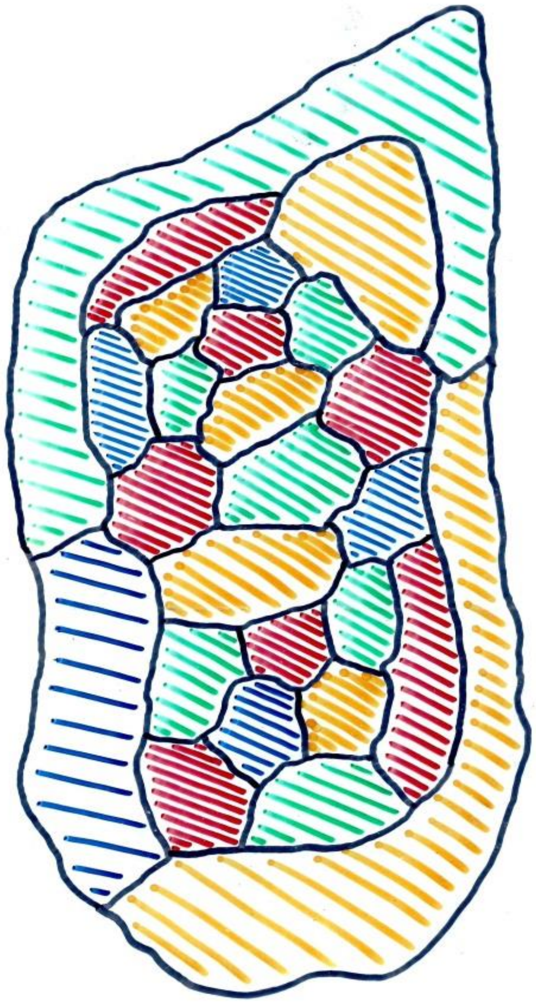
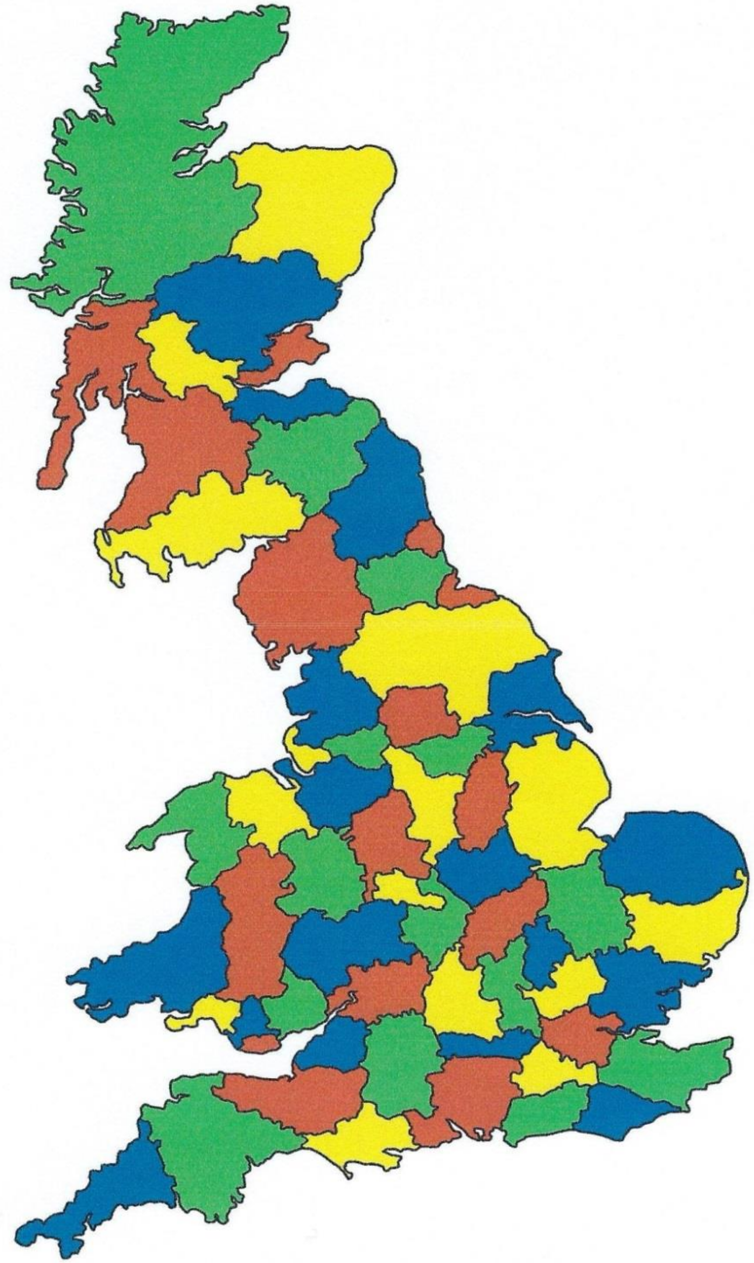
6. Colouring maps

Robin Wilson

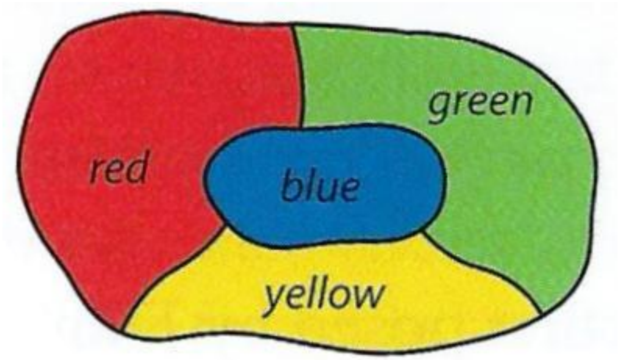


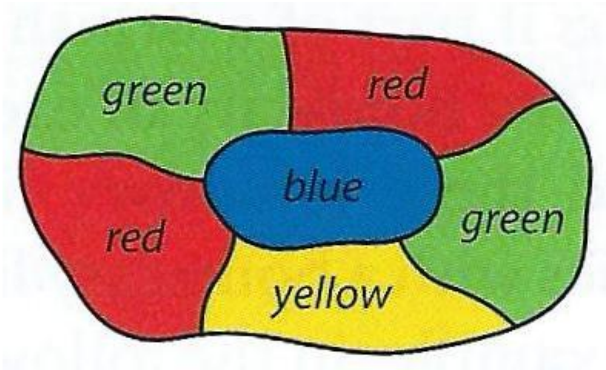


The four-colour problem

Can every map be coloured with four colours so that neighbouring countries are coloured differently?

We certainly need four colours for *some* maps . . .

