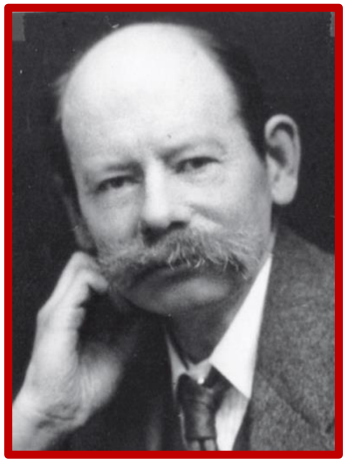
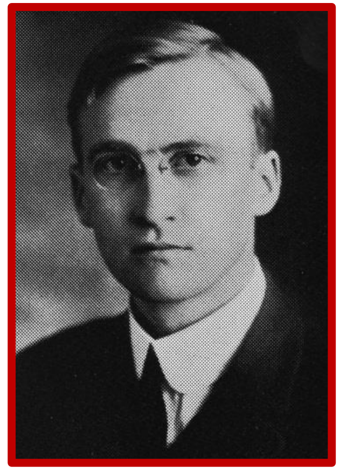
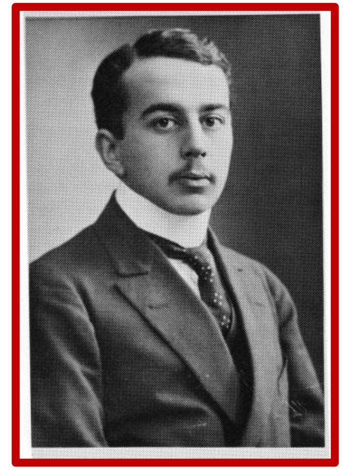
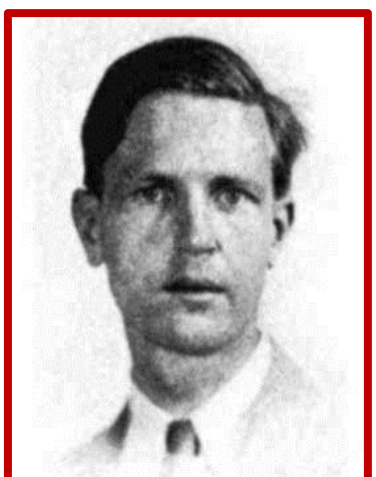
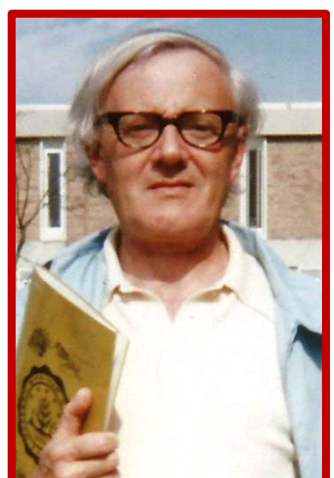
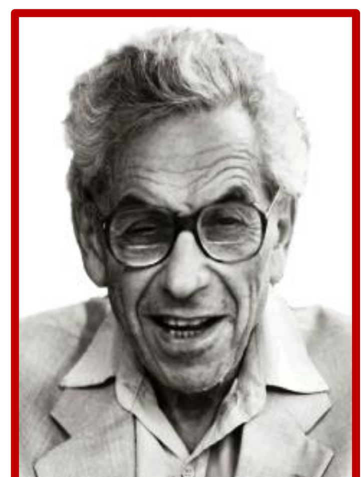
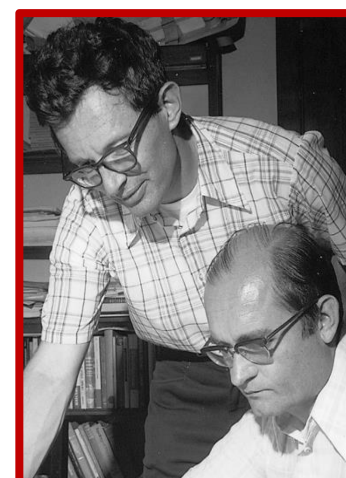
7/8. A Century of Graph Theory

A ‘whistle-stop tour’ with Robin Wilson of graph theory milestones and personalities from 1890 to 1990,





Graph theory: 1840–1890

1852: The 4-colour problem is posed
1879: Kempe ‘proves’ the 4-colour theorem
1880: Tait introduces edge-colourings

1855–57: Kirkman and Hamilton on cycles
1871: Hierholzer on Eulerian graphs

1845: Kirchhoff introduces spanning trees
1857–75: Cayley counts trees and molecules
1878: Sylvester’s chemistry and ‘graphs’
1889: Cayley’s n^{n-2} theorem

1861: Listing’s topological complexes

Four themes

A. Colouring maps and graphs
   (Four-colour theorem, Heawood conjecture)

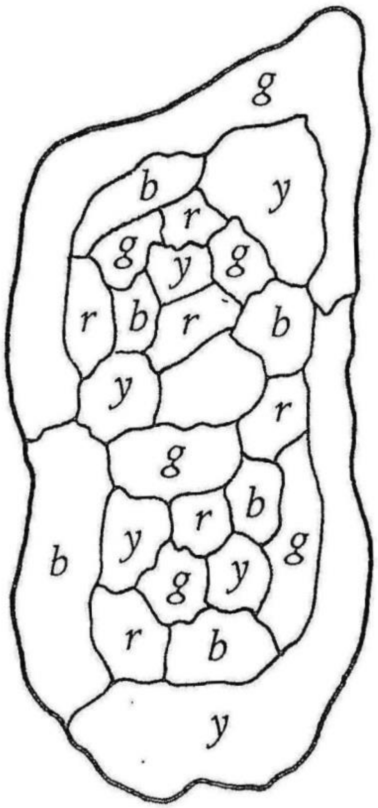
B. The structure of graphs

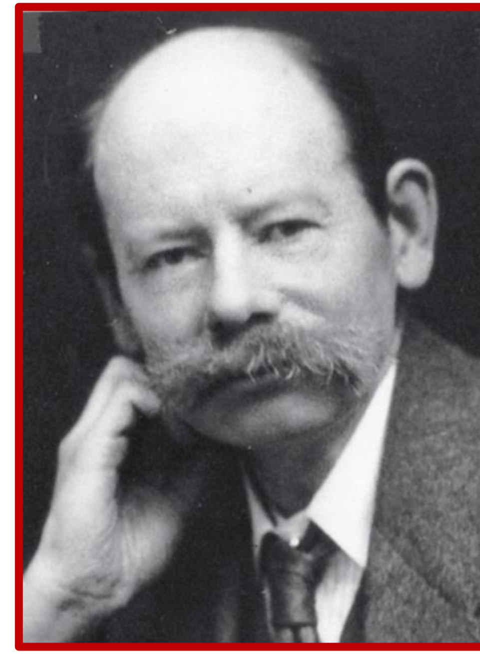
C. Algorithms

D. The development of graph theory as a subject

A 1890: Percy Heawood

Map-colour theorem

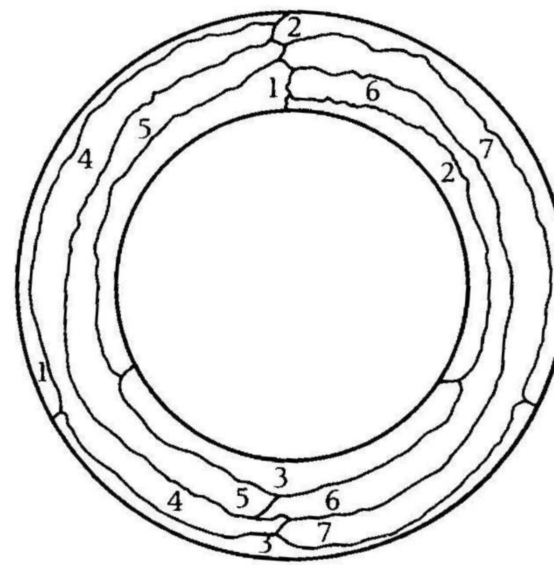




Heawood pointed out the error in Kempe's 'proof' of the four-colour theorem,

salvaged enough to prove the five-colour theorem,

and showed that, for maps on a g-holed torus (for $g \geq 1$), $[\frac{1}{2}(7 + \sqrt{(1 + 48g)})]$ colours are sufficient



A 1891: Lothar Heffter

Ueber das problem der Nachbargebiete

For $g > 1$, Heawood didn't prove that

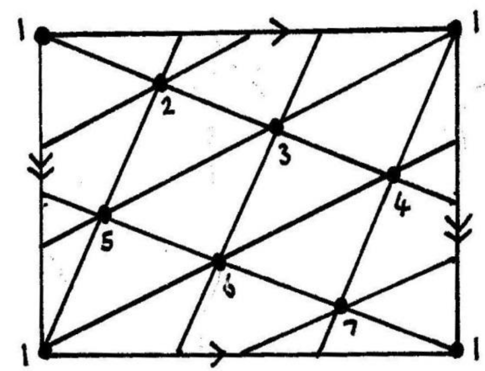
$[{}^{1}/_{2}(7 + \sqrt{(1 + 48g)})]$ colours may actually be needed

Heffter noticed the omission and asked (equivalently):

What is the least genus for n neighbouring regions on the surface? For $n \geq 7$ it's at least $\{{}^{1}/_{12}(n - 3)(n - 4)\}$

Heffter proved this for $n \leq 12$ and some other values

He also 'dualized' the problem to embedding complete graphs on a surface: what's the least genus g for the graph K_n?

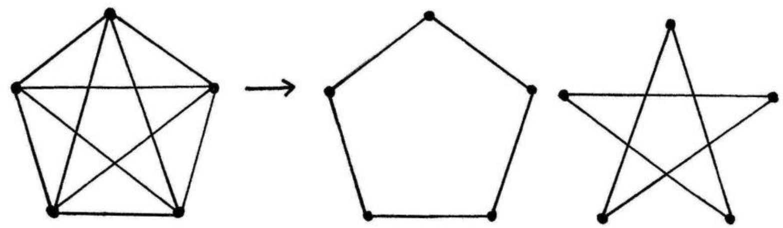


K_7 on a torus

B 1891/1898: Julius Petersen

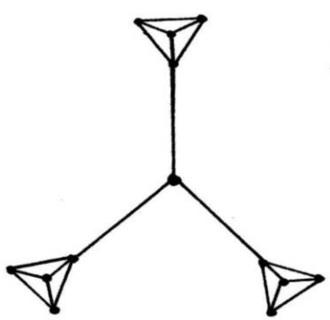
Die Theorie der regulären Graphs

When can you factorize a regular graph into regular 'factors' of given degree r?

