Polynomial algorithm to compute the toughness of graphs with bounded treewidth

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Toughness

Definition

Let t be a positive real number. A graph G is called t-tough, if

$$c(G - S) \leq \frac{|S|}{t}$$

for any cutset S of G.
The toughness of G, denoted by $\tau(G)$, is the largest t for which G is t-tough, taking $\tau(K_n) = \infty$ for all $n \geq 1$.

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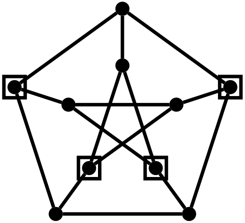

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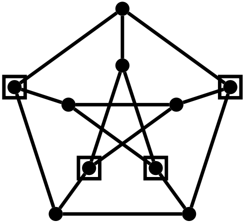

The Petersen graph is 4/3-tough.

A cycle is 1-tough.

Toughness

In other words: for a non-complete, connected graph G,

$$\tau(G) = \min_{S \text{ cutset}} \frac{|S|}{c(G-S)}.$$

S is called a **tough set** if it gives the ratio $\tau(G)$.

Observation

For a non-complete, connected graph G on n vertices the toughness $\tau(G)$ is a rational number $\frac{p}{q}$ with $1 \le p, q \le n$.

Proof.

Clearly $1 \le |S| \le n-2$ and $2 \le c(G-S) \le n-1$. □

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Complexity of toughness

Let t be an arbitrary positive rational number and consider the following problem.

***t*-TOUGH**

Instance: A graph G,

Question: Is it true that $\tau(G) \geq t$?

Theorem (Bauer, Hakimi, Schmeichel, 1990)

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Theorem (Bauer, van den Heuvel, Morgana, Schmeichel, 1998)

1-TOUGH *is coNP-complete for r-regular graphs for all $r \geq 3$.*

Complexity of toughness in special graph classes

Theorem (Kratsch, Lehel, Müller, 1996)

The problem **1**-TOUGH *is coNP-complete for bipartite graphs.*

Theorem (GY. K., K. Varga, 2022)

For any positive rational number $t \leq 1$ the problem ***t***-TOUGH *remains coNP-complete for bipartite graphs.*

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Deciding the toughness remains coNP-hard in many other special graph classes.

Famous open cases: planar graphs, chordal graphs

Complexity of toughness in special graph classes

For some other special graph classes there are polynomial algorithms to compute the toughness:

- Claw-free graphs (Matthews, Sumner, 1984)
- Split graphs (Kratsch, Lehel, Müller, 1996; Woeginger, 1998)
- Interval graphs (Kratsch, Kloks, Müller, 1994)
- $2K_2$-free graphs (Broersma, Patel, Pyatkin, 2014)
- some other more special classes ...

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Main result

Theorem

There exists an algorithm to compute the toughness of a graph G width running time $\mathcal{O}\left(n^3 \cdot \mathrm{tw}(G)^{2\mathrm{tw}(G)}\right)$, where n is the number of vertices in G and $\mathrm{tw}(G)$ is the treewidth of G.

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The toughness can be computed in polynomial time for graphs width bounded treewidth.

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Toughness is FPT parameterized with treewidth.

Tree decomposition

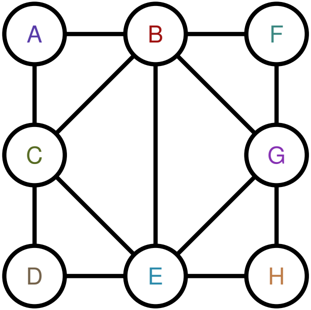
Let $G = (V, E)$ be a graph. Let $(X_t)_{t \in V(T)}$ be a family of vertex sets $X_t \subseteq V$ (bags) indexed by the nodes of a tree T. The pair $(T, \{X_t \mid t \in V(T)\})$ is a tree decomposition of G if it satisfies the following conditions:

- $\bigcup_{t \in V(T)} X_i = V$;
- for every edge $e = vw \in E$ there is a $t \in V(T)$ with $v, w \in X_t$;
- if $i, j, k \in V(T)$ and node j is on the path in T between nodes i and k, then $X_i \cap X_k \subseteq X_j$.