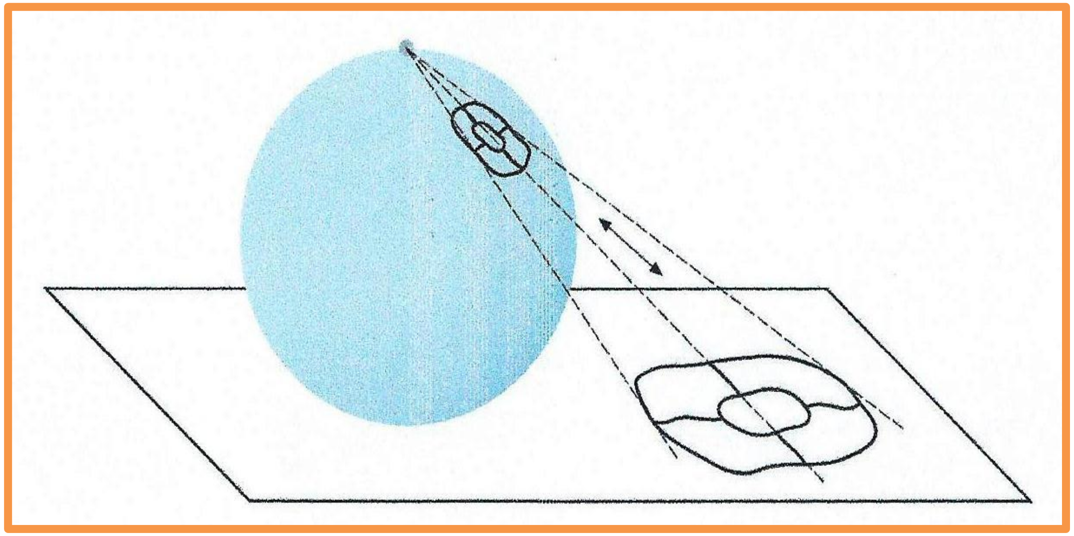




four neighbouring countries . . . but not here

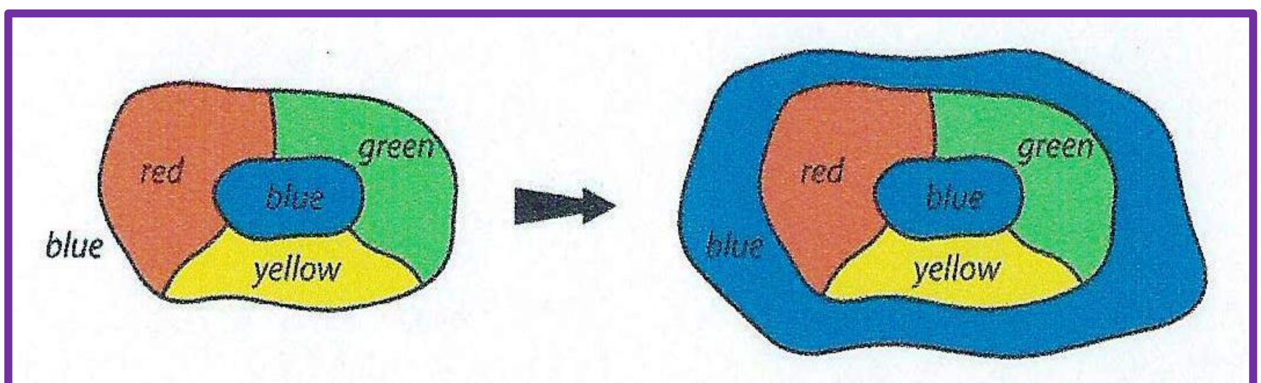
. . . but do four colours suffice for *all* maps?

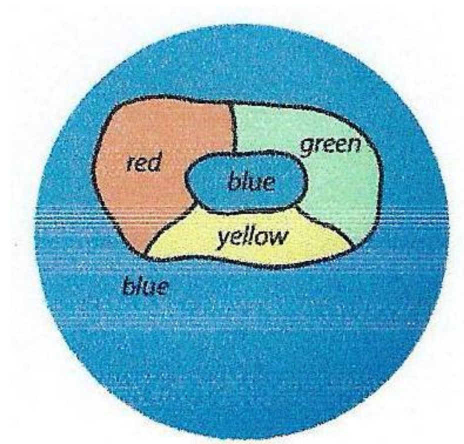
Two observations



The map can be on a plane or a sphere

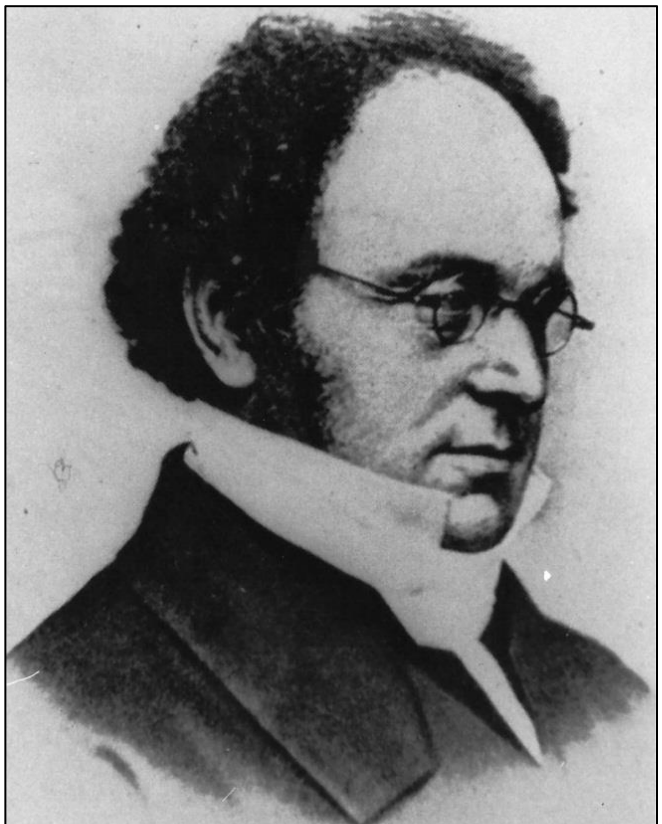
It doesn't matter whether we include the outside region