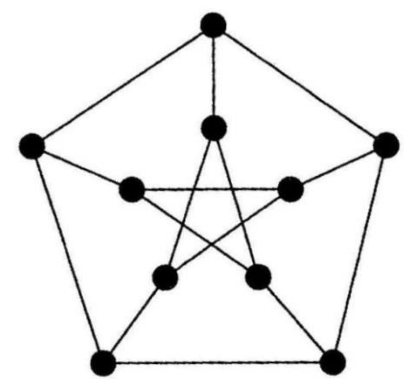


K_5 has a '2-factorization',
as does every regular graph of even degree

Sylvester:
this graph has
no 1-factor

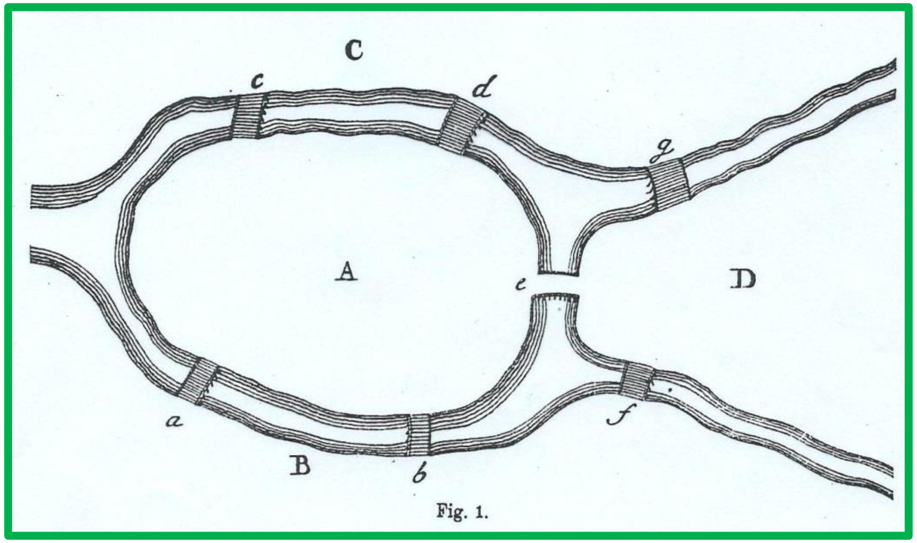


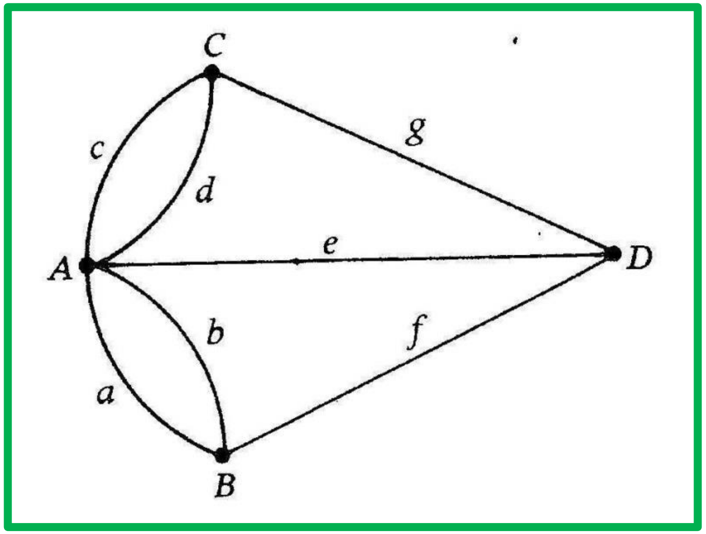
The Petersen graph
splits into
a 2-factor and
a 1-factor, but
not three 1-factors



B 1892: W. W. Rouse Ball

Mathematical Recreations and Problems



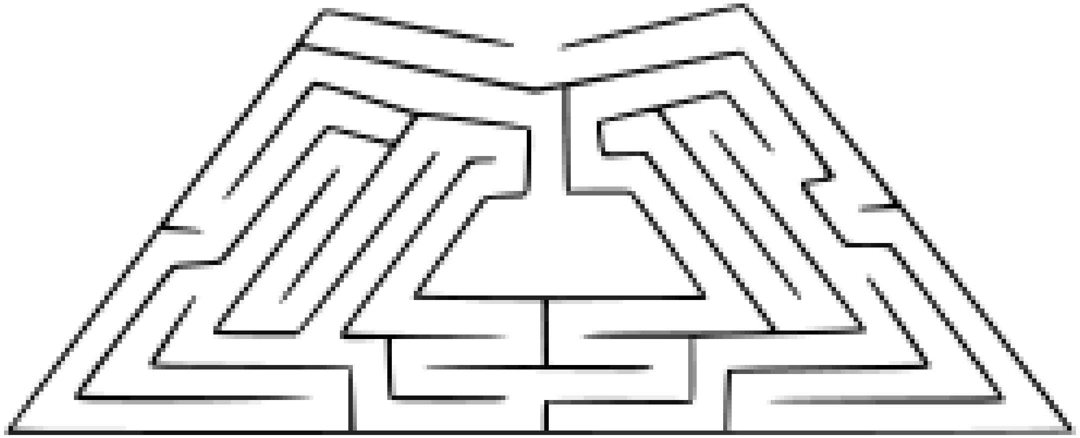


Solving the Königsberg bridges problem corresponds to drawing the right-hand picture without repeating any line or lifting your pen from the paper

Euler did NOT draw such a picture

C 1895: Gaston Tarry

Le problème des labyrinthes



Tarry's rule: don't return along a passage which led to a junction for the first time unless you can't do otherwise.

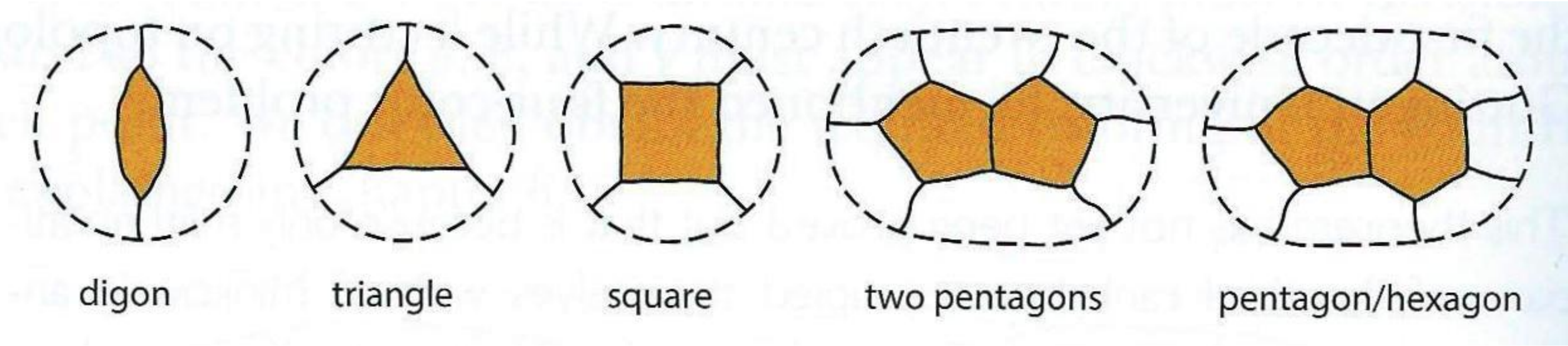
He also gave a practical method for carrying this out.

A 1904: Paul Wernicke

Über den kartographischen Vierfarbensatz

Kempe: Every cubic map on the plane contains a digon, triangle, square or pentagon

Wernicke: Every cubic map on the plane contains at least one of the following configurations:



They form an unavoidable set: every map must contain at least one of them

B 1907: M. Dehn & P. Heegaard

Analysis situs

Encyklopädie der Mathematische Wissenschaften

First comprehensive study of complexes, following on from ideas of Kirchhoff, Listing and Poincaré

Their opening section was on Liniensysteme (graphs) constructed from 0-cells (vertices) and 1-cells (edges)

This work was later continued by Oswald Veblen in a paper on Linear graphs (1912) and in an American Mathematical Society Colloquium Lecture series in 1916

A 1910: Heinrich Tietze

Einige Bemerkungen über das Problem des Kartenfärbens auf einseitigen Flächen

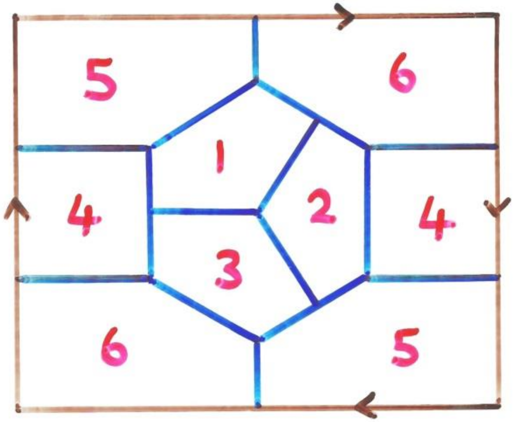
One-sided surfaces: on a Möbius band or projective plane,

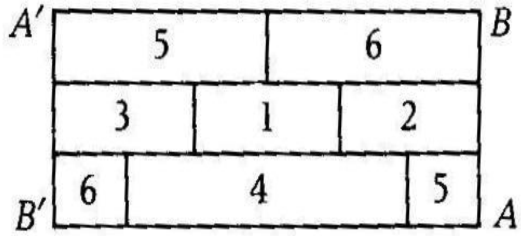
every map can be coloured with 6 colours

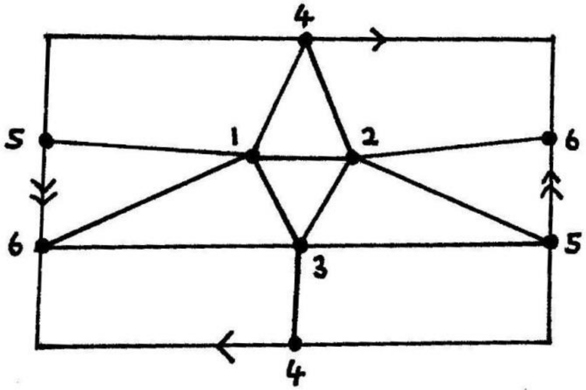
so at most 6 neighbouring regions can be drawn

