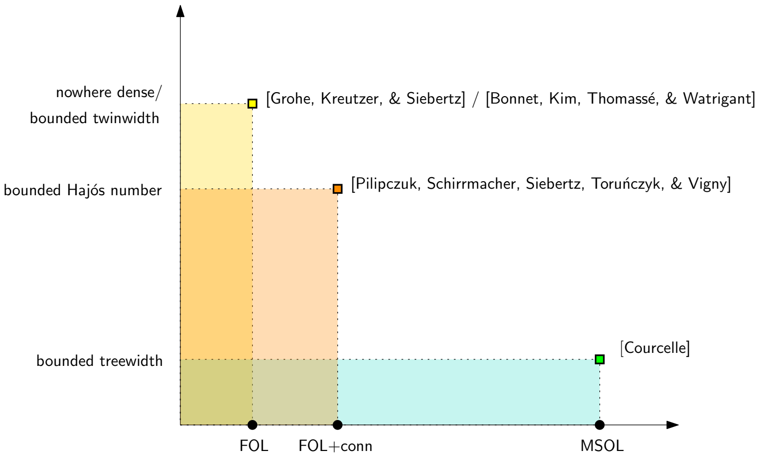


Deciding FOL+conn properties is FPT on graphs of bounded Hajós number*.
[Pilipczuk, Schirrmacher, Siebertz, Toruńczyk, & Vigny, 2022]

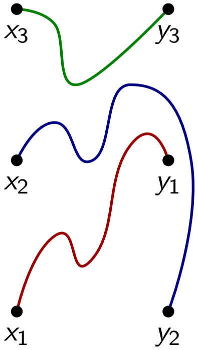
* *Hajós number* = maximum h for which G contains a subdivision of K_h as a subgraph.

FOL+DP

[Schirrmacher, Siebertz, & Vigny, 2021]

Additional predicate $\mathrm{dp}_k(x_1, y_1, \ldots, x_k, y_k)$:

There are pairwise vertex-disjoint paths between x_i and y_i, for every $i \in \{1, \ldots, k\}$.



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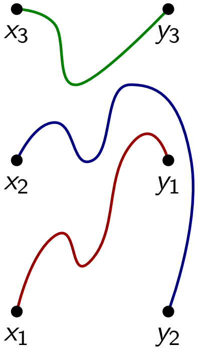

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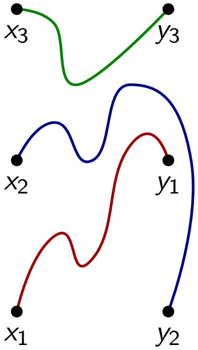

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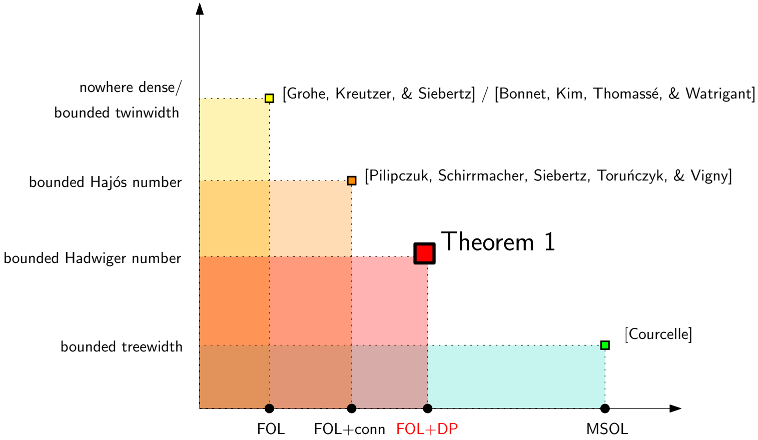


Theorem 1

Deciding FOL+DP properties is FPT on graphs of bounded Hadwiger number*.

* *Hadwiger number* = maximum r for which G contains K_r as a minor.

Indicative (meta) problems (whose standard parameterizations) automatically classified in FPT.

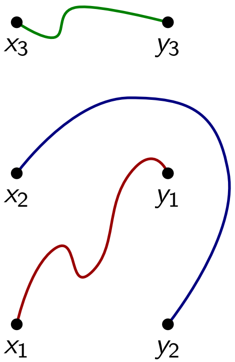
- MINOR CONTAINMENT,
- TOPOLOGICAL MINOR CONTAINMENT,
- CYCLABILITY,
- UNORDERED LINKABILITY,
- ORDERED LINKABILITY,
- $\mathcal{F}$-MINOR-DELETION,
- $\mathcal{F}$-TOPOLOGICAL MINOR-DELETION,
- $\mathcal{F}$-CONTRACTION DELETION (for bounded genus graphs),
- ANNOTATED $\mathcal{F}$-$\preceq$-DELETION,
- SUBSET $\mathcal{F}$-$\preceq$-DELETION,
- φ-DELETION,
- φ-AMALGAMATION,
- $\mathcal{L}$-φ-REPLACEMENT,
- φ-ELIMINATION DISTANCE,
- φ-RECONFIGURATION.