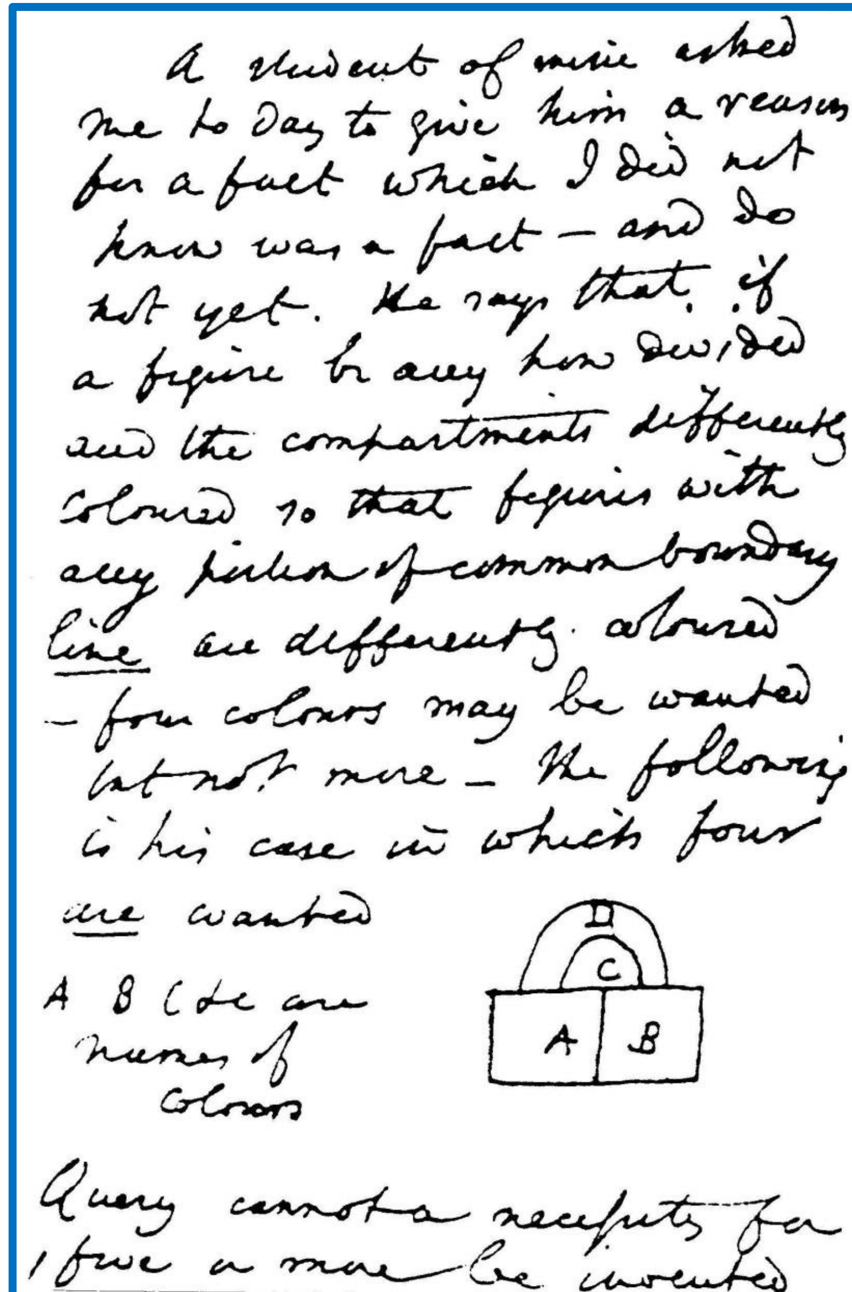




De Morgan's letter to W. R. Hamilton

23 October 1852



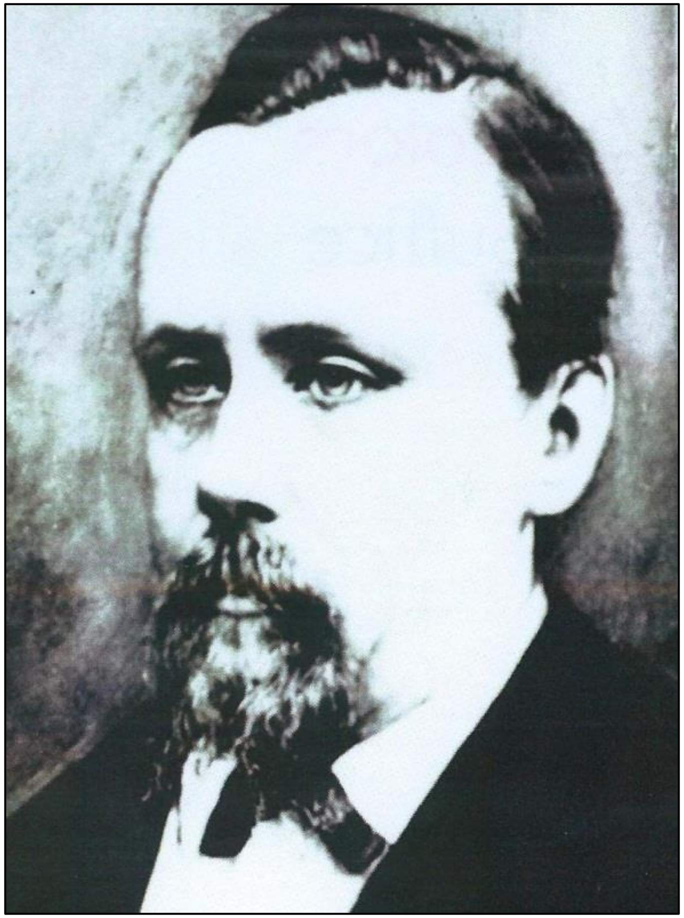


A student of mine asked me to day to give him a reason for a fact which I did not know was a fact — and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured — four colours may be wanted but not more — the following is his case in which four are wanted

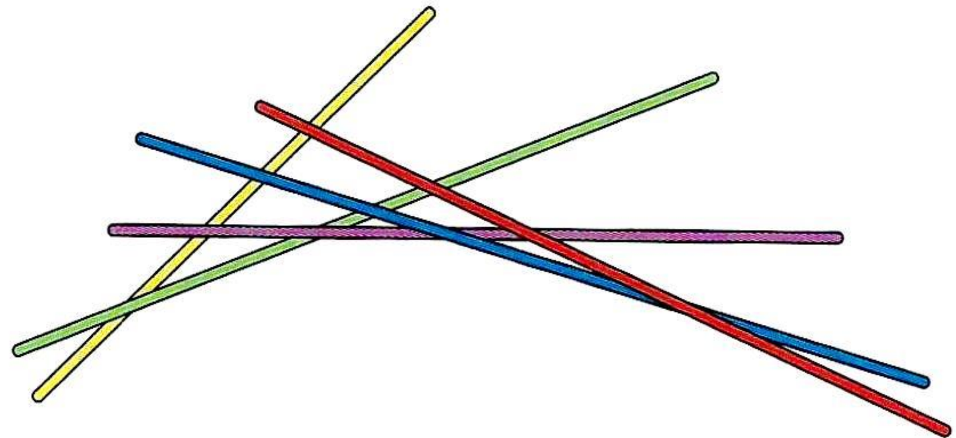
A B C D are names of colours

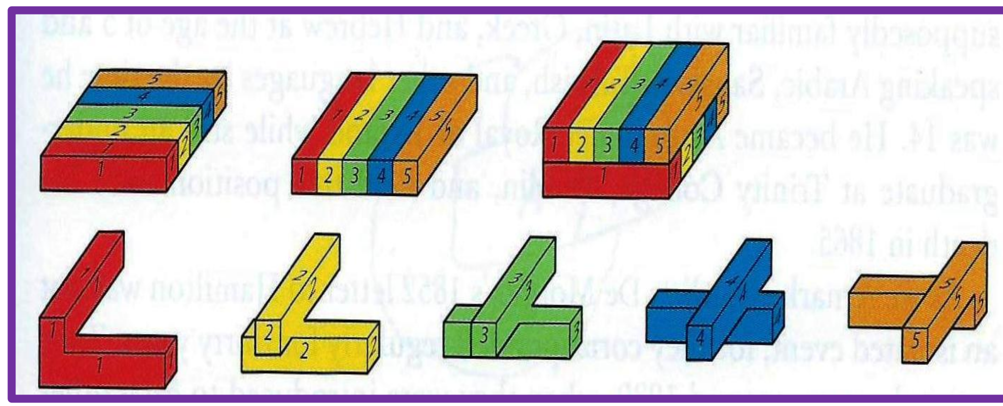
Query cannot a necessity for five or more be invented

Francis and Frederick Guthrie



Francis Guthrie





Frederick Guthrie, 1880

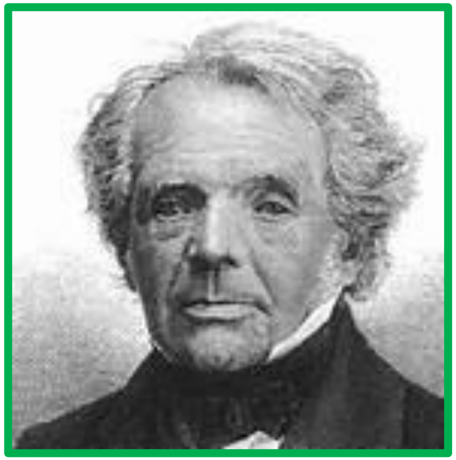
no analogue in three dimensions

The first appearance in print:

F. G. in *The Athenaeum*, June 1854

Tinting Maps.—In tinting maps, it is desirable for the sake of distinctness to use as few colours as possible, and at the same time no two conterminous divisions ought to be tinted the same. Now, I have found by experience that *four* colours are necessary and sufficient for this purpose,—but I cannot prove that this is the case, unless the whole number of divisions does not exceed five. I should like to see (or know where I can find) a general proof of this apparently simple proposition, which I am surprised never to have met with in any mathematical work. F. G.

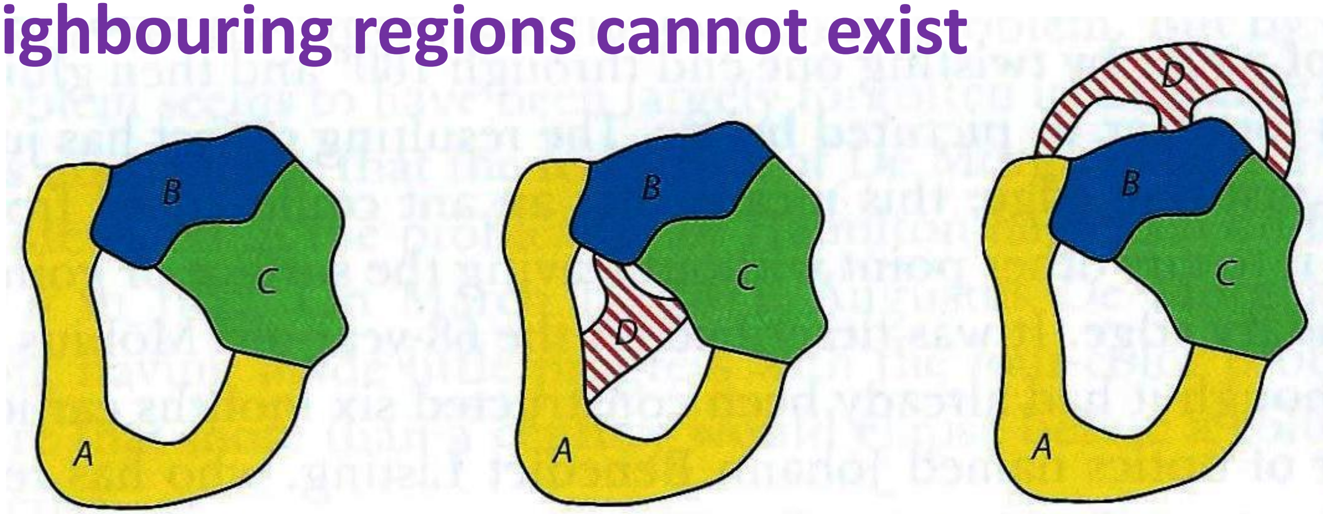
Möbius and the five princes (c.1840)



A king on his death-bed:

‘My five sons, divide my land among you, so that each part has a border with each of the others.’