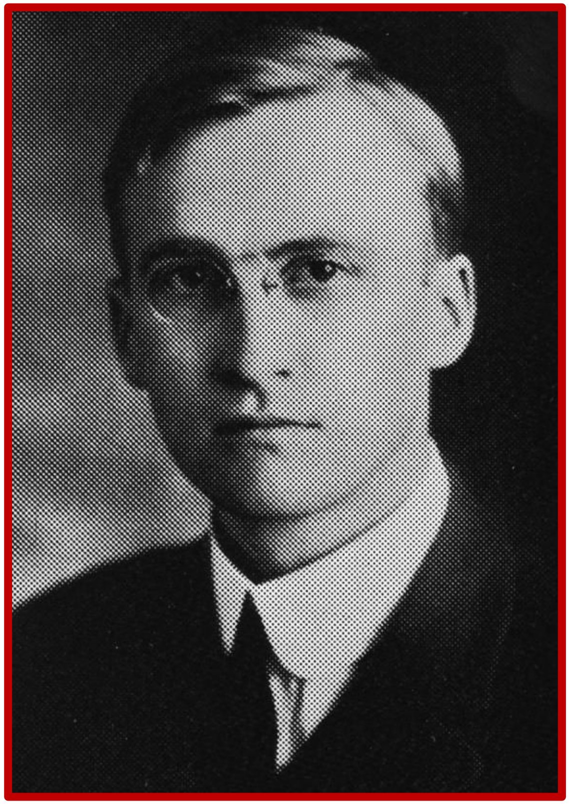
Klein bottle: 7 colours are needed (Franklin, 1934)

Tietze also obtained analogues of the formulas of Heawood and Heffter









A 1912: G. D. Birkhoff

A determinant formula for the number of ways of coloring a map

The number of ways is always
a polynomial in the number of colours,
now called the **chromatic polynomial**

The degree is the number of countries and the coefficients alternate in sign: Birkhoff obtained a formula for them

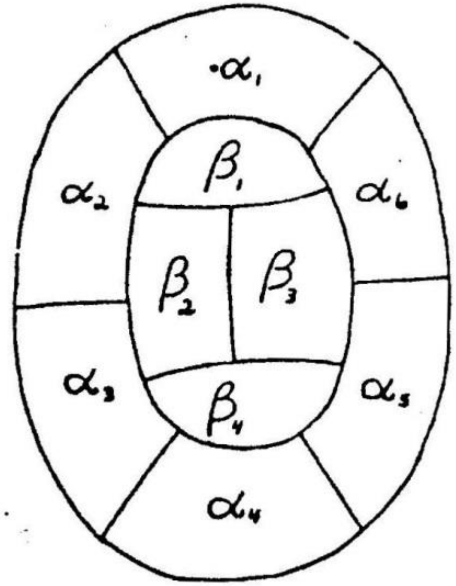
Related work by Birkhoff (1930), Whitney (1932),
and in a major paper by Birkhoff and D. C. Lewis (1944)

A 1913: G. D. Birkhoff

The reducibility of maps

A configuration of countries in a map is **reducible** if any 4-colouring of the rest of the map can be extended to the configuration

So irreducible configurations can't appear in minimal counter-examples to the 4-colour theorem



Kempe: digons, triangles and squares are reducible

Birkhoff: so is the 'Birkhoff diamond'

B 1916: Dénes König

Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre

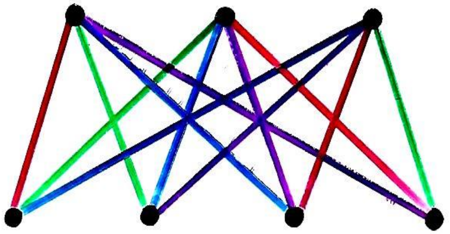
[also in Hungarian and French]

A graph is bipartite ↔ every cycle has even length

Every k-regular bipartite graph splits into k 1-factors

(proved earlier by E. Steinitz for configurations)

Interpretation for matching/marriage



So if each vertex of a bipartite graph has degree ≤ k, then its edges can be coloured with k colours

B 1918: Heinz Prüfer

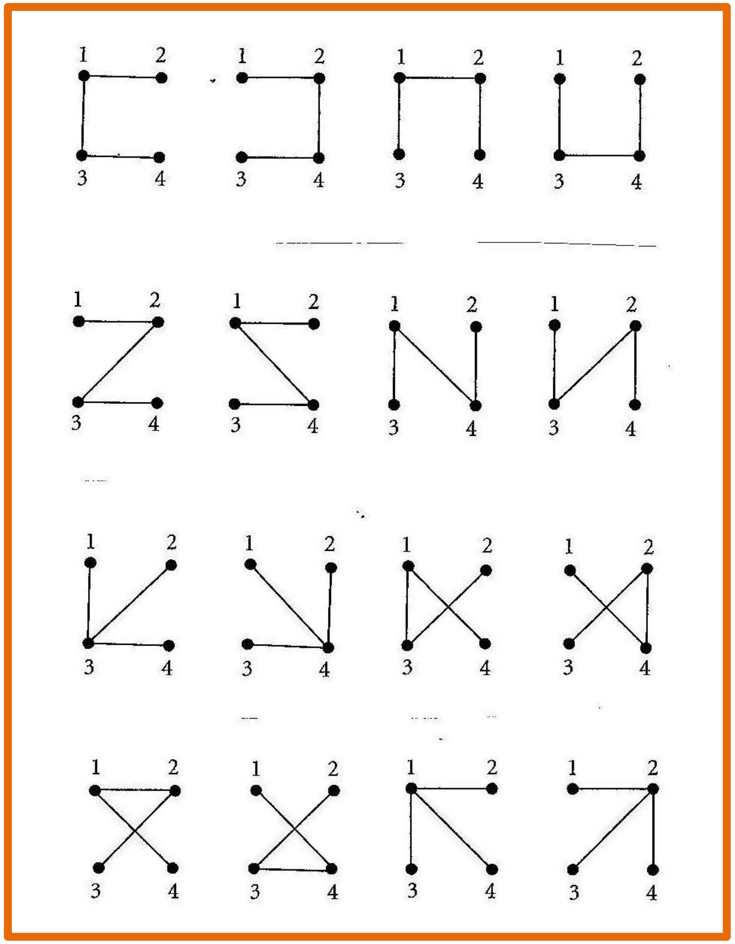
Neuer Beweis eines Satzes über Permutationen

First correct proof of Cayley's 1889 result:

There are n^{n-2} labelled trees on n vertices

or

K_n has n^{n-2} spanning trees



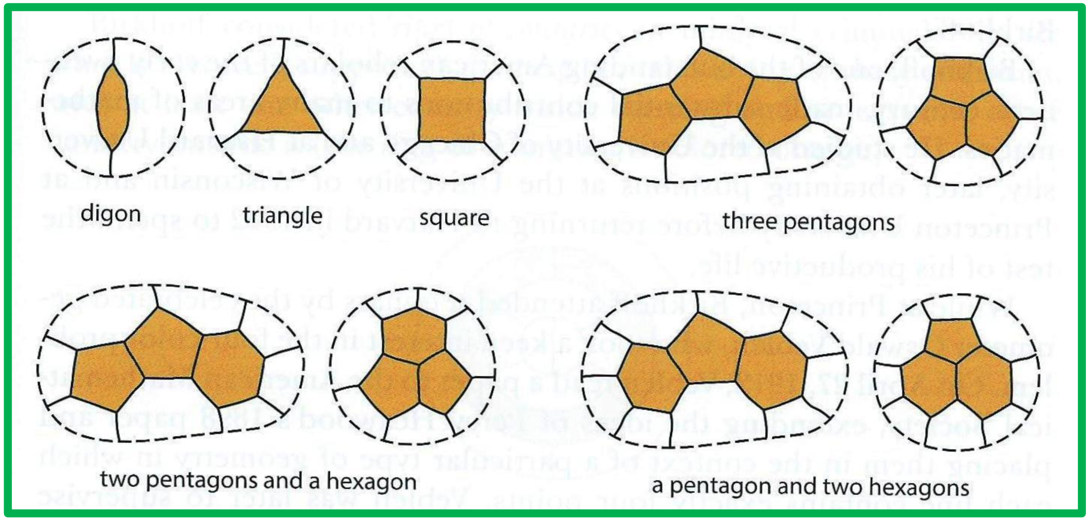
It uses the idea of associating a Prüfer sequence $(a_1, a_2, \ldots, a_{n-2})$ with each tree.

A 1922: Philip Franklin

The four color problem

Every cubic map with no digons, triangles or squares has at least 12 pentagons.

A new unavoidable set:



Any counter-example has at least 25 countries

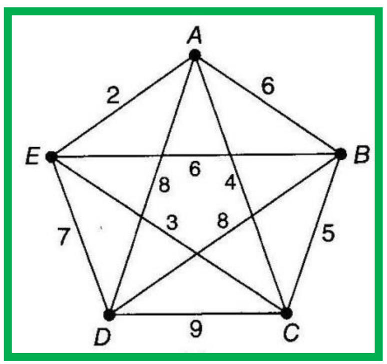
Further unavoidable sets were found by Henri Lebesgue (1940)

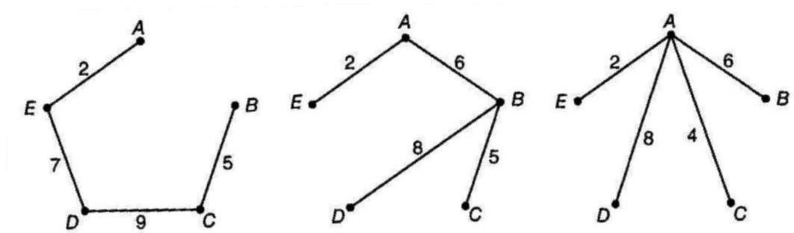
C 1924: Otakar Borůvka

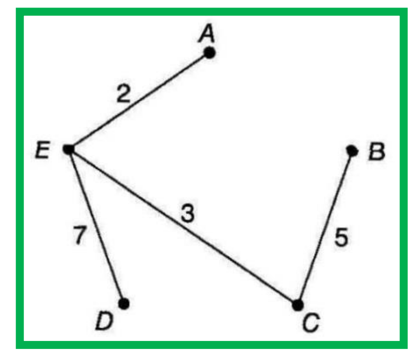
[On a certain minimal problem]

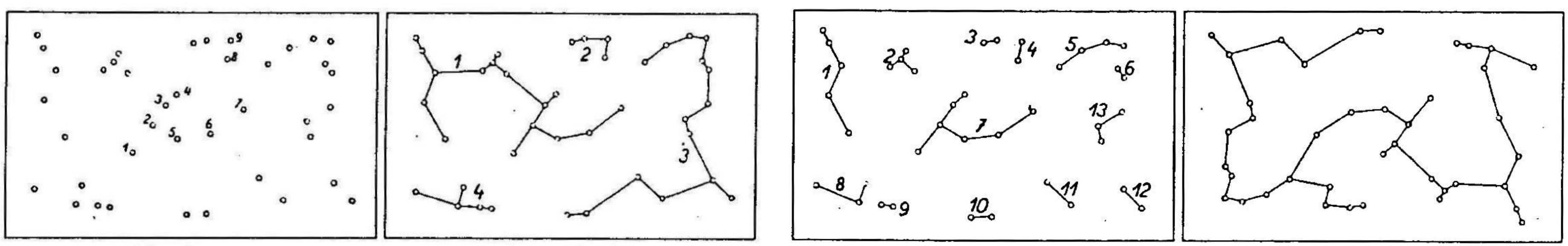
Minimum connector problem: In a weighted graph, find the spanning tree of shortest length.