FOL+SDP

Scattered disjoint paths predicates:

$s\text{-dp}_k(x_1, y_1, \ldots, x_k, y_k)$

There are pairwise vertex-disjoint paths between x_i and y_i, for every $i \in \{1, \ldots, k\}$
s.t. no two vertices of two distinct paths are within distance $\leq s$.



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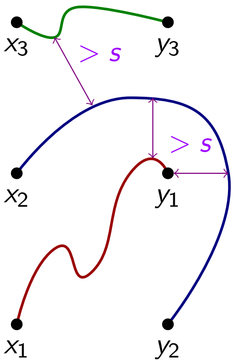

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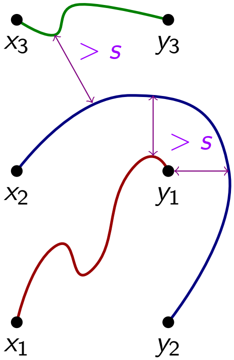
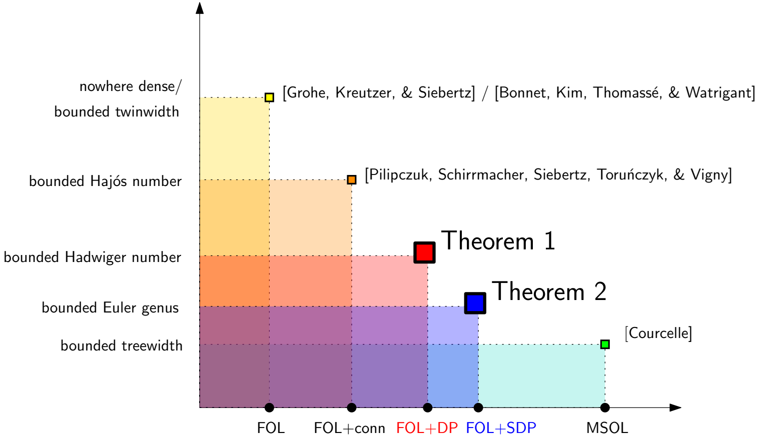


$\text{dp}_k(x_1, y_1, \ldots, x_k, y_k) = 0\text{-dp}_k(x_1, y_1, \ldots, x_k, y_k)$



Theorem 2

Deciding FOL+SDP properties is FPT on graphs of bounded Euler genus.

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- INDUCED TOPOLOGICAL MINOR CONTAINMENT,
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- φ-DELETION,
- φ-AMALGAMATION,
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- φ-ELIMINATION DISTANCE,
- φ-RECONFIGURATION.

Sketch of proof: .

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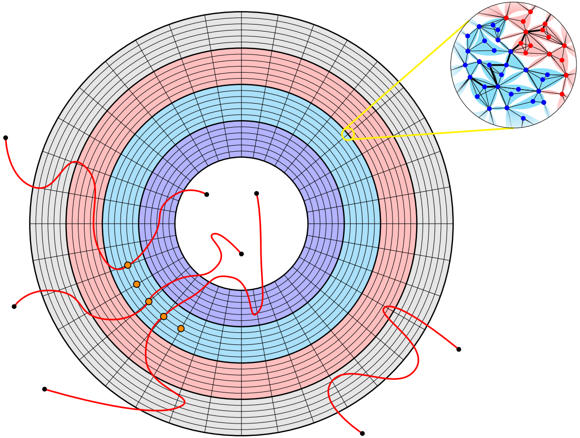
Irrelevant vertex technique

Detect vertex whose removal does not affect the existence of solution.
Build reduction rule to simplify instance.

[Robertson & Seymour, 1994]



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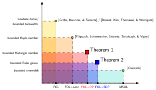
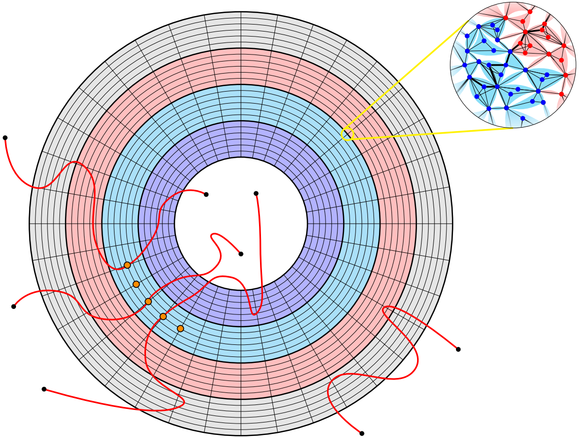
Abstraction of irrelevant vertex technique for graphs of bounded Hadwiger number / Euler genus to two **AMTs**.