Möbius’s problem has no solution: five neighbouring regions cannot exist



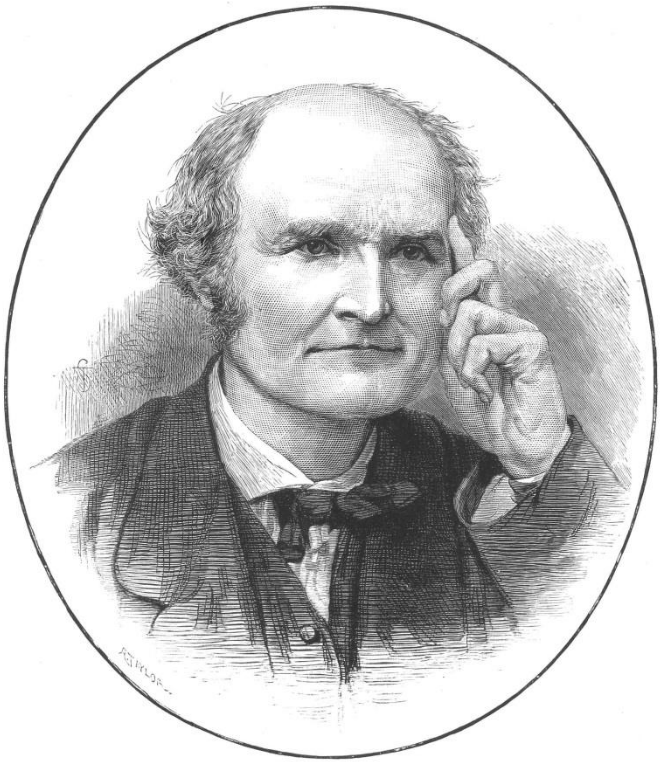
Some logic . . .

A solution to Möbius's problem would give us a 5-coloured map:

'5 neighbouring regions exist' implies that
'the 4-colour theorem is false'
and so
'the 4-colour theorem is true' implies that
'5 neighbouring regions don't exist'
BUT
'5 neighbouring regions don't exist' does NOT imply that 'the 4-colour theorem is true'

So Möbius did NOT originate the 4-colour problem.

Arthur Cayley revives the problem

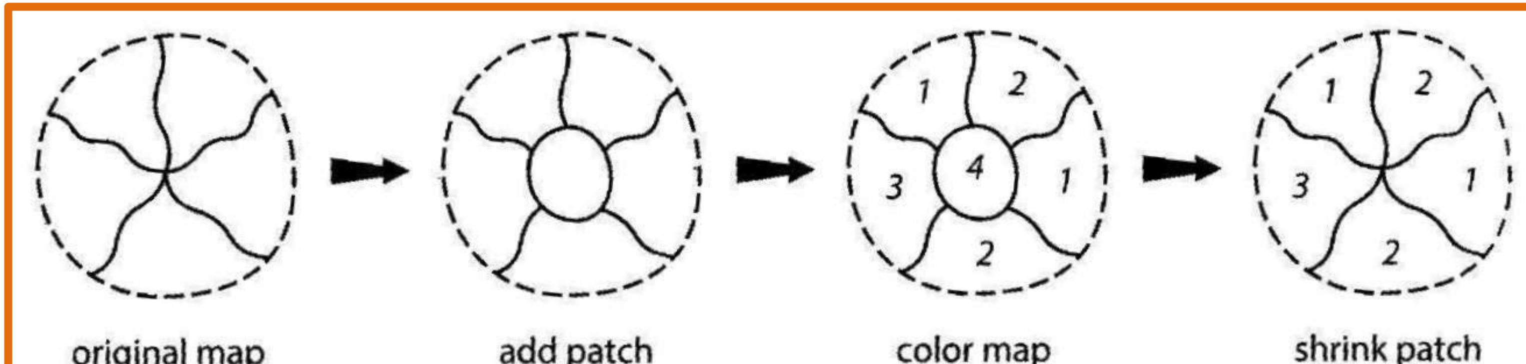


13 June 1878

London Mathematical Society

Has the problem been solved?

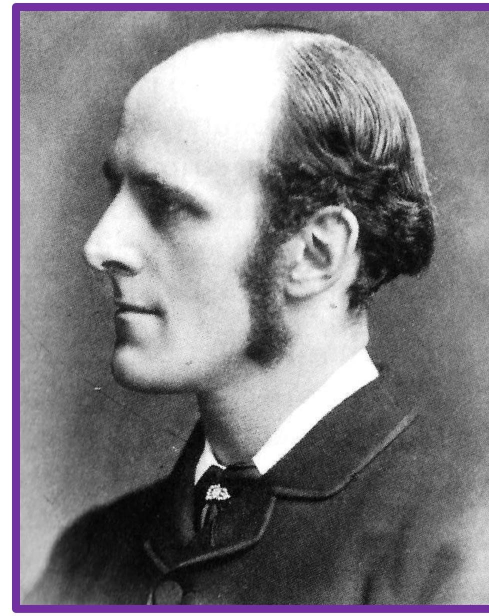
1879: short paper: we need consider only 'cubic' maps (3 countries at each point)



A. B. Kempe 'proves' the theorem

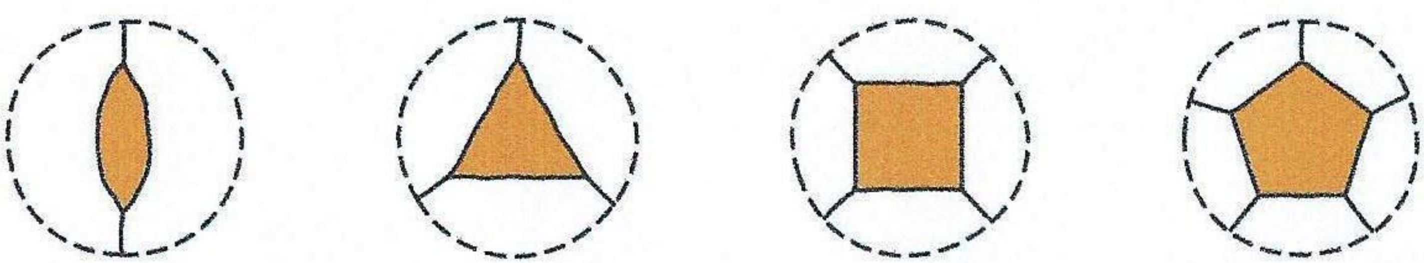
On the geographical problem of the four colours

American Journal of Mathematics, 1879



From Euler's polyhedron formula:

Every map contains a digon, triangle, square or pentagon



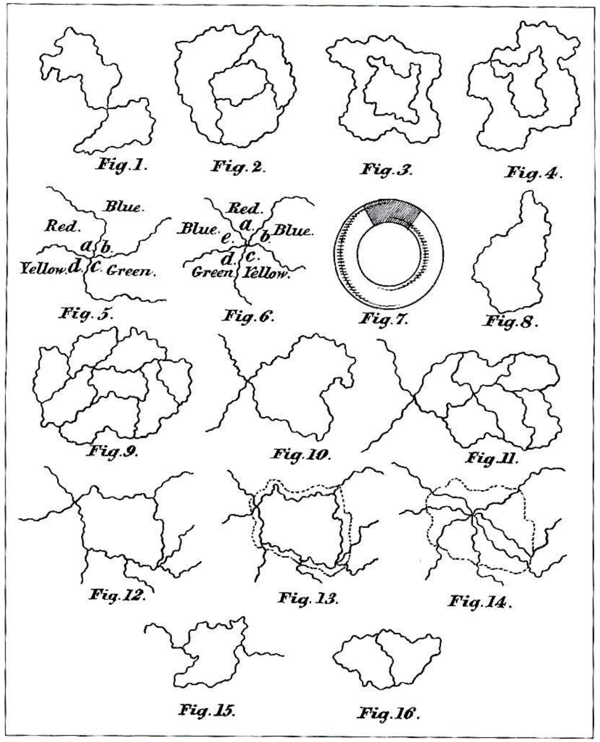
Kempe's paper (1879)

On the geographical problem of the four colours

On the Geographical Problem of the Four Colours.