Cayley: if there are n vertices, there are n^{n-2} spanning trees.







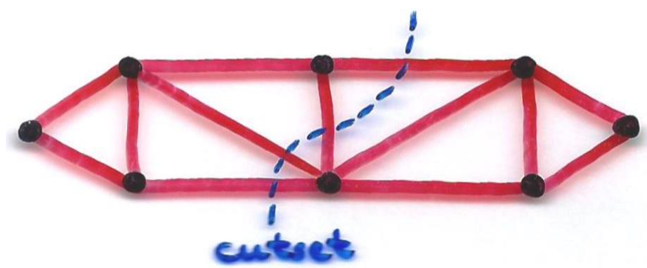


Also solved by V. Jarnik (1930), and by J. B. Kruskal (1954) and R. C. Prim (1957).

B 1927: Karl Menger

Zur allgemeinen Kurventheorie

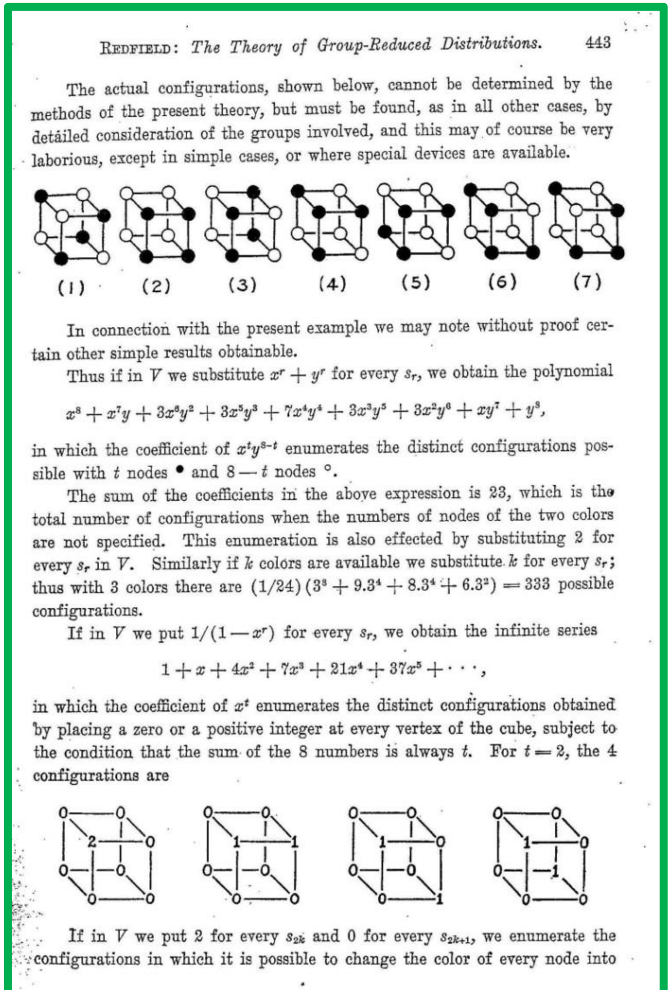
On a problem in analytic topology:
in graph theory terms
it's a **minimax** connectivity theorem:
the **max number** of disjoint paths between two vertices = the **min number** of vertices / edges we must remove to separate the graph



— equivalent to König's theorem (1916) and Hall's 'marriage' theorem (1935)

B 1927: J. Howard Redfield

The theory of group-reduced distributions



REDFIELD: *The Theory of Group-Reduced Distributions.* 443

The actual configurations, shown below, cannot be determined by the methods of the present theory, but must be found, as in all other cases, by detailed consideration of the groups involved, and this may of course be very laborious, except in simple cases, or where special devices are available.

(1) (2) (3) (4) (5) (6) (7)

In connection with the present example we may note without proof certain other simple results obtainable.

Thus if in V we substitute $x^r + y^r$ for every s_r, we obtain the polynomial

$$x^8 + x^7y + 3x^6y^2 + 3x^5y^3 + 7x^4y^4 + 3x^3y^5 + 3x^2y^6 + xy^7 + y^8,$$

in which the coefficient of x^ty^{8-t} enumerates the distinct configurations possible with t nodes • and $8 - t$ nodes ∘.

The sum of the coefficients in the above expression is 23, which is the total number of configurations when the numbers of nodes of the two colors are not specified. This enumeration is also effected by substituting 2 for every s_r in V. Similarly if k colors are available we substitute k for every s_r; thus with 3 colors there are $(1/24)(3^8 + 9.3^4 + 8.3^4 + 6.3^2) = 333$ possible configurations.

If in V we put $1/(1 - x^r)$ for every s_r, we obtain the infinite series

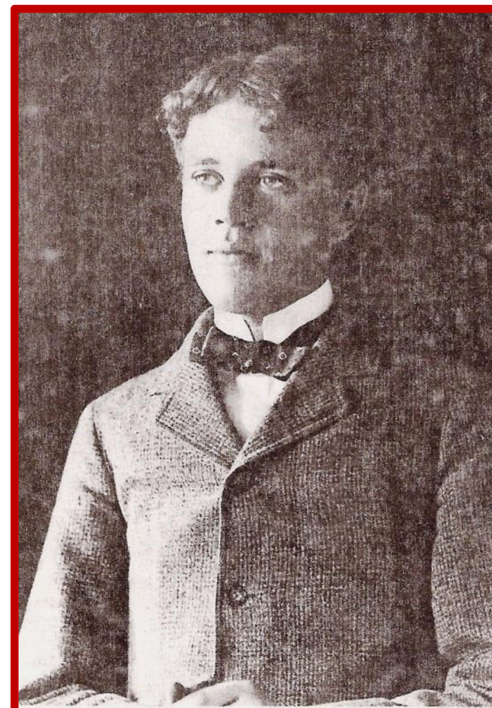
$$1 + x + 4x^2 + 7x^3 + 21x^4 + 37x^5 + \cdots,$$

in which the coefficient of x^t enumerates the distinct configurations obtained by placing a zero or a positive integer at every vertex of the cube, subject to the condition that the sum of the 8 numbers is always t. For $t = 2$, the 4 configurations are

If in V we put 2 for every s_{2k} and 0 for every s_{2k+1}, we enumerate the configurations in which it is possible to change the color of every node into

Counting under symmetry, counting simple graphs

(symmetrical aliorative dyadic relation-numbers)



B 1930: F. P. Ramsey

On a problem in formal logic

'Ramsey's theorem' for sets

→ 'Ramsey graph theory'

[Erdős, Harary, Bollobás, etc.]

Example: Six people at a party

Among any six people, there must be three friends or three non-friends.

18 people needed for four friends/non-friends.

How many are needed for five?

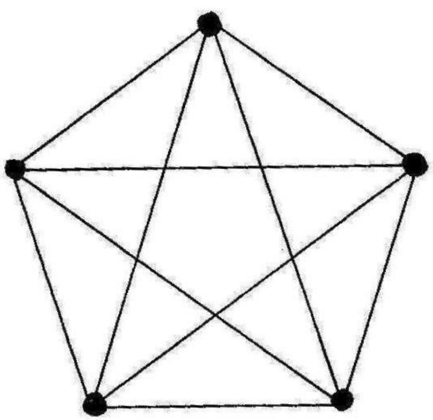
So every red/blue colouring of the edges of K_6 gives us either a red triangle or a blue triangle.

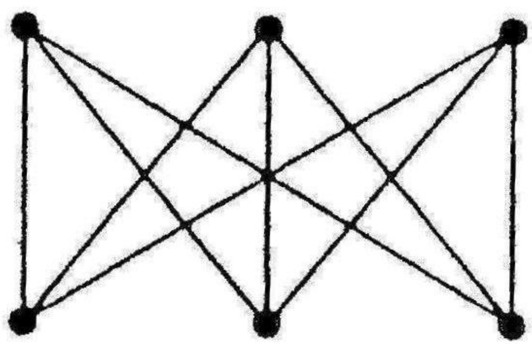
With k colours, how many vertices do we need to guarantee a given graph of one colour?

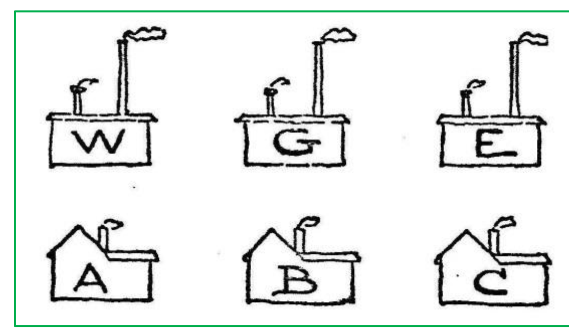
1930: Kasimierz Kuratowski

Sur le problème des courbes gauches en topologie

A graph is planar if and only if it doesn't contain K_5 or $K_{3,3}$



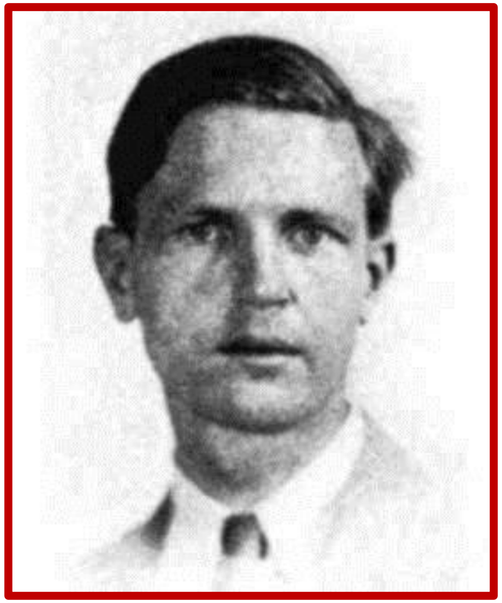




Proved independently by O. Frink & P. A. Smith

The utilities puzzle of Sam Loyd

B 1931–1935: Hassler Whitney



1931: Non-separable and planar graphs
1931: The coloring of graphs

1932: A logical expansion in mathematics
1932: Congruent graphs and the connectivity of graphs