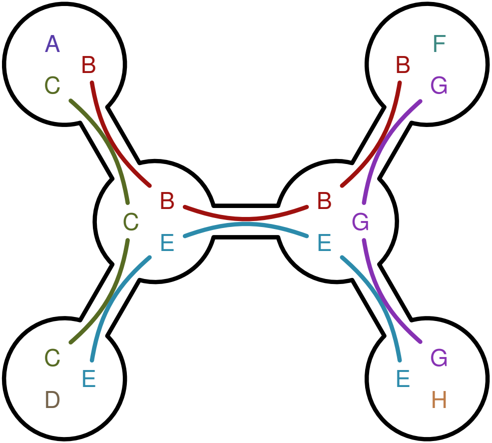
The width of the tree decomposition is $\max_{t \in V(T)} |X_t| - 1$.
The treewidth of a G is the minimum width of a tree decomposition of G.

Tree decomposition

Graph G



Tree decomposition (T, X)



Bounded treewidth

Theorem (Bodlaender, 1996)

A tree decomposition with width $\mathrm{tw}(G)$ *can be constructed in* $\mathrm{tw}(G)^{\mathcal{O}(\mathrm{tw}(G)^3)} \cdot n$ *time.*

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Bounded treewidth

Theorem (Bodlaender, 1996)

A tree decomposition with width $\mathrm{tw}(G)$ *can be constructed in* $\mathrm{tw}(G)^{\mathcal{O}(\mathrm{tw}(G)^3)} \cdot n$ *time.*

Many NP-hard problems are FPT parameterized with treewidth, so they are solvable in polynomial time if the treewidth is bounded.

Trees

Graphs with $\mathrm{tw}(G) = 1$ are trees.

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Lemma

The toughness of a tree is $1/\Delta(G)$.

Proof.

Every tough set is a single vertex with maximum degree. $\square$

Series parallel graphs

Graphs with $\mathrm{tw}(G) = 2$ are series parallel graphs.

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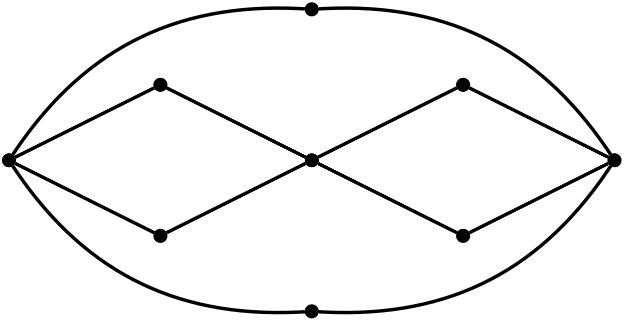

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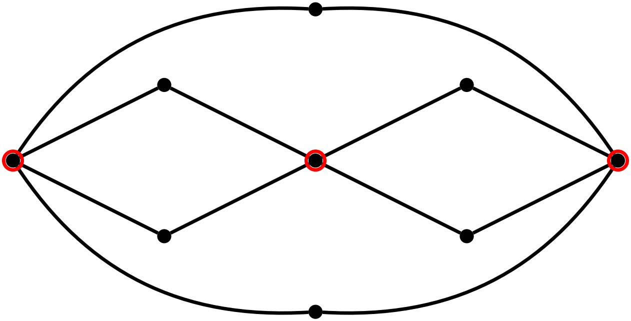

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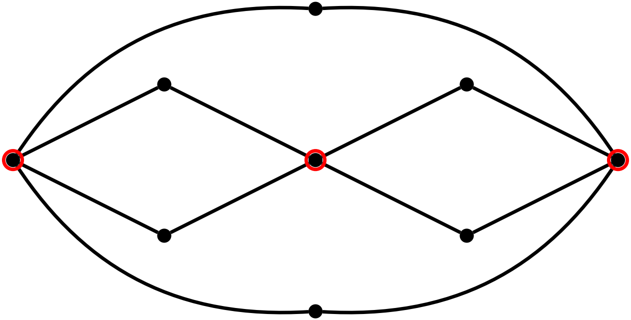


A polynomial algorithm can be designed using dynamic programming on the series-parallel decomposition tree.