BY A. B. KEMPE, B. A., *London, England.*

IF we examine any ordinary map, we shall find in general a number of lines dividing it into districts, and a number of others denoting rivers, roads, etc. It frequently happens that the multiplicity of the latter lines renders it extremely difficult to distinguish the boundary lines from them. In cases where it is important that the distinction should be clearly marked, the artifice has been adopted by map-makers of painting the districts in different colours, so that the boundaries are clearly defined as the places where one colour ends and another begins; thus rendering it possible to omit the boundary lines altogether. If this clearness of definition be the sole object in view, it is obviously unnecessary that non-adjacent districts should be painted different colours; and further, none of the clearness will be lost, and the boundary lines can equally well be omitted, if districts which merely meet at one or two points be painted the same colour. (See Fig. 1.)

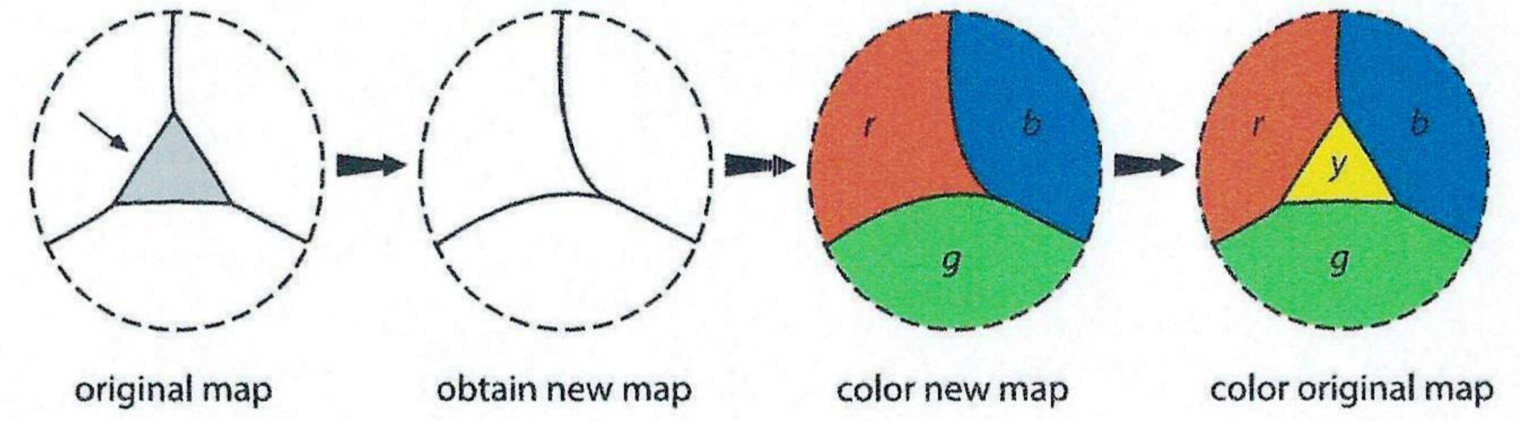


Kempe's proof 1: digon or triangle

Every map can be 4-coloured

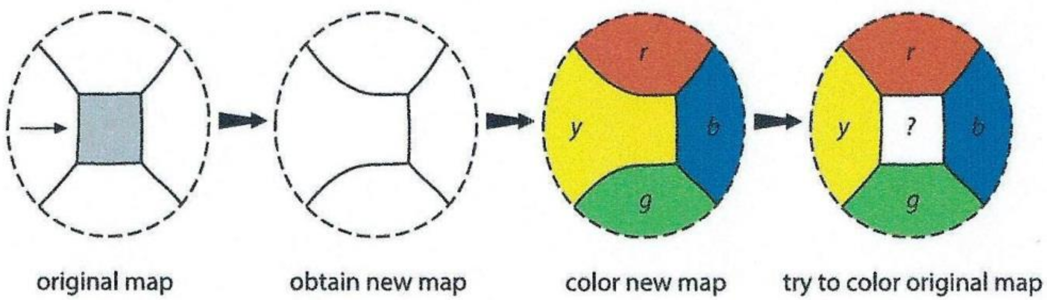
Assume not, and let M be a map with the smallest number of countries that cannot be 4-colored.

If M contains a digon or triangle T, remove it, 4-colour the resulting map, reinstate T, and colour it with any spare colour.
This gives a 4-colouring for M: contradiction

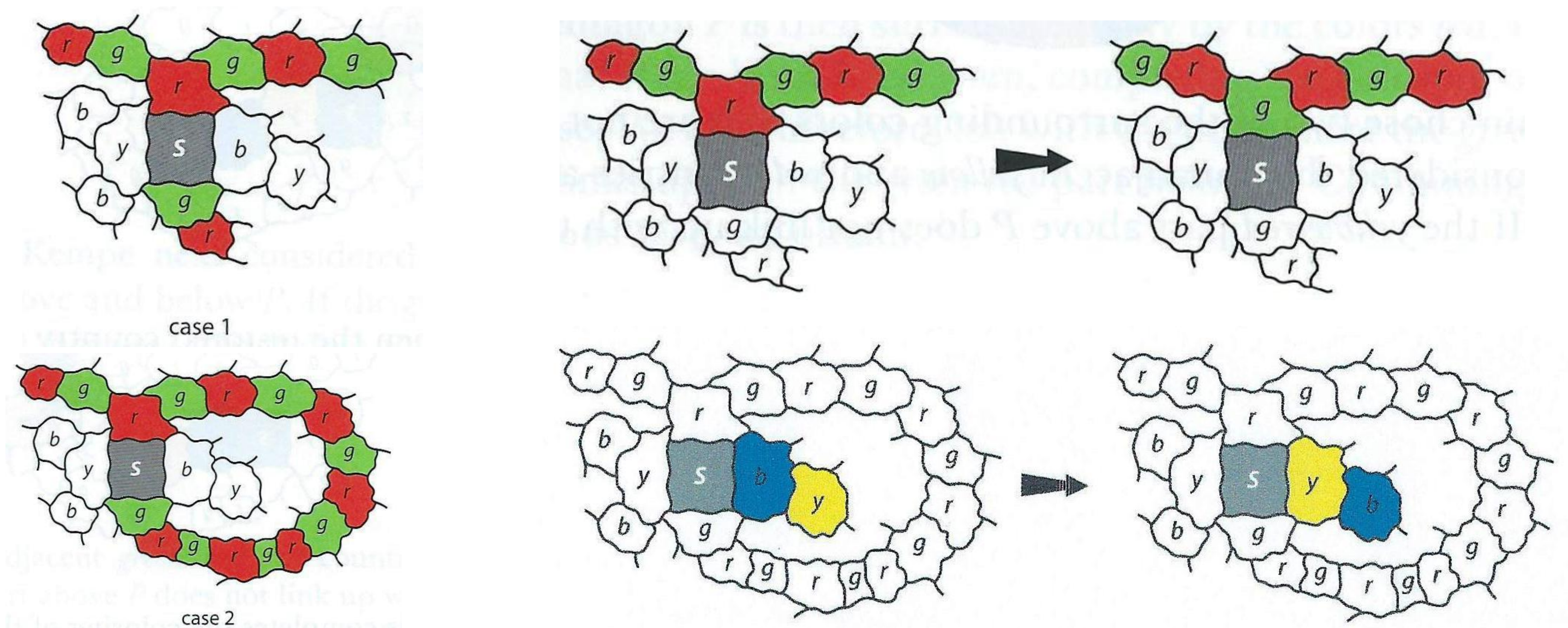


Kempe's proof 2: square

If the map M contains a square S, try to proceed as before:

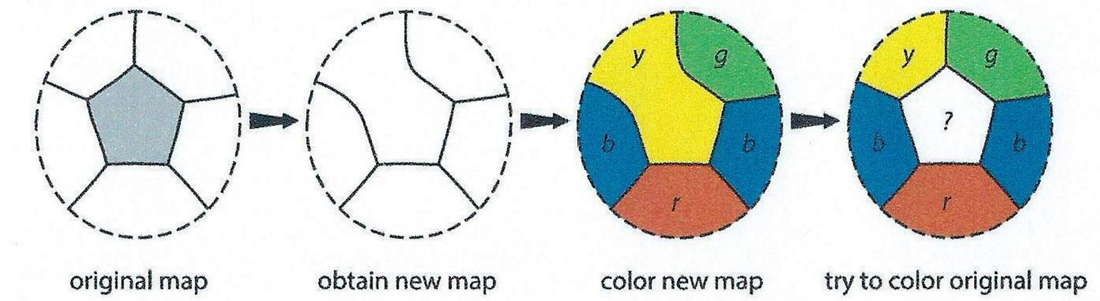


Are the red and green countries joined? Two cases:

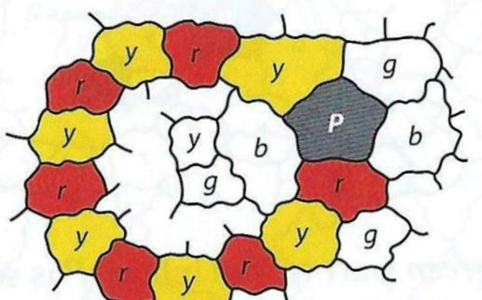


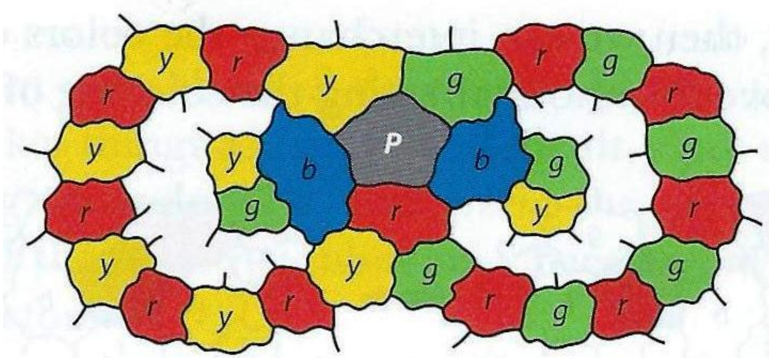
Kempe's proof 3: pentagon

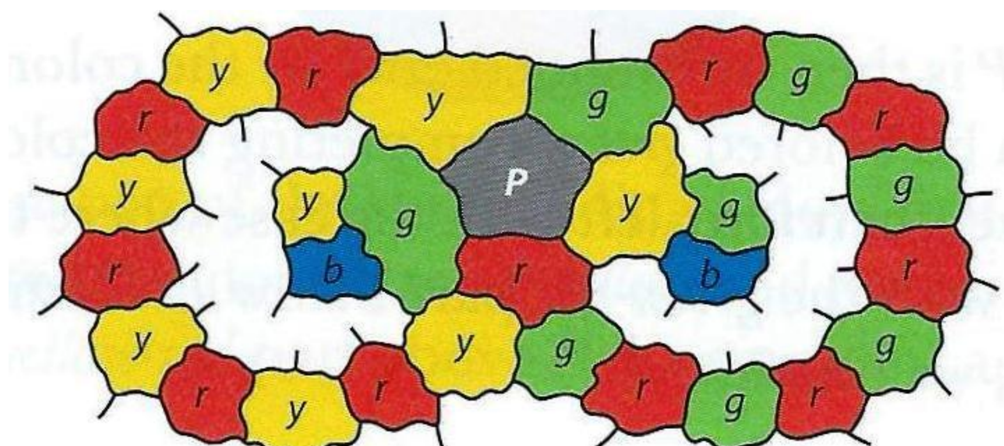
If the map M contains a pentagon P:



Carry out TWO 'Kempe interchanges' of colour:

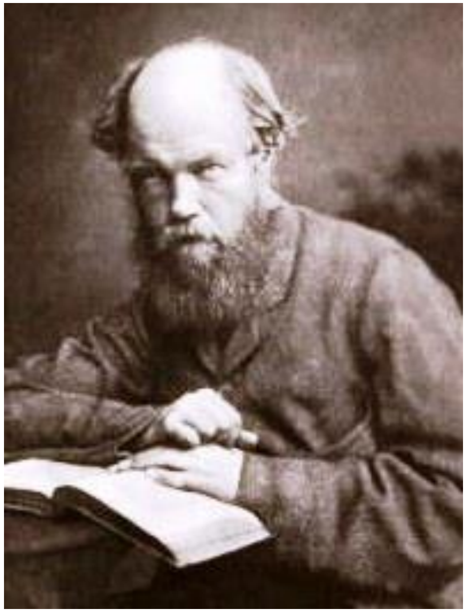






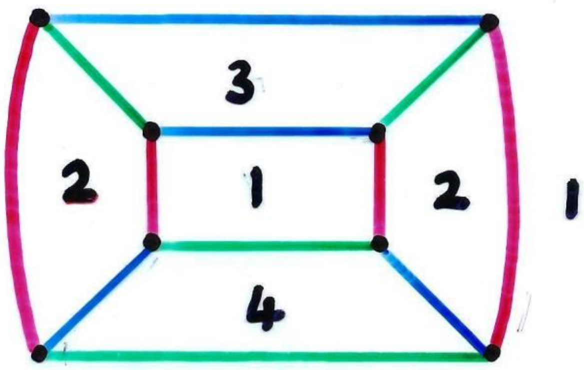
P. G. Tait, 1880

Remarks on the colouring of maps



For a cubic map:
instead of 4-colouring the countries, colour the boundary edges.
At each meeting point all three colours appear.

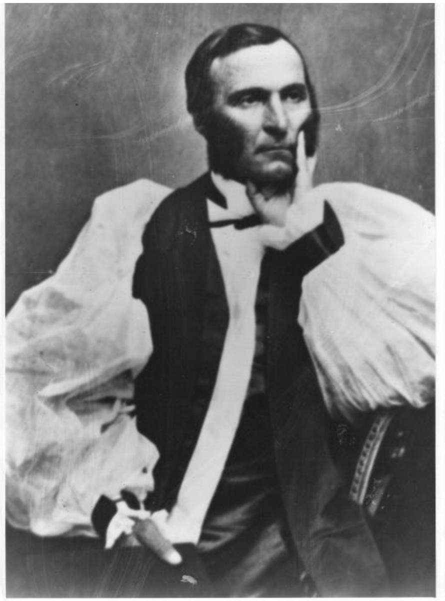
1–2 or 3–4
1–3 or 2–4
1–4 or 2–3



4-colouring the countries ↔ 3-colouring the edges

The problem becomes popular . . .

Lewis Carroll turned the problem into a game for two people . . .