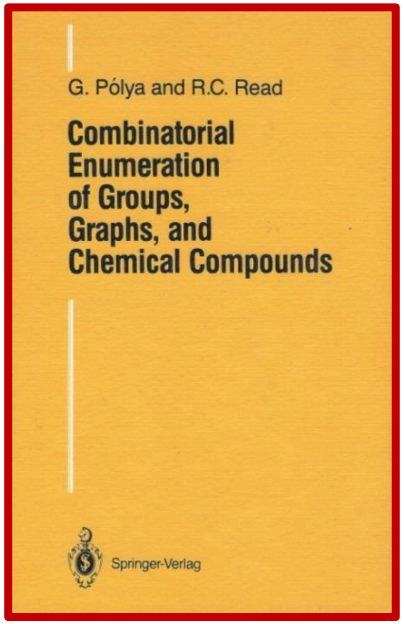
1933: A set of topological invariants for graphs
1933: 2-isomorphic graphs
1933: On the classification of graphs

1935: On the abstract properties of linear dependence
(on ‘matroids’)

B 1935–37: Georg Pólya

Kombinatorische Anzahlbestimmungen für Gruppen, Graphen, und chemische Verbindungen

On enumerating graphs and chemical molecules (the orbits under a group of symmetries) using the cycle structure of the group

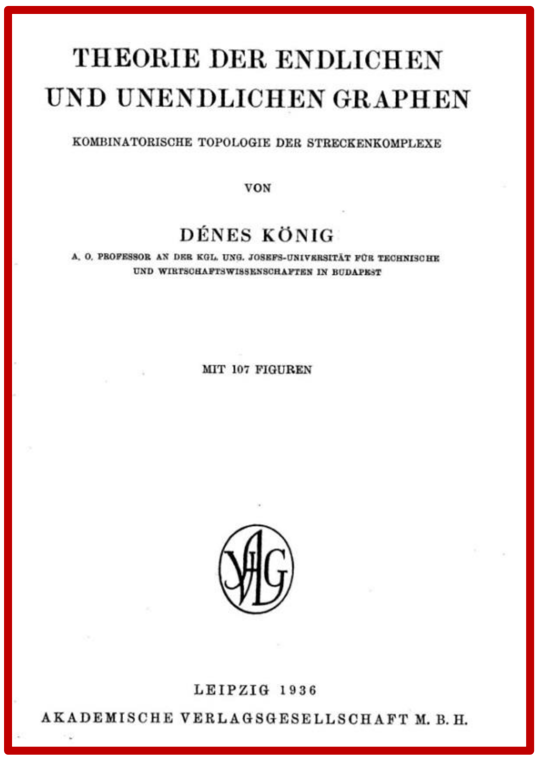


Later work on graph enumeration by Otter, de Bruijn, Harary, Read, Robinson, etc.

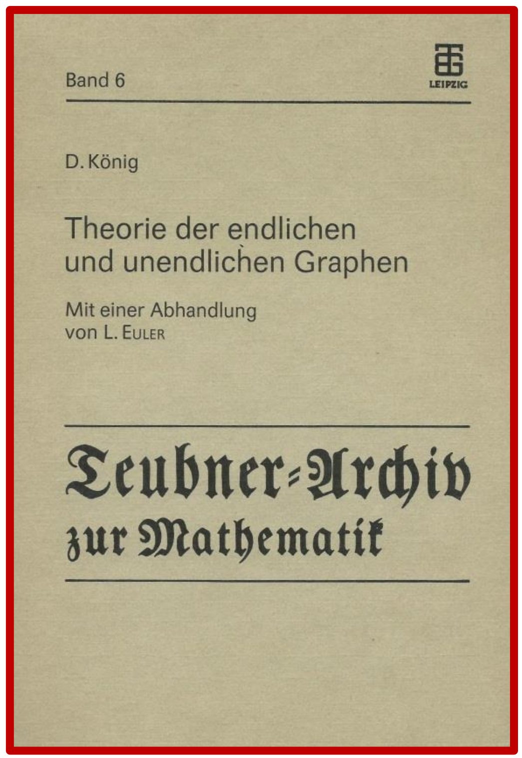
D 1936: Dénes König

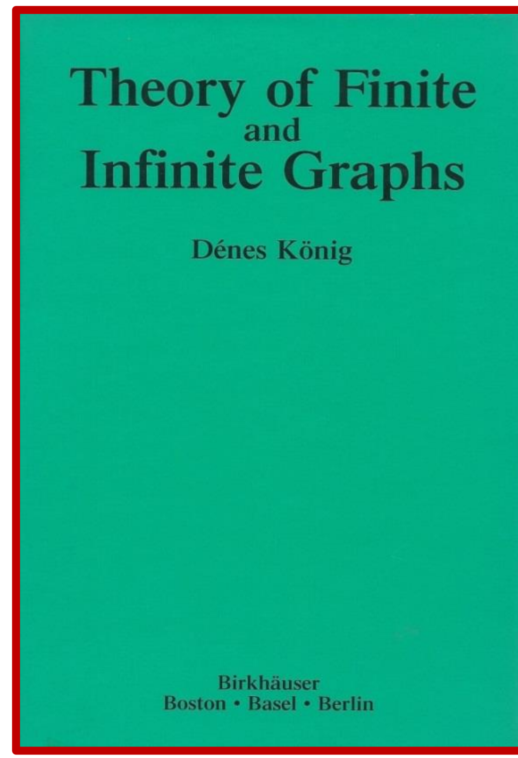
Theorie der endlichen und unendlichen Graphen

The 'first textbook on graph theory'



THEORIE DER ENDLICHEN
UND UNENDLICHEN GRAPHEN
KOMBINATORISCHE TOPOLOGIE DER STRECKENKOMPLEXE
VON
DÉNES KÖNIG
A. O. PROFESSOR AN DER KGL. UNG. JOSEFS-UNIVERSITÄT FÜR TECHNISCHE UND WIRTSCHAFTSWISSENSCHAFTEN IN BUDAPEST
MIT 107 FIGUREN
LEIPZIG 1936
AKADEMISCHE VERLAGSGESELLSCHAFT M. B. H.

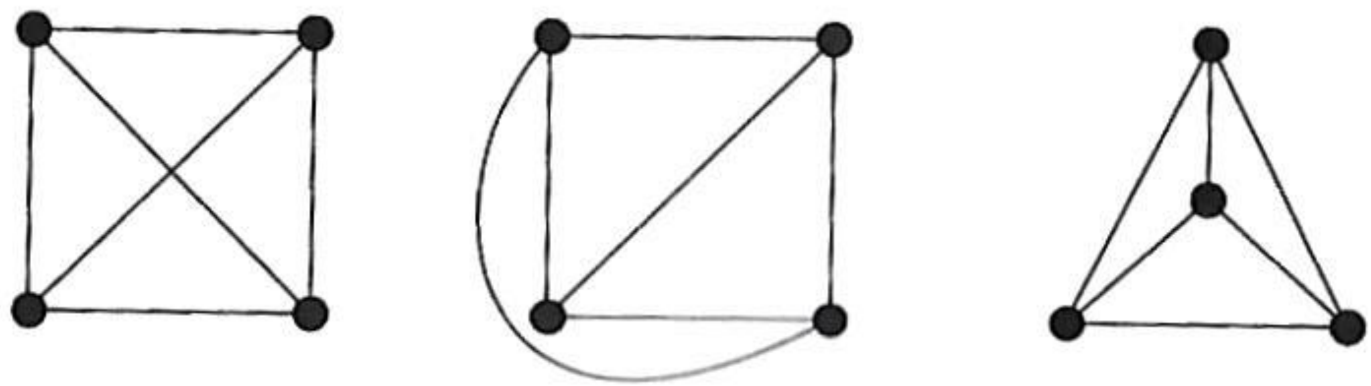




B 1937/1948 K. Wagner / I. Fáry

Über eine Eigenschaft der ebenen Komplexe

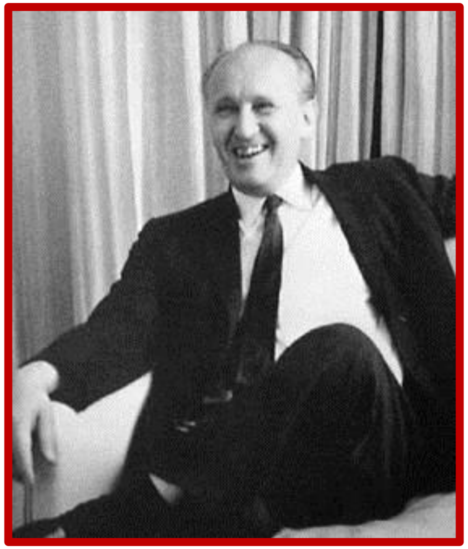
On straight line representation of planar graphs



Every simple planar graph can be drawn in the plane using only straight lines

B 1940: P. Turán

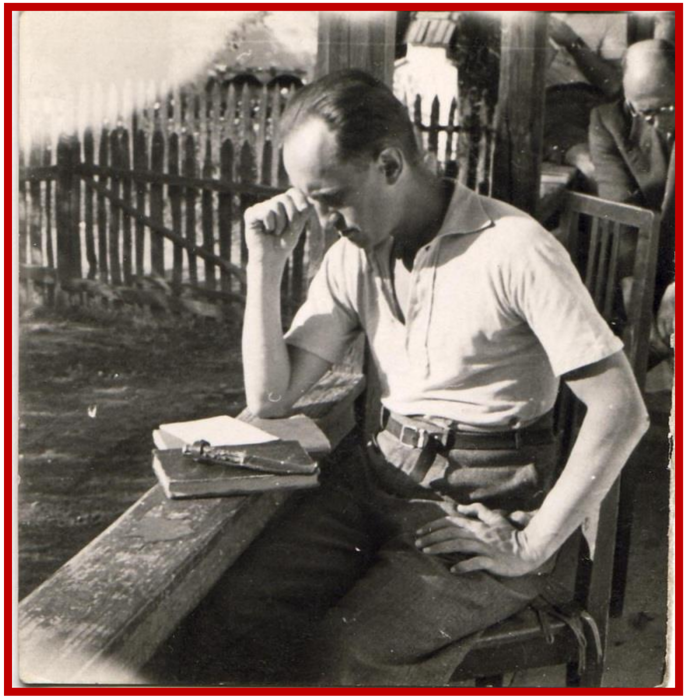
Eine Extremalaufgabe aus der Graphentheorie



