A splitter theorem for directed graphs

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Koper, September 20th 2022

Generating strongly 2-connected graphs

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Generation of graphs

The class of all 2-connected graphs can be constructed from cycles by adding internally disjoint paths (ears). [Whitney '32]

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The class of all 3-connected graphs can be constructed from K_4 by 3-augmentations. [Barnette, Grünbaum '69, Titov '75]

Seymour's splitter theorem

Theorem [Seymour '80]

If G and H are 3-connected graphs such that H is a proper minor of G, then there exists a 3-connected graph K such that K is a minor of G and one of the following statements holds:

- K can be obtained from H by 3-expansion or edge addition, or
- H and K are wheels and $|V(H)| + 1 = |V(K)|$.

Generation-Corollary

The class of 3-connected graphs can be constructed from the set of wheels by 3-expansions and edge-additions.

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The class of 3-connected graphs can be constructed from the set of wheels by 3-expansions and edge-additions.

Split-Corollary

The class of graphs with a proper K_5 minor is disjoint from the class of graphs excluding $K_{3,3}$ as a minor.

From graphs to directed graphs

The class of all 2-connected graphs can be constructed from cycles by adding internally disjoint paths (ears).

The class of all strongly connected digraphs can be constructed from directed cycles by internally disjoint directed "paths" (start and end vertex may be the same).

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What about strongly 2-connected digraphs?

Decompositions along 1-separations

Every digraph can be decomposed along separations of order one into a tree of strongly 2-connected digraphs.

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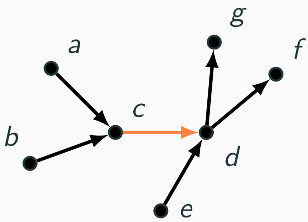
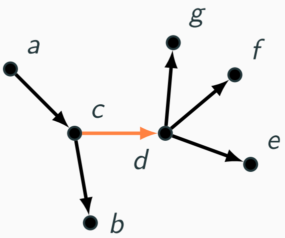
Every digraph can be decomposed along separations of order one into a tree of strongly 2-connected digraphs.

Thus, constructing the strongly 2-connected digraphs allows us to construct all digraphs.

Butterfly minors





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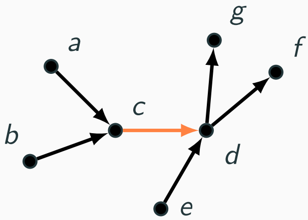
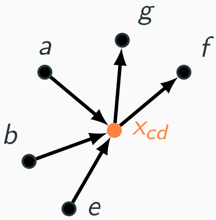
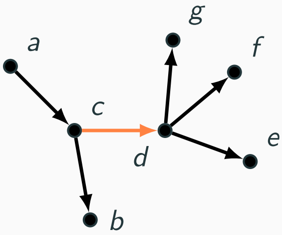
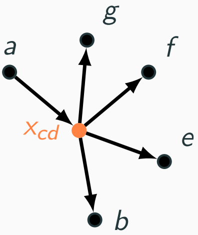
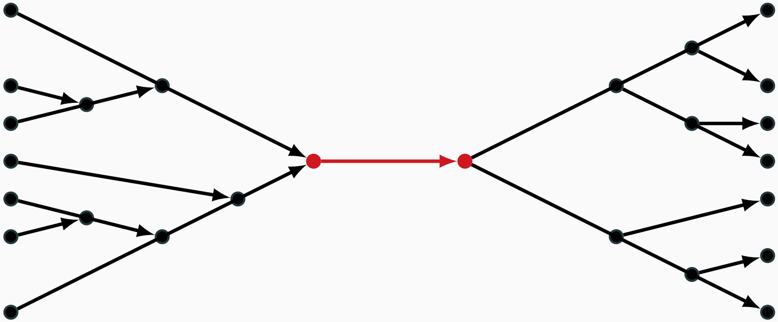