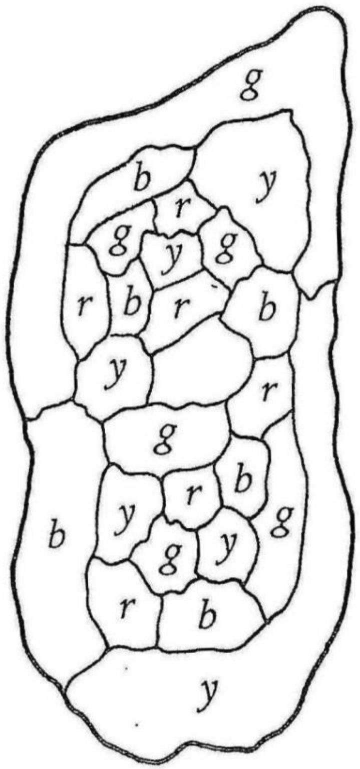


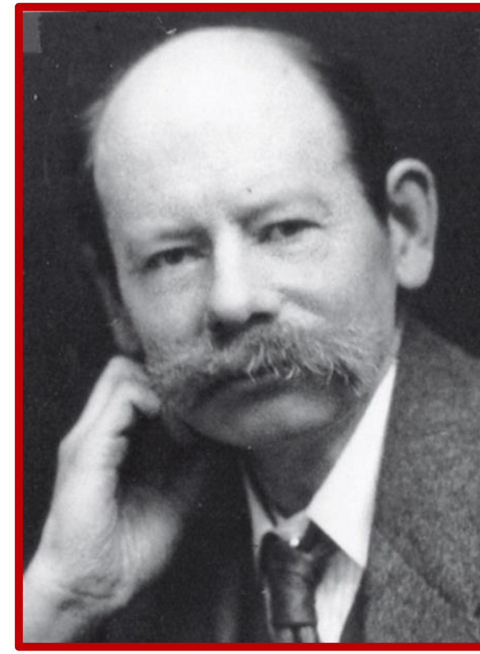
1886: J. M. Wilson, Headmaster of Clifton College, set it as a challenge problem for the school

1887: . . . and then sent it to the *Journal of Education*

. . . who in 1889 published a 'solution' by Frederick Temple, Bishop of London

1890: Percy Heawood

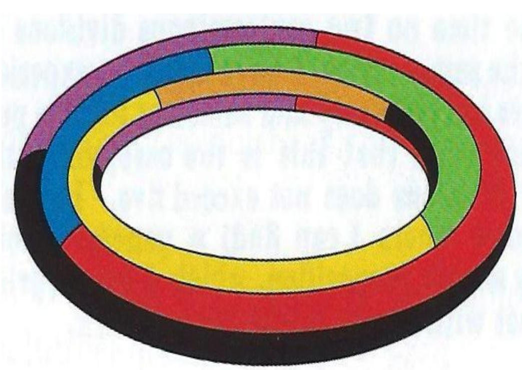




Map-colour theorem

Heawood pointed out the error in Kempe's 'proof' of the four-colour theorem,

and salvaged enough to prove the five-colour theorem.

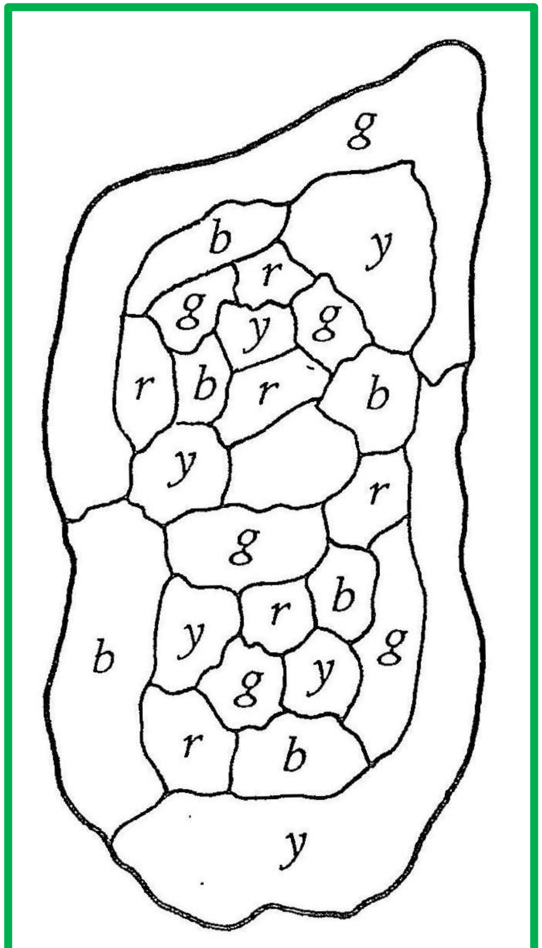


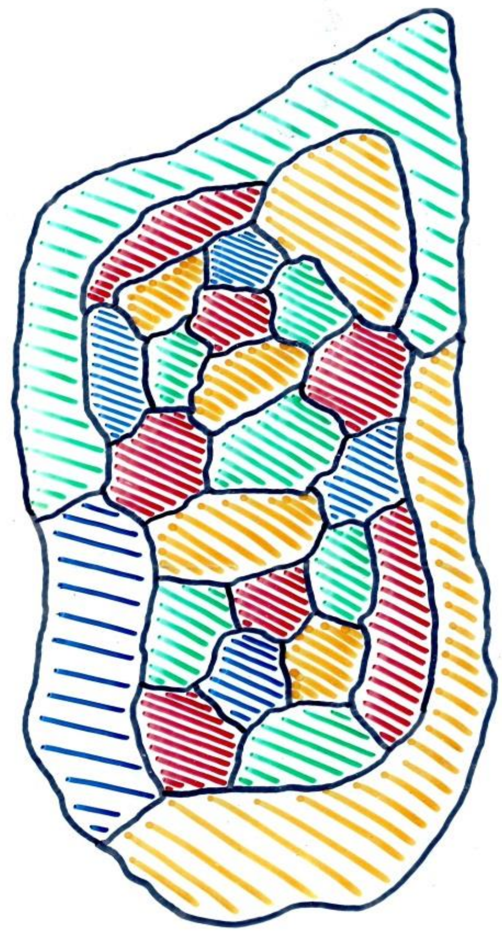
He also showed that, for maps on a g-holed torus (for $g \geq 1$), $[{}^{1}/{}_{2}\{7 + \sqrt{(1 + 48g)}\}]$ colours suffice.

[for a torus: $g = 1$: number = 7]

Heawood's example 1

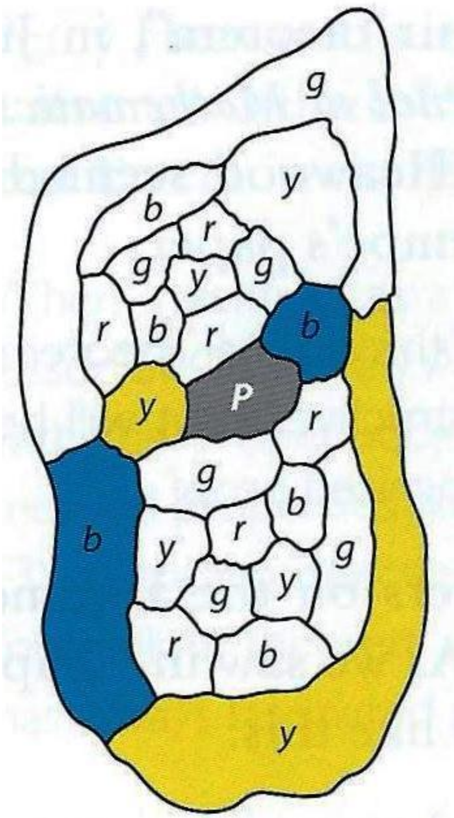
You cannot do two Kempe interchanges at once . . .