Extremal graph theory

A graph with n vertices
and no triangles
has ≤ $[n^2/4]$ edges

[proved earlier by W. Mantel (1907)]



[Turán also studied the 'brick factory
problem' on crossing numbers
of bipartite graphs]

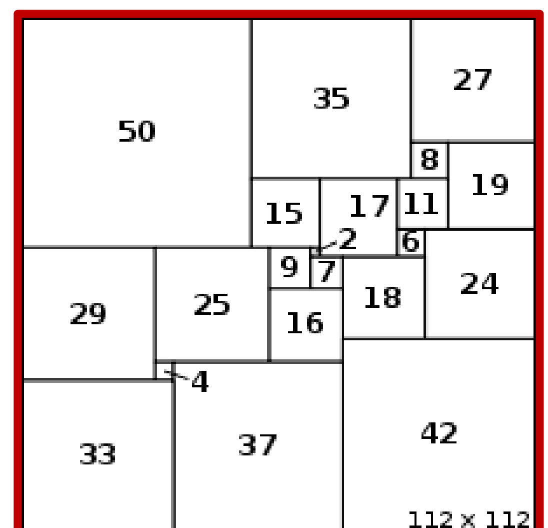
A 1941: R. L. Brooks

On colouring the nodes of a network

Vertex-colourings:

If G is a connected graph with maximum degree k, then its vertices can be coloured with at most k + 1 colours, with equality for odd complete graphs and odd cycles

Brooks was one of the team of Brooks, Stone, Smith and Tutte who used directed graphs to 'square the square' in 1940

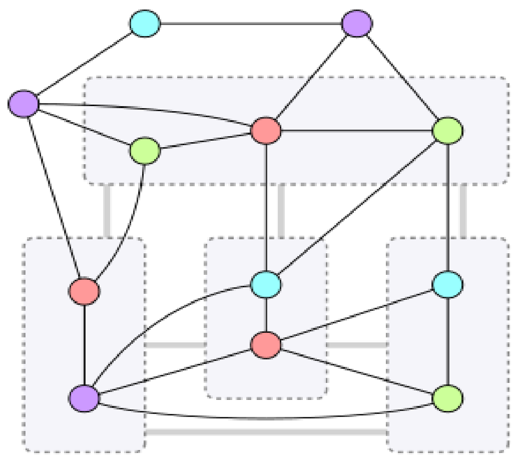


B 1943: Hugo Hadwiger

Über eine Klassifikation der Streckencomplexe

Hadwiger's conjecture

Every connected graph with chromatic number k can be contracted to K_k



Hadwiger: conjecture true for k ≤ 4

Wagner (1937): true for k = 5 ↔ four-colour theorem

Robertson, Seymour and Thomas (1993): true for k = 6 (also uses four-colour theorem)

Still unproved in general

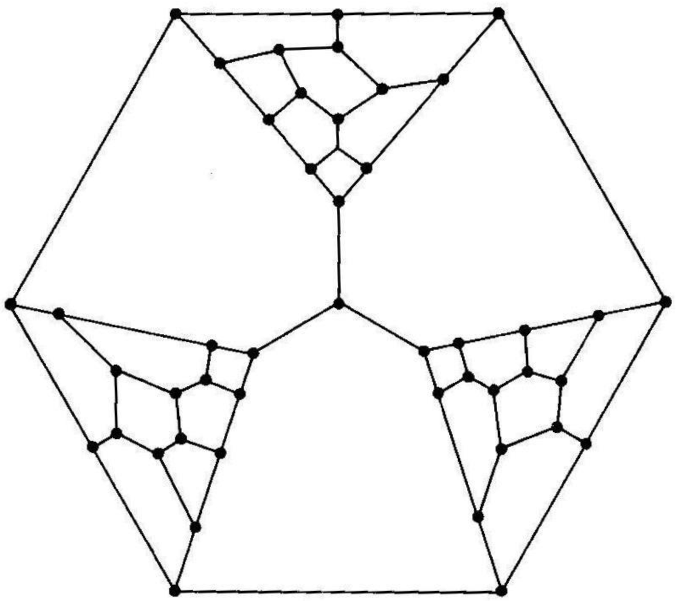
B 1946: W. T. Tutte

On Hamilton circuits

Tait's conjecture (1880):

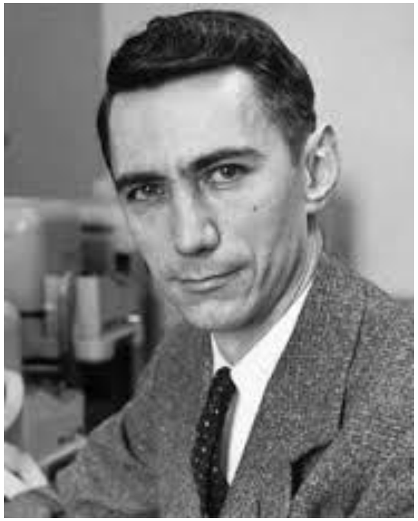
Every cubic polyhedral graph has a Hamiltonian cycle

'It mocks alike at doubt and proof'



False: Tutte produced an example with 46 vertices

In 1947 Tutte found a condition for a graph to have a 1-factor (extended to r-factors in 1952)



A 1949: Claude E. Shannon

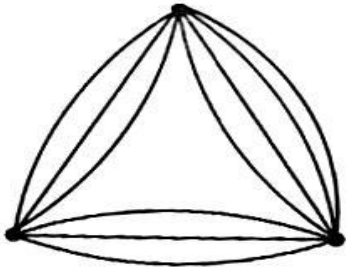
A theorem on coloring the lines of a network

On a problem arising from the colour-coding of wires in an electrical unit, such as relay panels, where the emerging wires at each point must be coloured differently.

Theorem: The lines of any network can be properly coloured with at most [3m/2] colours,

where m = max number of lines at a junction.

This number is necessary for some networks.



B 1952: Gabriel Dirac

Some theorems on abstract graphs

Sufficient conditions for a graph G to be Hamiltonian

Dirac (1952): If G has n vertices, and if the degree of each vertex is at least $^1/_2$n, then G is Hamiltonian

Ore (1960): If deg(v) + deg(w) ≥ n for all non-adjacent vertices v and w, then G is Hamiltonian

Dirac also wrote on 'critical graphs'

[Later Hamiltonian results by Pósa, Chvátal, Bondy, etc.]

C Algorithms from the 1950s/1960s

Assignment problem

H. Kuhn (1955)

Network flow problems

L. R. Ford & D. R. Fulkerson (1956)

Minimum connector problem

J. B. Kruskal (1956) and R. E. Prim (1957)

Shortest path problem

E. W. Dijkstra (1959)

‘Chinese postman problem’

Kwan Mei-Ko (= Meigu Guan) (1962)

B 1959: P. Erdős & A. Rényi

On random graphs I

Probabilistic graph theory

G(n, m) model (Erdős–Rényi)

Take a random graph with n vertices and m edges.

How many components does it have?

How big is its largest component?

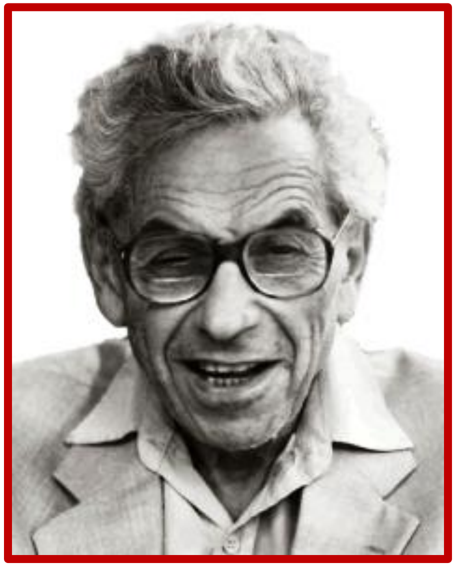
What is the probability that it is connected?

G(n, p) model (E. N. Gilbert)

Take n vertices and add edges at random with probability p.

How big is its largest component?

When does the graph become connected?



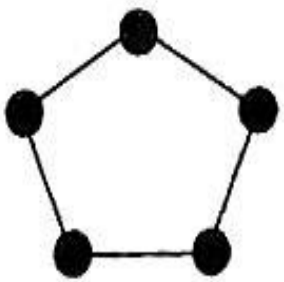
1960: A. J. Hoffman and R. R. Singleton

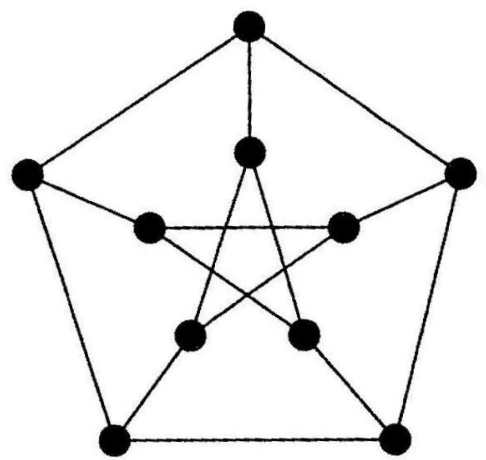
On Moore graphs with diameters 2 and 3

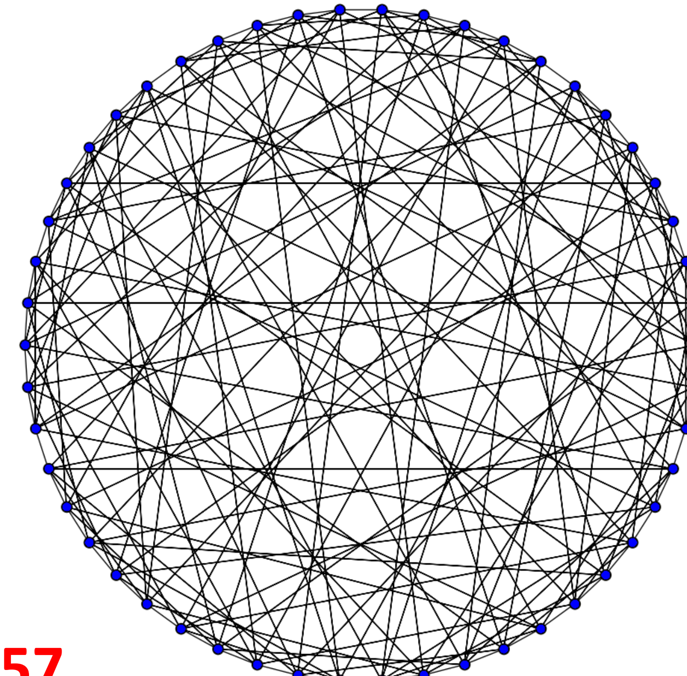
Let G be regular of degree d and have n vertices.