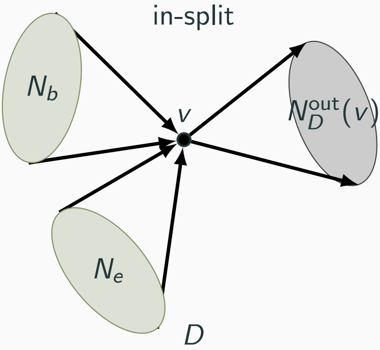


Butterfly minor models



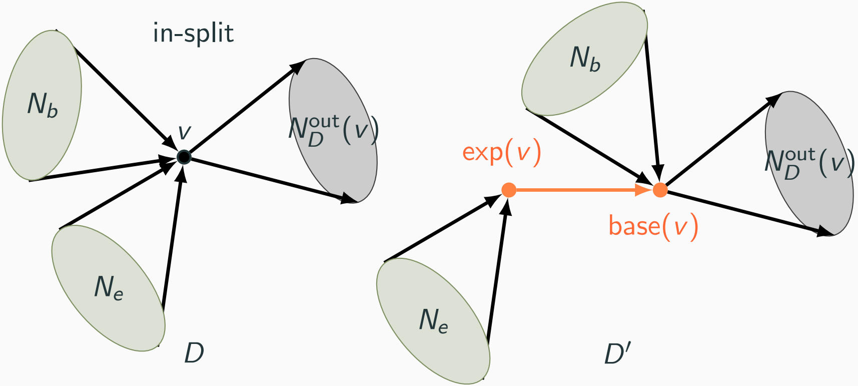
Let H and D be digraphs. We call a subgraph H' of D an H-expansion, if H' is a minor model of H.

splits



$|N_b| \geq 1$, $|N_e| \geq 2$

splits



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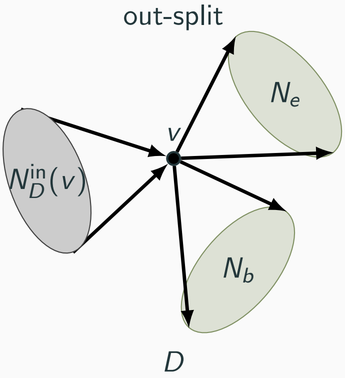

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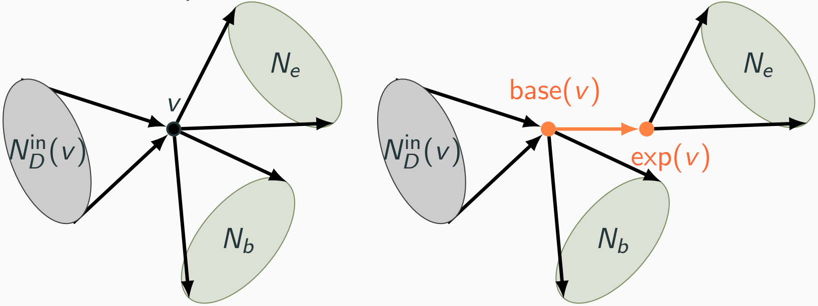
splits

out-split



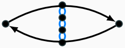
D D'

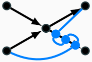
$|N_b| \geq 1$, $|N_e| \geq 2$

Augmentations

A_1) basic augmentation 

A_2) chain augmentation 

A_3) collarette augmentation 

A_4) bracelet augmentation 

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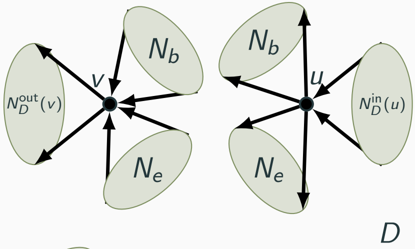