Nice tree decomposition

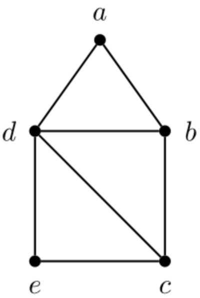
A rooted tree decomposition $(T, \{X_t : t \in T\})$ of a graph G is nice if every node $t \in V(T) \setminus$ root is of one of the following four types:

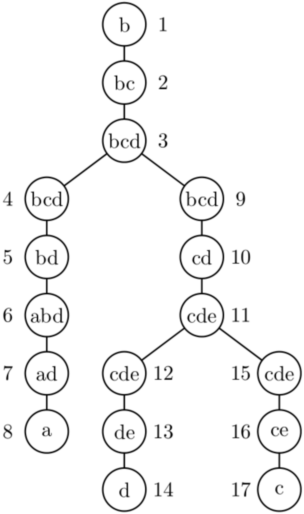
- Leaf: no children and $|X_t| = 1$.
- Introduce: a unique child t' and $X_t = X_{t'} \cup \{v\}$ with $v \notin X_{t'}$.
- Forget: a unique child t' and $X_t = X_{t'} \setminus \{v\}$ with $v \in X_{t'}$.
- Join: two children t_1 and t_2 with $X_t = X_{t_1} = X_{t_2}$.

Theorem (Bodlaender, Kloks, 1996)

A tree decomposition $(T, \{X_t : t \in T\})$ of width $\mathrm{tw}(G)$ of an n-vertex graph G can be transformed in time $\mathcal{O}\left(\mathrm{tw}(G)^2 \cdot n\right)$ into a nice tree decomposition of G of width $\mathrm{tw}(G)$ and $4n$ nodes.

Nice tree decomposition





The algorithm

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- The total size of information for one vertex of t is $\mathcal{O}\left(n \cdot \mathrm{tw}(G)^{\mathrm{tw}(G)}\right)$.

The algorithm

- For the root r of the tree compute:

$$\tau(G) = \min\left\{ \frac{s}{\textsc{Mnc}(r, s, Q, \mathcal{P})} \,\middle|\, 0 \le s < n; \textsc{Mnc}(r, s, Q, \mathcal{P}) \ge 2 \right\}$$

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How to compute $\textsc{Mnc}(t, s, Q, \mathcal{P})$?

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- Join: hardest case

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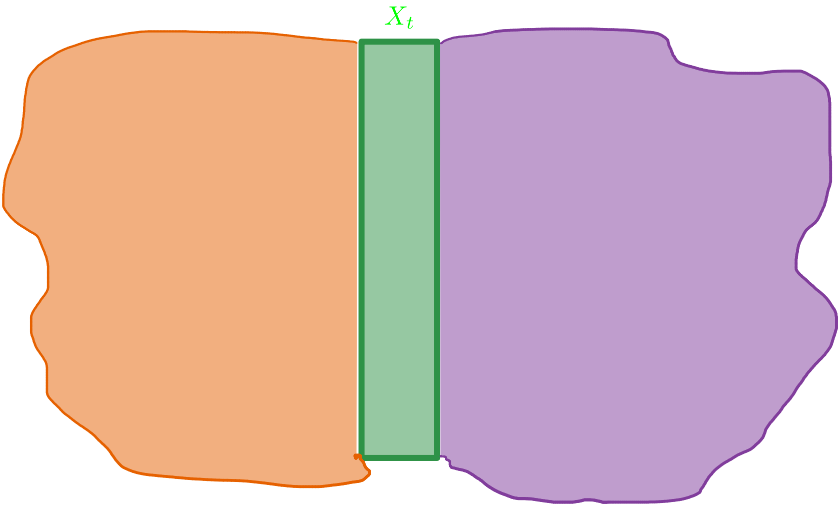