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- **Flat walls framework**

[Sau, **S.**, & Thilikos, 2021]
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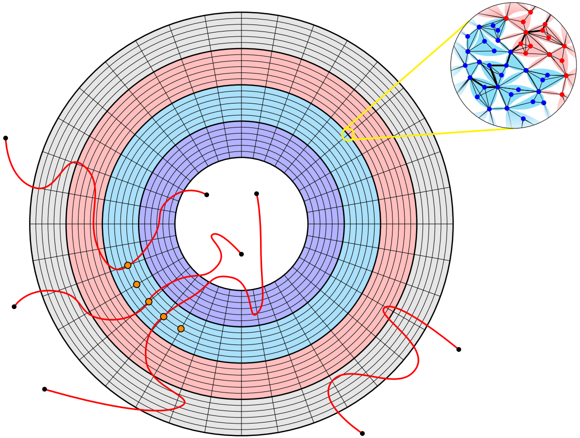

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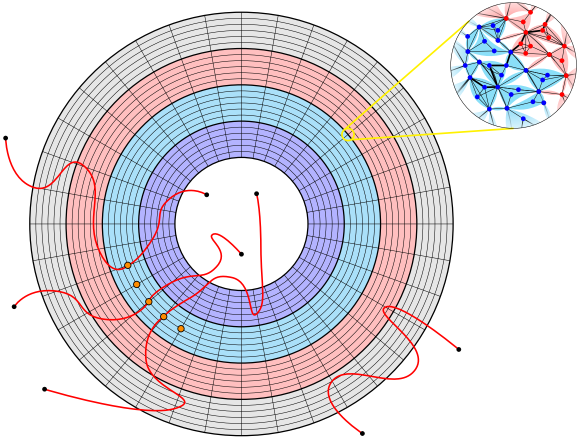
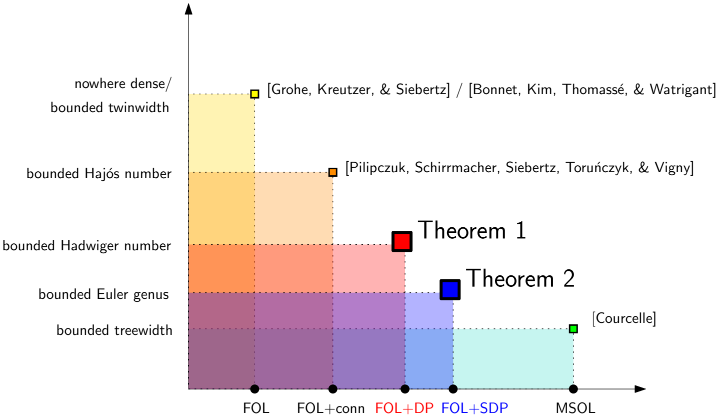
- **Combing Linkages in Annuli**

[Golovach, **S.**, & Thilikos, 2022]



Conclusion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Theorem 1
Theorem 2
[Courcelle]
FOL
FOL+conn
FOL+DP
FOL+SDP
MSOL

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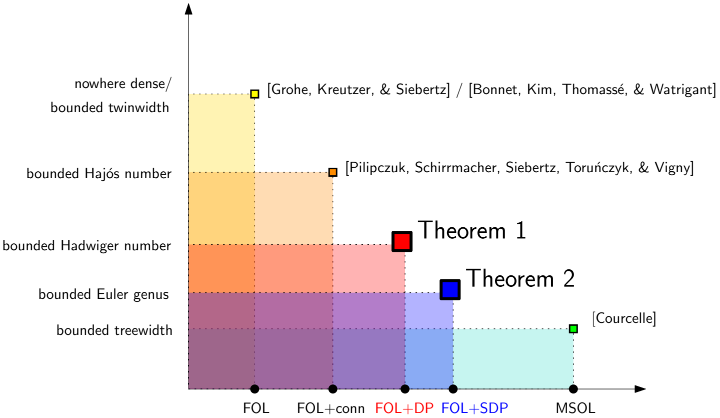

Research directions:

- Extend **Theorem 1** to graphs of bounded Hajós number.
- Enhance FOL+(S)DP with additional properties on paths.

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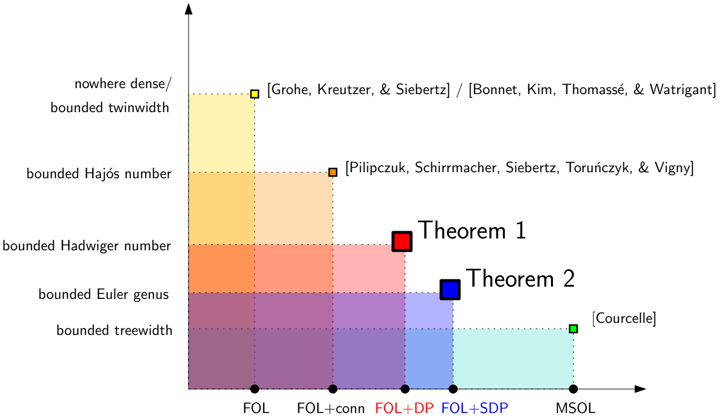

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