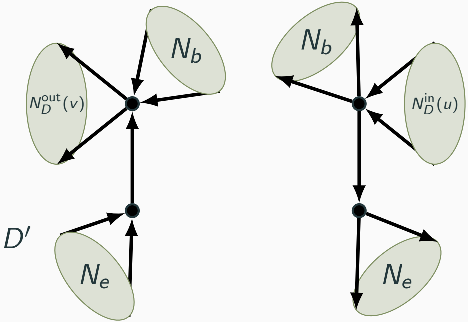

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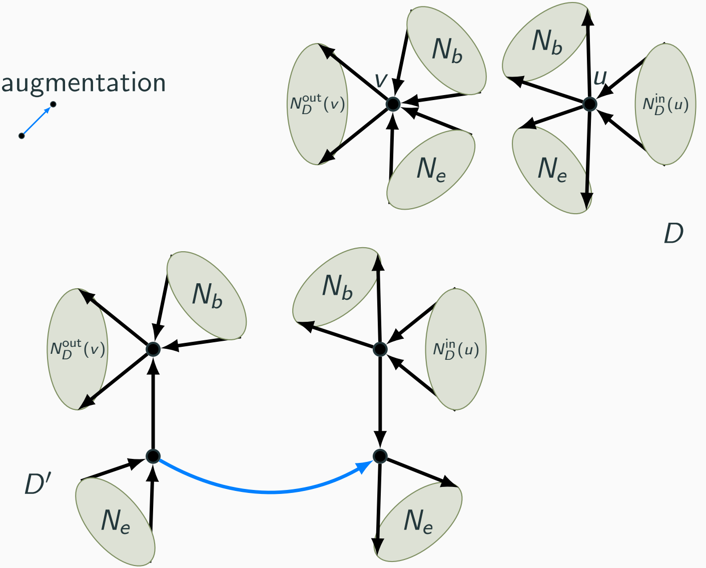
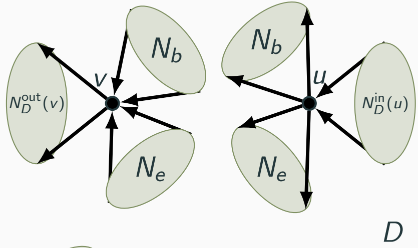
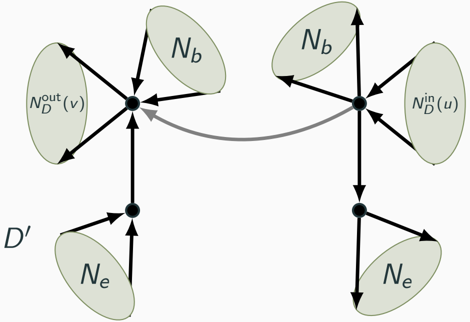


Augmentations

A_2) chain augmentation





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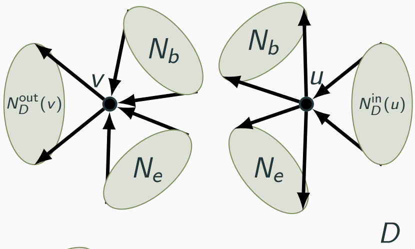
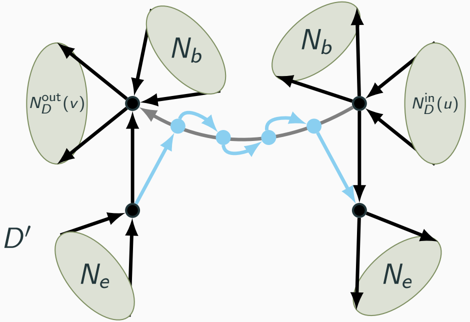
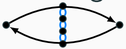
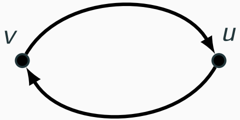




Augmentations

A_3) collarette augmentation





D

D'

Augmentations

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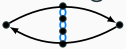
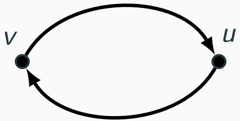
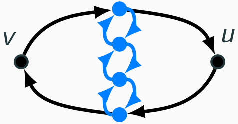




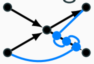
D

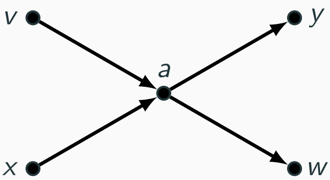


D'

Augmentations

A_4) bracelet augmentation





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D'

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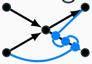
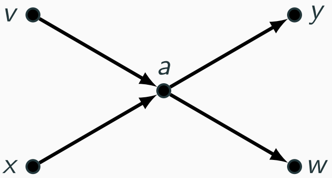
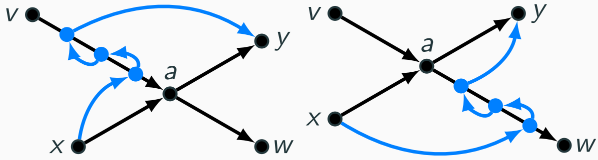




D

D'

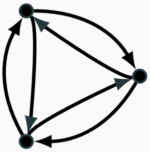


A splitter theorem for directed graphs

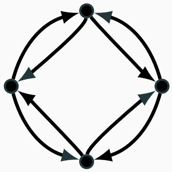
Directed splitter theorem

If D and H are strongly 2-connected digraphs and H' is an H-expansion in D, then there exists a strongly 2-connected digraph K such that D is a K-expansion and K is an H-augmentation.