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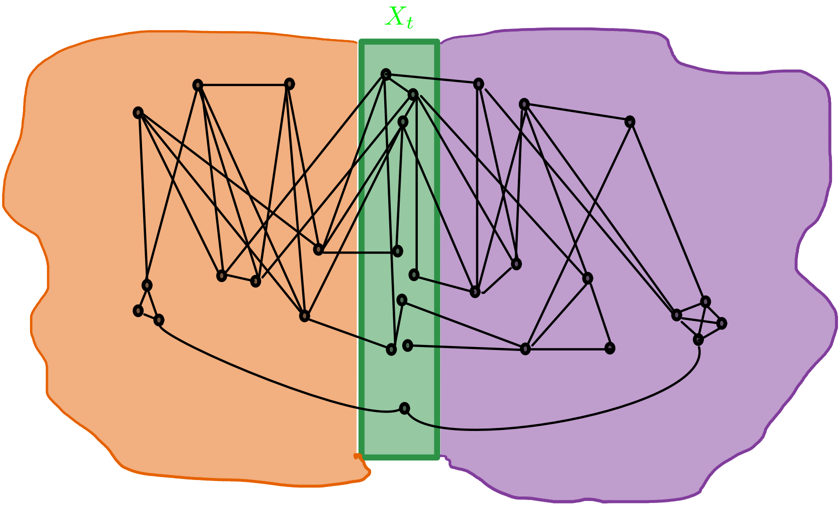

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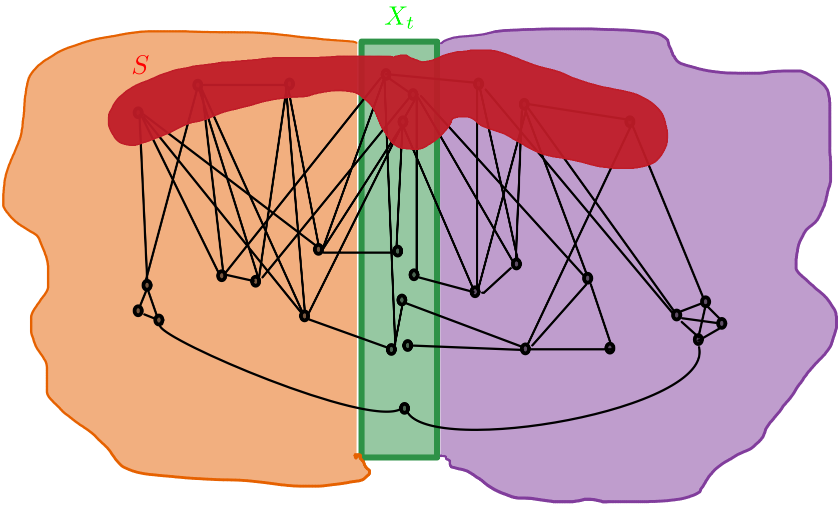

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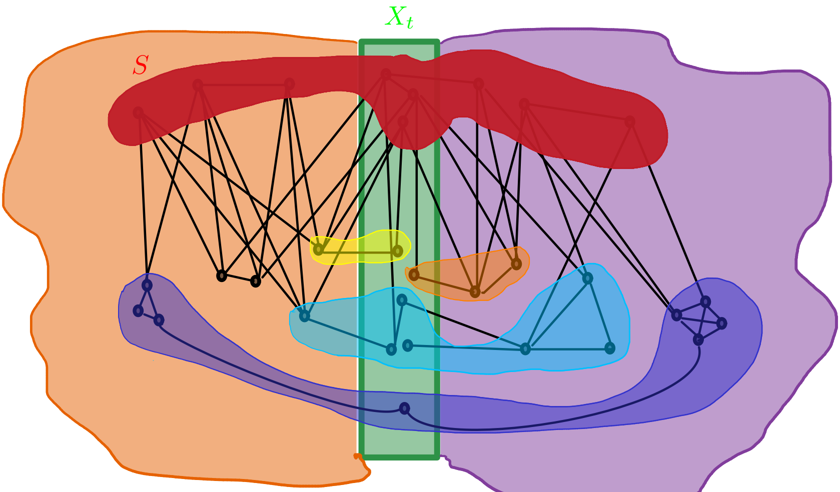