Then $n \le 1 + d \sum (d - 1)^{i-1}$.

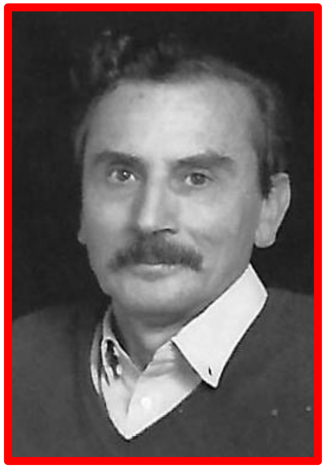
If equality holds, G is a **Moore graph**.







For diameter 2,
d = 2, 3, 7, and possibly 57



D Graph theory texts

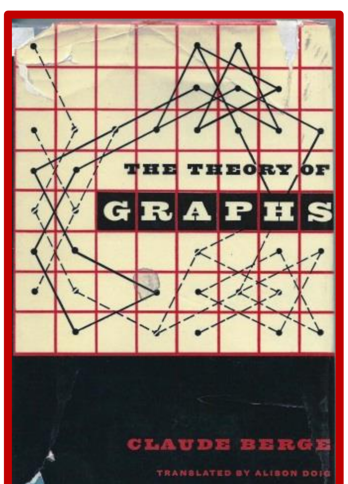
Claude Berge: *Theorie des Graphes et ses Applications* (1958)

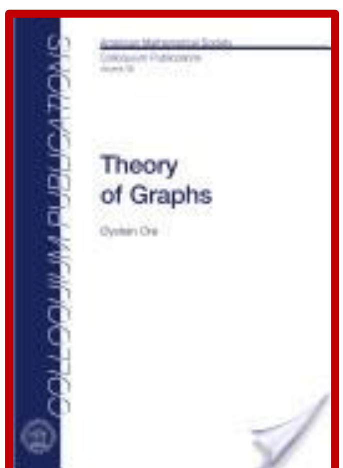
Oystein Ore: *Theory of Graphs* (1962)

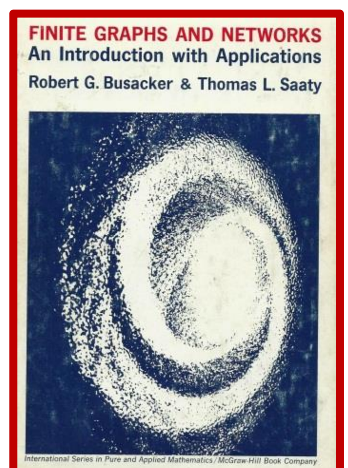
R. G. Busacker & T. L. Saaty: Finite graphs and networks (1965)

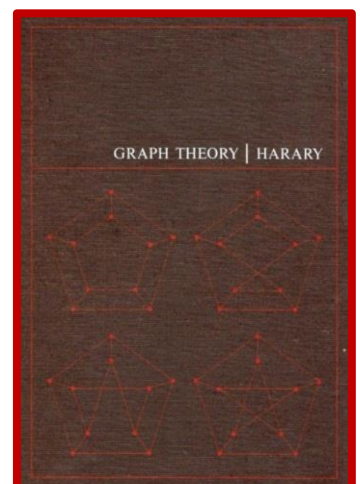
Frank Harary: *Graph Theory* (1969)

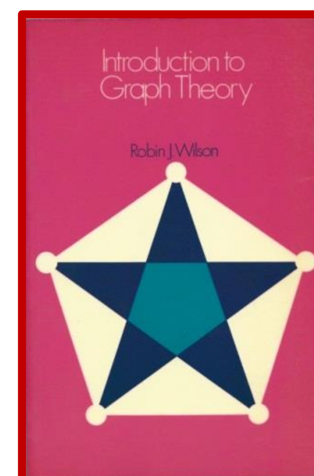
Robin Wilson: *Introduction to Graph Theory* (1972)







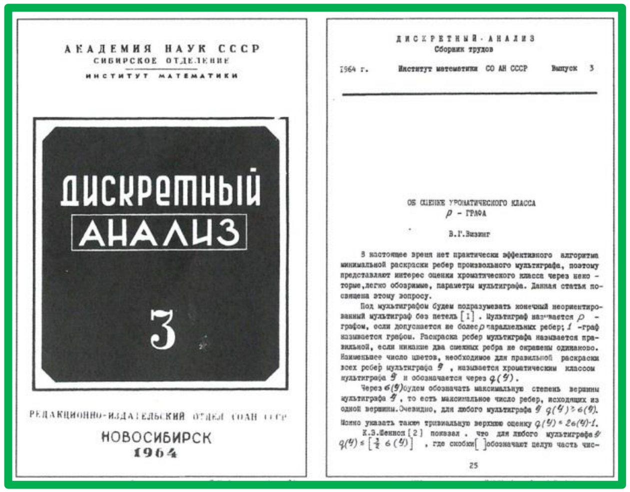




A   1964: V. G. Vizing

On an estimate of the chromatic class of a p-graph

(in Russian)



АКАДЕМИЯ НАУК СССР
СИБИРСКОЕ ОТДЕЛЕНИЕ
ИНСТИТУТ МАТЕМАТИКИ

ДИСКРЕТНЫЙ АНАЛИЗ

3

РЕДАКЦИОННО-ИЗДАТЕЛЬСКИЙ ОТДЕЛ СОАН СССР
НОВОСИБИРСК
1964

ДИСКРЕТНЫЙ АНАЛИЗ
Сборник трудов

1964 г. Институт математики СО АН СССР Выпуск 3

ОБ ОЦЕНКЕ ХРОМАТИЧЕСКОГО КЛАССА
p – ГРАФА

В.Г.Визинг

В настоящее время нет практически эффективного алгоритма минимальной раскраски ребер произвольного мультиграфа, поэтому представляют интерес оценки хроматического класса через некоторые, легко обозримые, параметры мультиграфа. Данная статья посвящена этому вопросу.

Под мультиграфом будем подразумевать конечный неориентированный мультиграф без петель [1]. Мультиграф называется p -графом, если допускается не более p параллельных ребер; 1 -граф называется графом. Раскраска ребер мультиграфа называется правильной, если никакие два смежных ребра не окрашены одинаково. Наименьшее число цветов, необходимое для правильной раскраски всех ребер мультиграфа $\mathcal{G}$, называется хроматическим классом мультиграфа $\mathcal{G}$ и обозначается через $q(\mathcal{G})$.

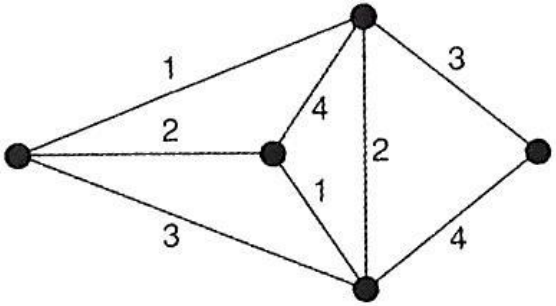
Через $\sigma(\mathcal{G})$ будем обозначать максимальную степень вершины мультиграфа $\mathcal{G}$, то есть максимальное число ребер, исходящих из одной вершины. Очевидно, для любого мультиграфа $\mathcal{G}$ $q(\mathcal{G}) \geq \sigma(\mathcal{G})$. Можно указать также тривиальную верхнюю оценку $q(\mathcal{G}) \leq 2\sigma(\mathcal{G})-1$.

К.Э.Шеннон [2] показал, что для любого мультиграфа $\mathcal{G}$ $q(\mathcal{G}) \leq \left[\frac{3}{2}\sigma(\mathcal{G})\right]$, где скобки [ ] обозначают целую часть чис-

25

If G is a graph with maximum degree Δ and at most p parallel edges, then its edges can be coloured with Δ + p colours.

Corollary: If G is simple, then its edges need either Δ or Δ + 1 colours.

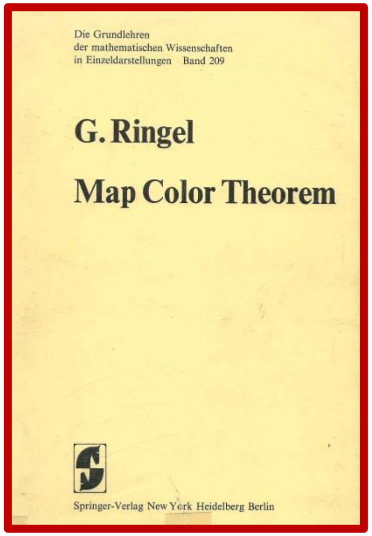


A 1968: G. Ringel & J. W. T. Youngs

Solution of the Heawood map-coloring problem

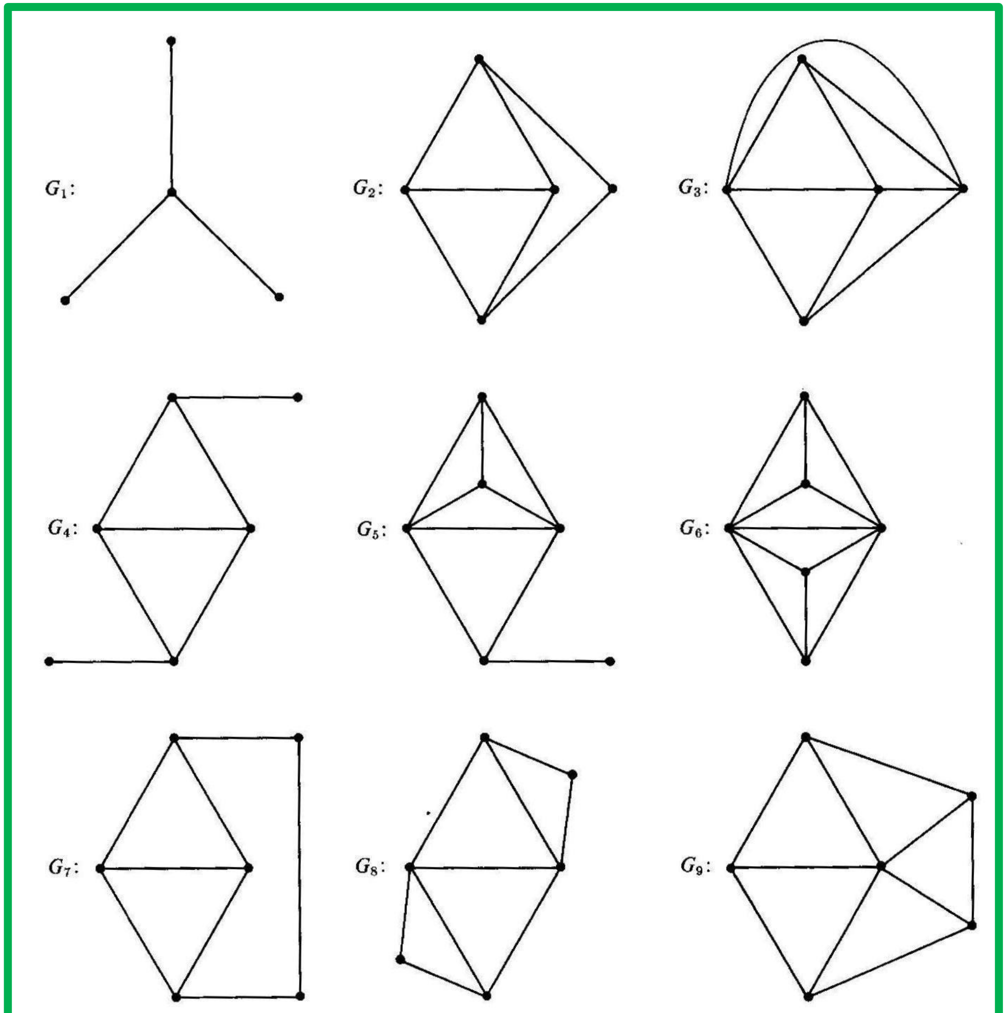
Ringel and Youngs reduced the drawing of K_n on a sphere with $\{^1/_{12}(n-3)(n-4)\}$ handles to twelve cases which they dealt with individually.