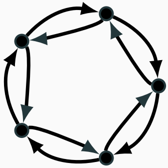
Base class $\mathcal{B}$



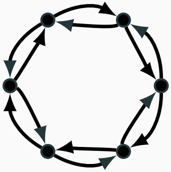
C_3



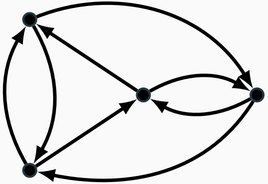
C_4



C_5



C_6



A_4

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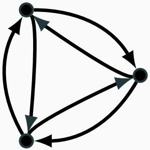

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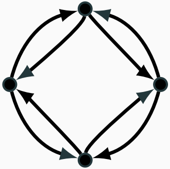
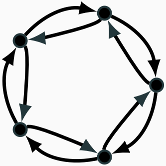


C_4



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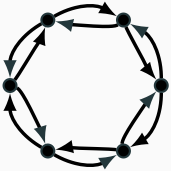

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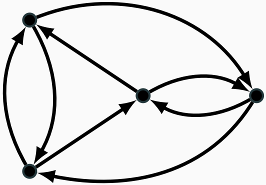

A_4

Base-Theorem [Wiederrecht '20]

Every strongly 2-connected digraph on at least 3 vertices contains a digraph from $\mathcal{B}$ as a butterfly minor.

Corollary

For every strongly 2-connected digraph D there exists a sequence $(D_0, \ldots, D_k)$ of strongly 2-connected digraphs such that $D_0 \in \mathcal{B}$, $D_k \cong D$, and
D_{i+1} is an augmentation of D_i for all $1 \leq i \leq k$.

$D_k \cong D$

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$$\begin{tikzcd}
D_0' \arrow[r, bend left, "\subseteq"] & D_k \cong D \\
D_0 \in \mathcal{B} \arrow[u, "\text{exp.}"] &
\end{tikzcd}$$

exists by Base-Theorem

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D_0 \in \mathcal{B} \arrow[u, "\text{exp.}"] \arrow[r, "\text{aug.}"'] & D_1 \arrow[u, "\text{exp.}"] &
\end{tikzcd}
$$

apply Directed Splitter Theorem to D and D_0

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\end{tikzcd}
$$

apply Directed Splitter Theorem to D and D_1

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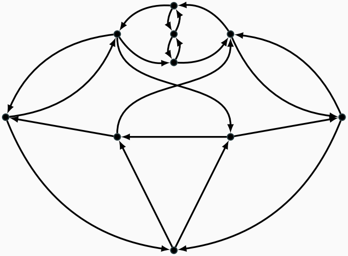
\end{tikzcd}
$$

example construction



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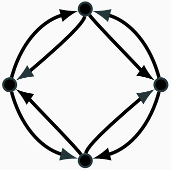

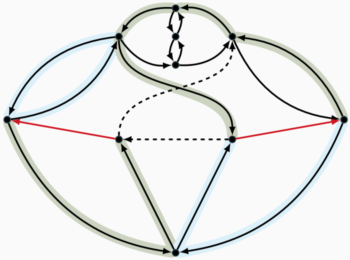

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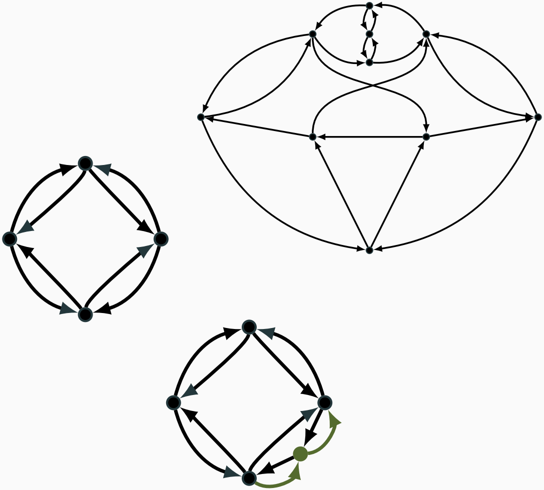

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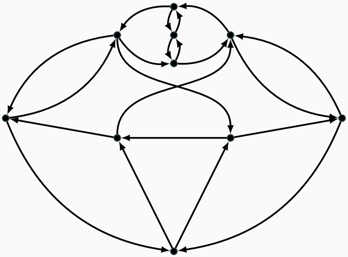

