One graph to rule them all

Marthe Bonamy

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Trees?

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Theorem (Alstrup, Dahlgaard, Knudsen '17)

Universal graph for trees on $O(n)$ vertices.

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Theorem (Alon '17)

Universal graph for all graphs on $(1 + o(1)) \cdot 2^{\frac{n-1}{2}}$ *vertices.*

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n^6: easy (5-degeneracy), n^4 (Kannan et al. '88), n^3 (Schnyder '89), $n^{2+o(1)}$ (Gavoille, Labourel '07).

A key lemma for planar graphs

Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood '19)

Every planar graph is the almost-induced subgraph of $P \boxtimes H$, where P is a path, and H a graph of treewidth 8.

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Theorem (Dujmović, Esperet, Joret, Walczak, Wood '19)

Planar graphs admit a nonrepetitive colouring with $O(1)$ colours.

('02 question by Alon, Grytczuk, Hałuszczak, Riordan)

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Theorem (Dujmović, Esperet, Joret, Gavoille, Micek, Morin '20)

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Lower bounds

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If $\mu_{\mathcal{H}}(n) = O(\log n)$ and $\mathcal{H}$ is hereditary, there is a universal graph for $\mathcal{H}$ on $2^{O(\log n)} = Poly(n)$ vertices.

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$\Rightarrow$ Every dense class can be optimally compressed.

Ruining the game

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Theorem (Hatami, Hatami '21)

No. (Sometimes need (almost) $2^{\sqrt{n}}$.)

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If $\mu_{\mathcal{H}}(n) < (\frac{1}{4} - \varepsilon) \cdot n$ and $\mathcal{H}$ is hereditary, there is a universal graph for $\mathcal{H}$ on $2^{n^{1-\frac{1}{d}} \cdot \log n}$ vertices, where $d = d_\varepsilon$.

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Thank you!

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