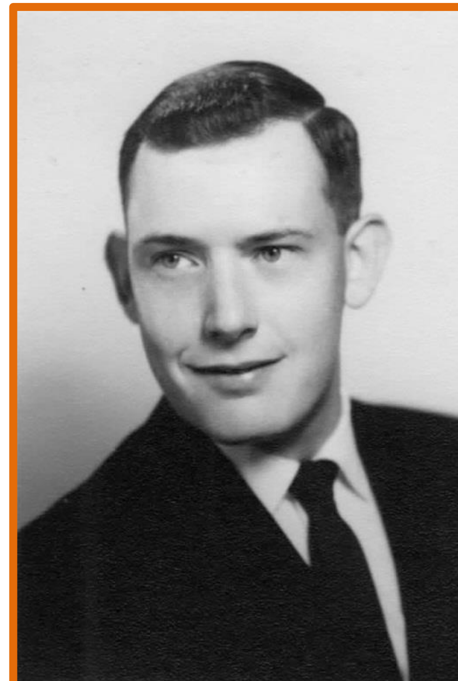
(The non-orientable case had been completed by Ringel in 1952.)



B 1968: Lowell Beineke

Derived graphs and digraphs





The nine forbidden subgraphs for line graphs

C 1970s: computational complexity

Efficiency of algorithms

P: 'easy' problems, solved in polynomial time

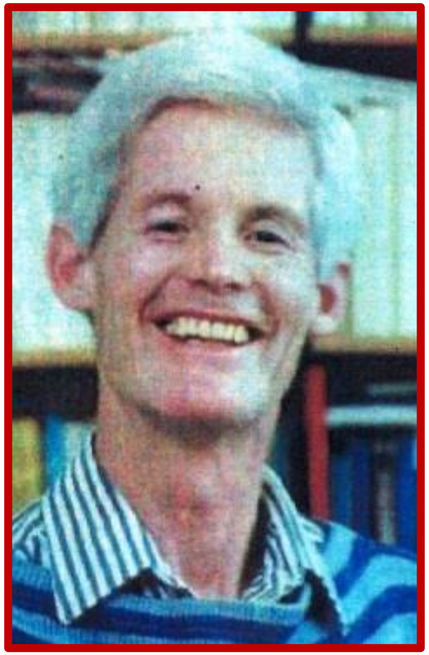
planarity algorithms (n), minimum connector problem (n^2)

NP: 'non-deterministic polynomial-time problems': any proposed solution can be checked in polynomial time

Clay millennium question: is P = NP?

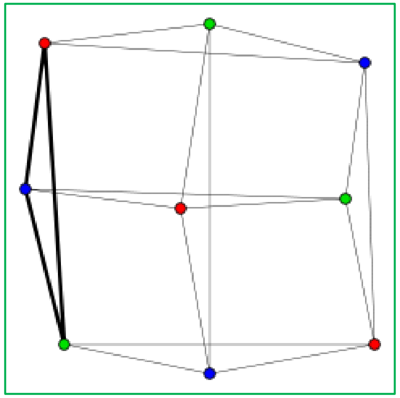
S. Cook (1971): The complexity of theorem-proving procedures

Every NP problem can be polynomially reduced to a single NP problem (the 'satisfiability problem')



B 1972: Laszló Lovász

A characterization of perfect graphs



A graph G is **perfect** if, for each induced subgraph, the chromatic number = the size of the largest clique

Berge graph (1963): neither G nor its complement has an induced odd cycle of length ≥ 5

Lovász (1972): Perfect graph theorem:
A graph is perfect if and only if its complement is perfect

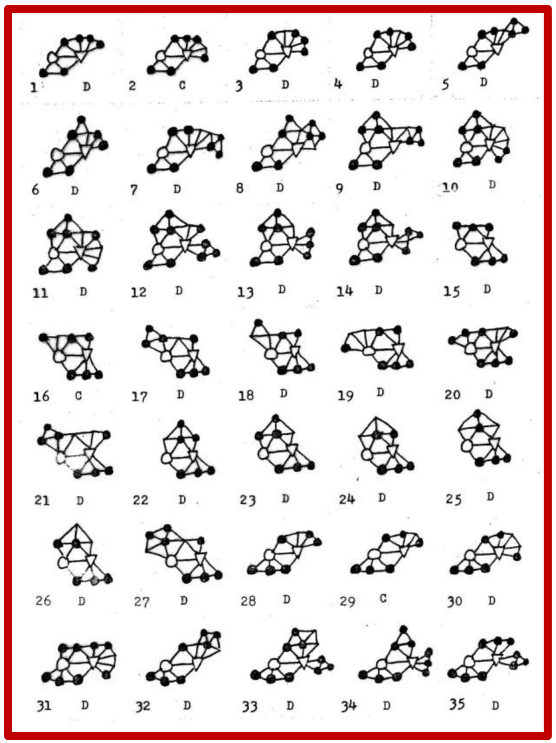
M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas (2006):
Strong perfect graph theorem:
Perfect graphs = Berge graphs

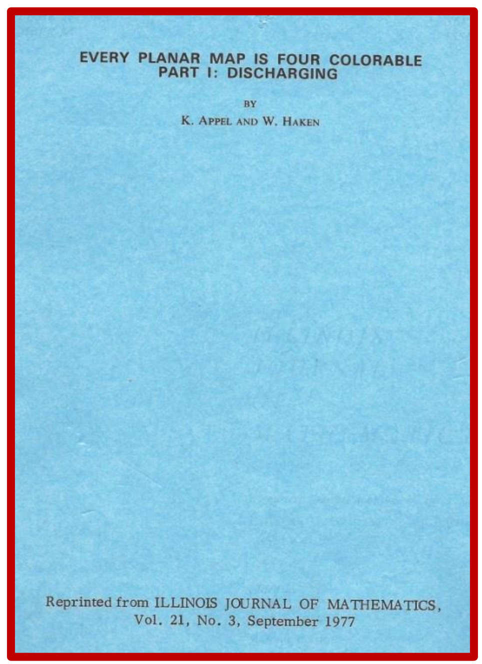
1976: K. Appel & W. Haken

Every planar map is four-colorable

H. Heesch: find an unavoidable set of reducible configurations

Using a computer Appel and Haken (and J. Koch) found an unavoidable set of 1936 reducible configurations (later 1482)



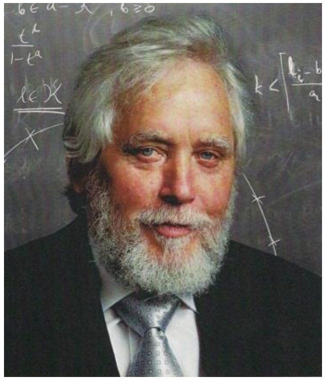


EVERY PLANAR MAP IS FOUR COLORABLE
PART I: DISCHARGING

BY

K. APPEL AND W. HAKEN

Reprinted from ILLINOIS JOURNAL OF MATHEMATICS,
Vol. 21, No. 3, September 1977



B 1978: Endre Szemerédi
Regular partitions of graphs

Szemerédi's regularity lemma:

Every large enough graph can be divided into subsets of around the same size so that the edges between different subsets behave almost randomly.

In other words: all graphs can be approximated by 'random-looking' graphs

1975: weaker version for bipartite graphs, relating to sets of integers with no k of them in arithmetic progression.

Generalised by Tim Gowers and others.

Szemerédi was awarded the 2012 Abel Prize.

B 1979: H. Glover & J. P. Huneke

The set of irreducible graphs for the projective plane is finite

How many 'forbidden subgraphs' are there for a surface?

Kuratowski (1930): for the sphere, just K_5 and $K_{3,3}$

Glover & Huneke (1979) (with D. Archdeacon & C. Wang): for the projective plane the number is 103

For the torus the number is unknown, but is ≥ 800

Robertson and Seymour (1984): The graph minor theorem
For *every* surface the number is finite

1994: Carsten Thomassen

Every planar graph is 5-choosable

Vizing (1975) and Erdős, Rubin and Taylor (1979) introduced the idea of a list-colouring.

Assign a list L(v) of colours to each vertex v of a graph G. A *list-colouring* of G is a colouring in which each vertex is assigned a colour from its list. If G has a list-colouring for every L with L(v)| = k for all v, then G is k-*list-colourable* or k-*choosable*.

Thomassen proved the above list version of Heawood's five-colour theorem, thereby answering a conjecture of Erdős, Rubin and Taylor and giving a good algorithm for the five-colour theorem.

Thomassen has settled many conjectures in graph theory, including a proof of Tutte's 'weak 3-flow conjecture'.

B 1983–2004: N. Robertson & P. Seymour

with co-workers R. Thomas, M. Chudnovsky, . . .

A succession of fundamental results that changed the face of graph theory:

- The graph minor theorem
- An improved proof of the 4-colour theorem
- The strong perfect graph conjecture
- Proof of the Hadwiger conjecture for K_6
- Every snark contains the Petersen graph

and many more . . .