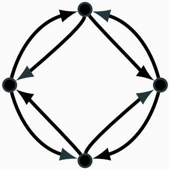

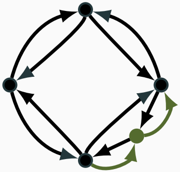

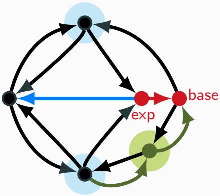

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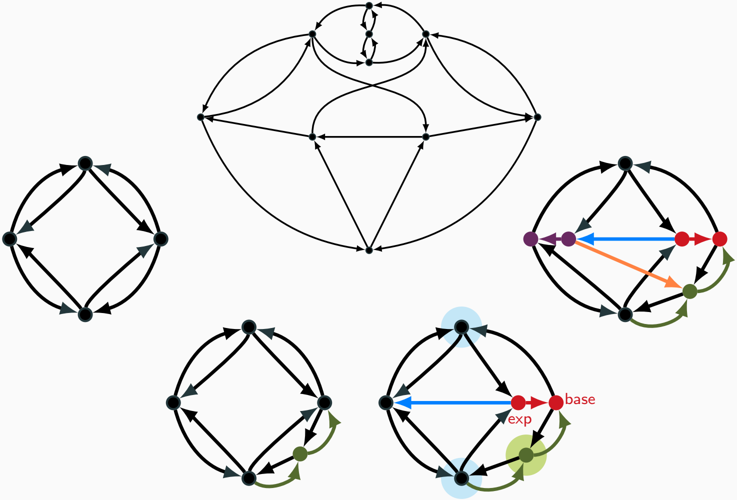

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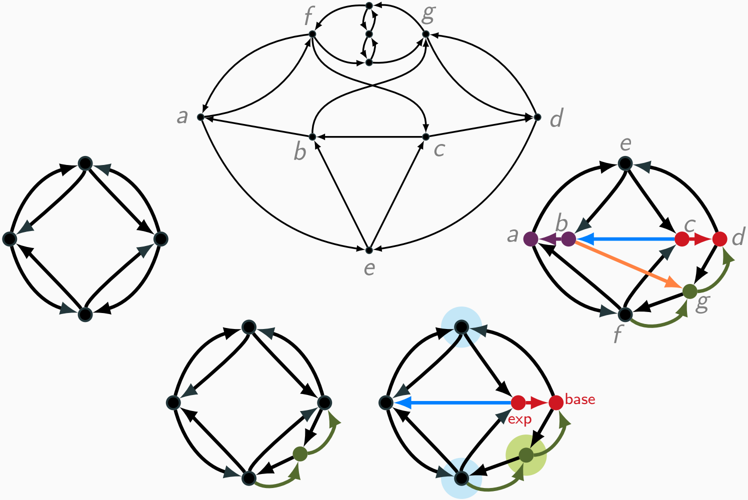

Proof sketch

If $D_i \ncong D$ yet, then there is an ear-path with respect to the minor model of D_i in D.

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What does our obtained ear look like?

- switching – we can replace a path in the minor model by the ear-path getting a new model for D_i
- augmenting – adding this ear-path immediately yields a valid model for a larger graph (D_{i+1})
- bad – can not be easily added as it would close a cycle in the minor model

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case 3: there is a vertex with non-trivial model and in- and out-degree 2

Proof sketch

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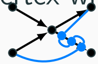
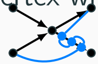
case 2: there is a chain 

case 3: there is a vertex with non-trivial model and in- and out-degree 2 

Proof sketch

case 1: there is an augmenting path 

case 2: there is a chain 

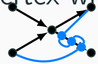
case 3: there is a vertex with non-trivial model and in- and out-degree 2 

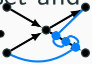
case 4: there is a vertex with non-trivial branch set and in- or out-degree at least 3

Proof sketch

case 1: there is an augmenting path 

case 2: there is a chain 

case 3: there is a vertex with non-trivial model and in- and out-degree 2 

case 4: there is a vertex with non-trivial branch set and in- or out-degree at least 3 switch   

Proof sketch

case 1: there is an augmenting path 

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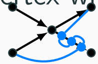
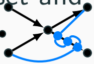
case 3: there is a vertex with non-trivial model and in- and out-degree 2 

case 4: there is a vertex with non-trivial branch set and in- or out-degree at least 3 switch   

case 5: trivial branch sets and a certain type of switching path

Proof sketch

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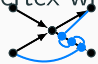
case 3: there is a vertex with non-trivial model and in- and out-degree 2 

case 4: there is a vertex with non-trivial branch set and in- or out-degree at least 3 switch 

case 5: trivial branch sets and a certain type of switching path switch non-trivial branch set

case 6: trivial branch sets and no such switching path

Proof sketch

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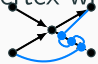
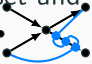
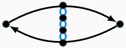
case 3: there is a vertex with non-trivial model and in- and out-degree 2 

case 4: there is a vertex with non-trivial branch set and in- or out-degree at least 3 switch   

case 5: trivial branch sets and a certain type of switching path switch non-trivial branch set

case 6: trivial branch sets and no such switching path choose ear carefully  

Open questions

- What would be a good definition of k-blocks for directed graphs?
  Can we characterise or construct them?
  What are minors enforcing such k-blocks?
- Can one prove splitter theorems for strong minors?
- What kind of classes can we split with our theorem?

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Thank you!