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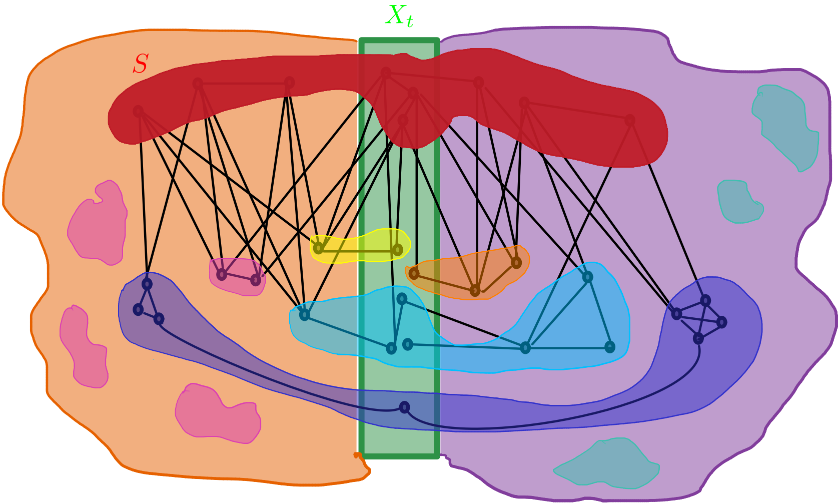

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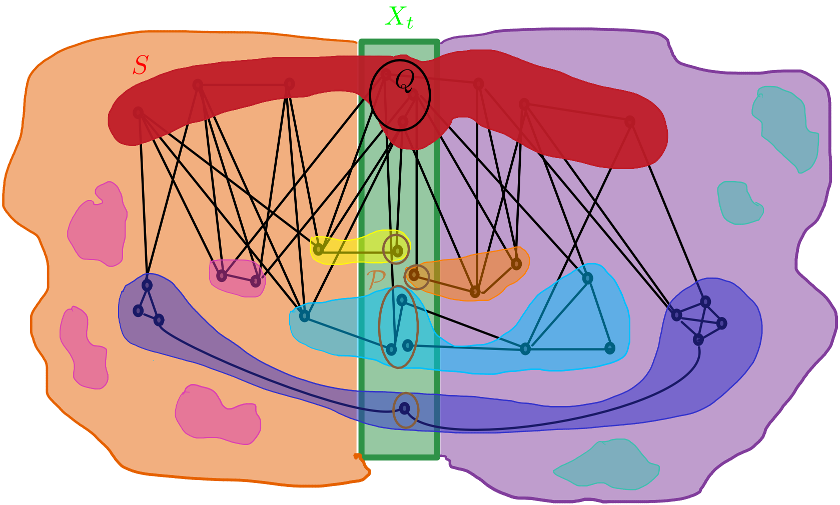

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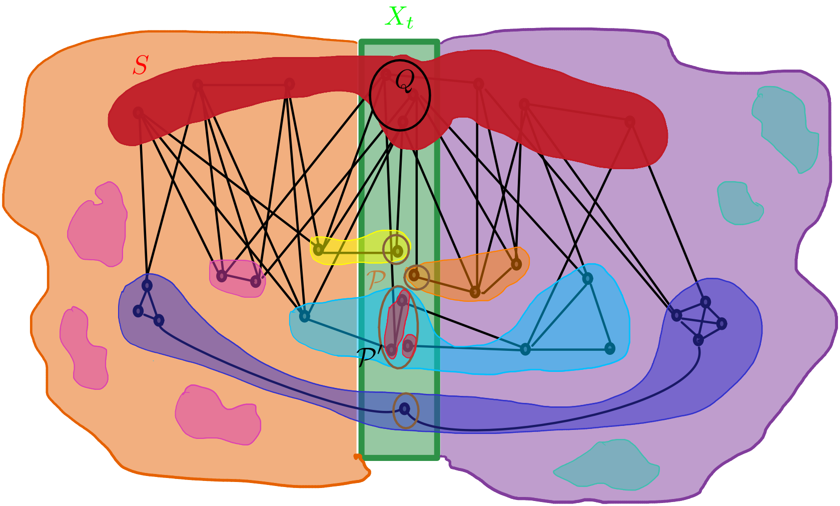

Running time

- Number of vertices in the tree: $\mathcal{O}(n)$
- Computing for Leafs: $\mathcal{O}(1)$
- Computing for Introduce, Forget: $\mathcal{O}\left(n \cdot \mathrm{tw}(G)^{\mathrm{tw}(G)}\right)$
- Computing for Join: $\mathcal{O}\left(n^2 \cdot \mathrm{tw}(G)^{2\mathrm{tw}(G)}\right)$
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Conjecture

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I believe that the methods invented by Bodlaender, Cygan, Kratsch and Nederlof (2013) will work here, too.

Open questions

Question

What is the complexity of t-TOUGH for

- *chordal graphs?*
- *planar graphs?*

The End