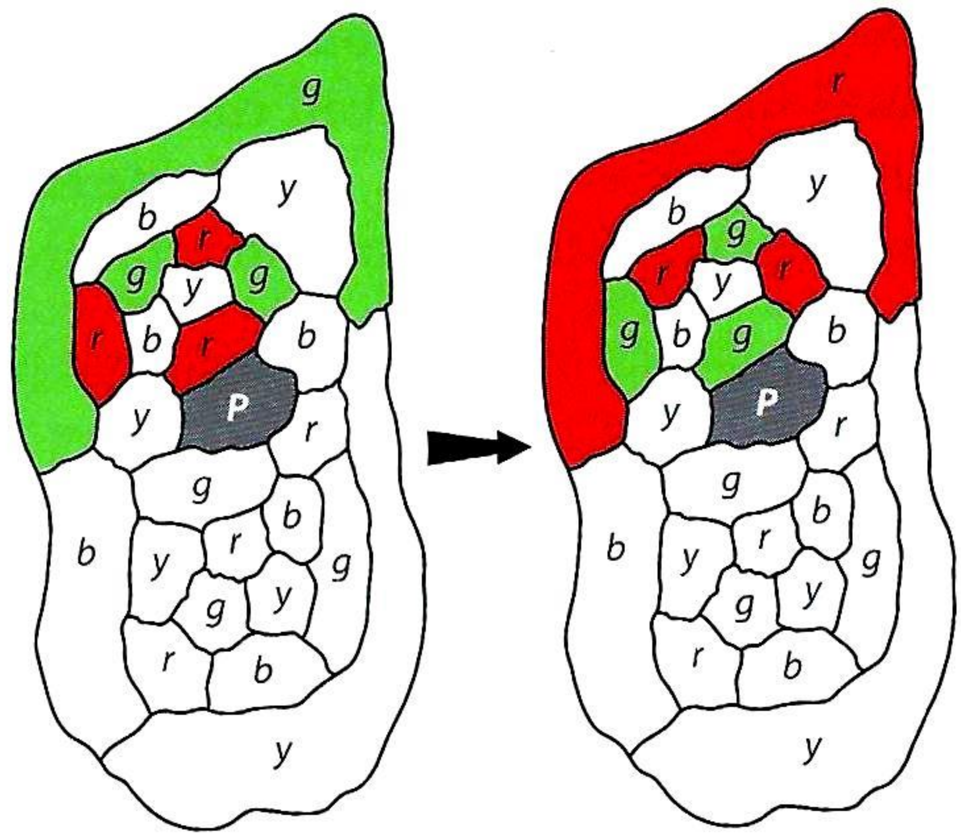




Heawood's example 2

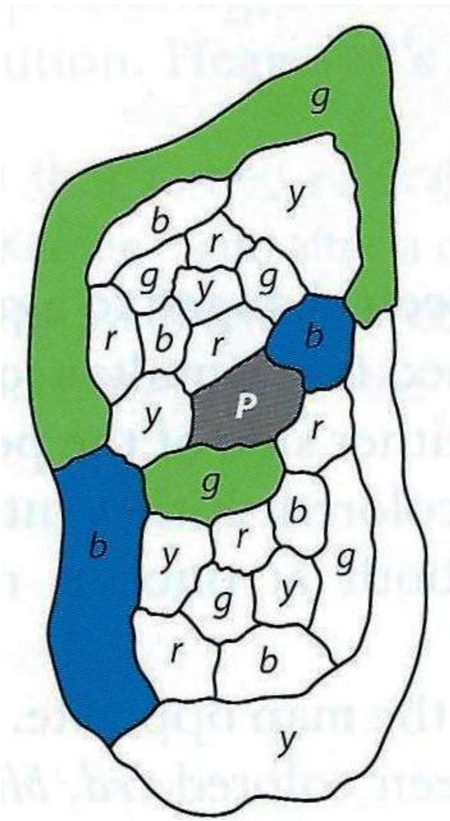
blue and yellow are connected . . .
so red and green are separated

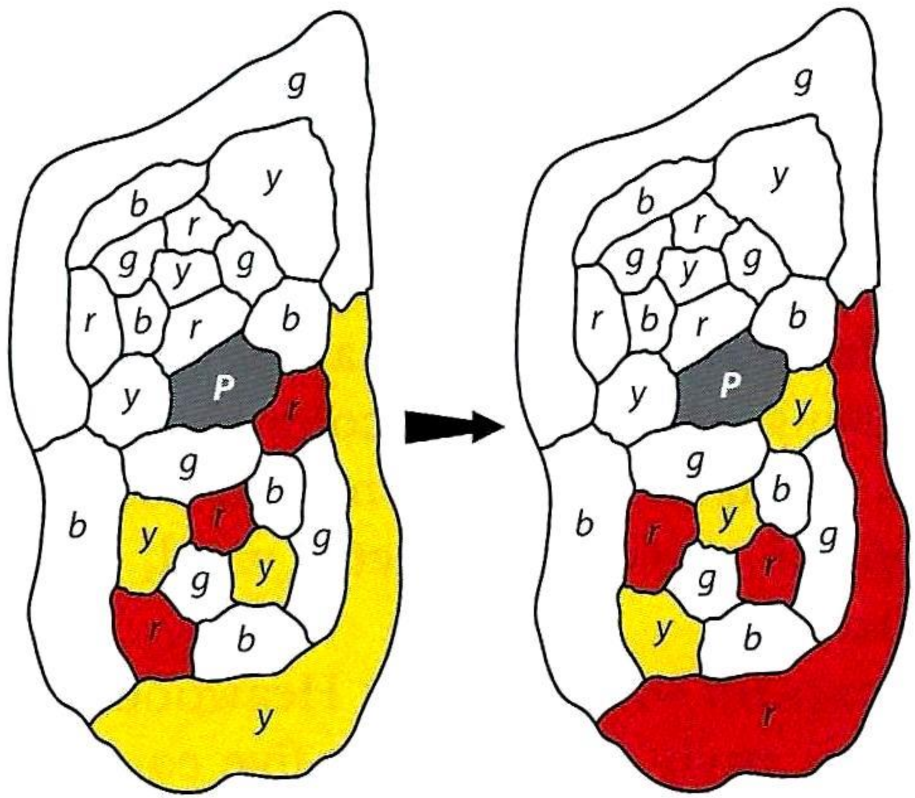




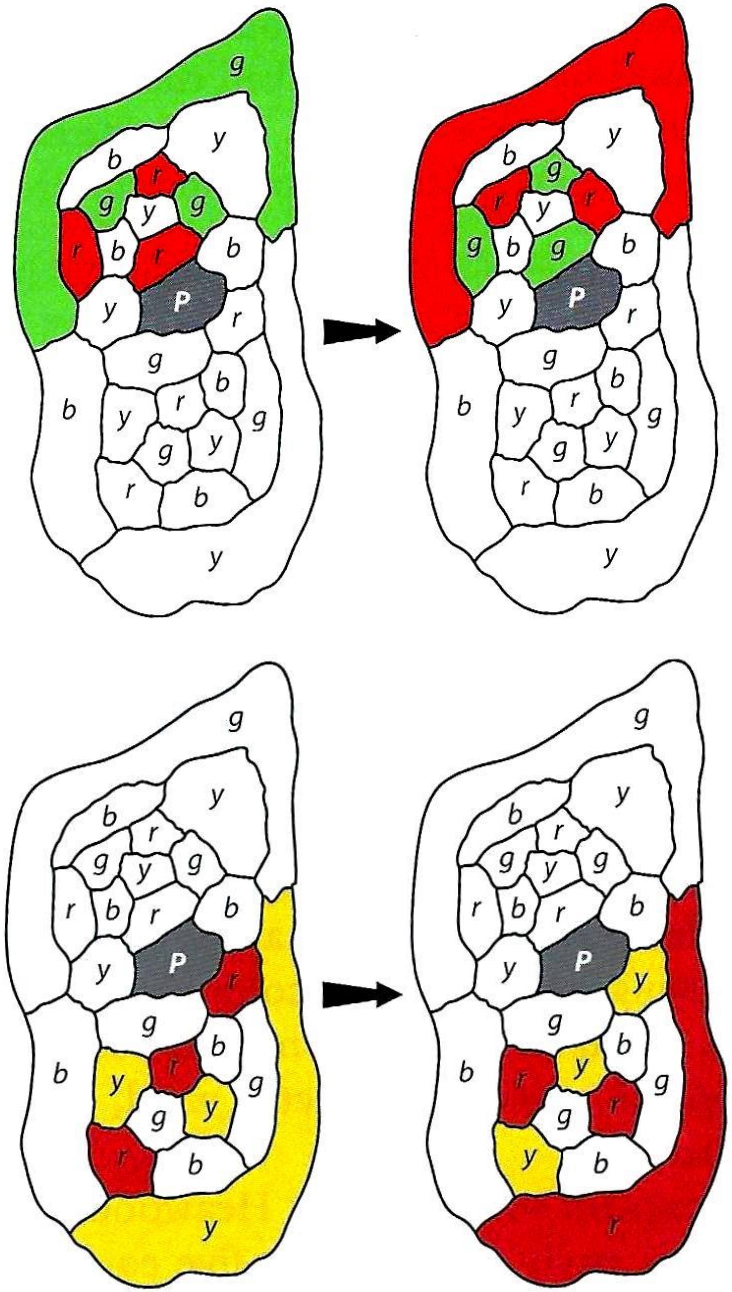
Heawood’s example 3

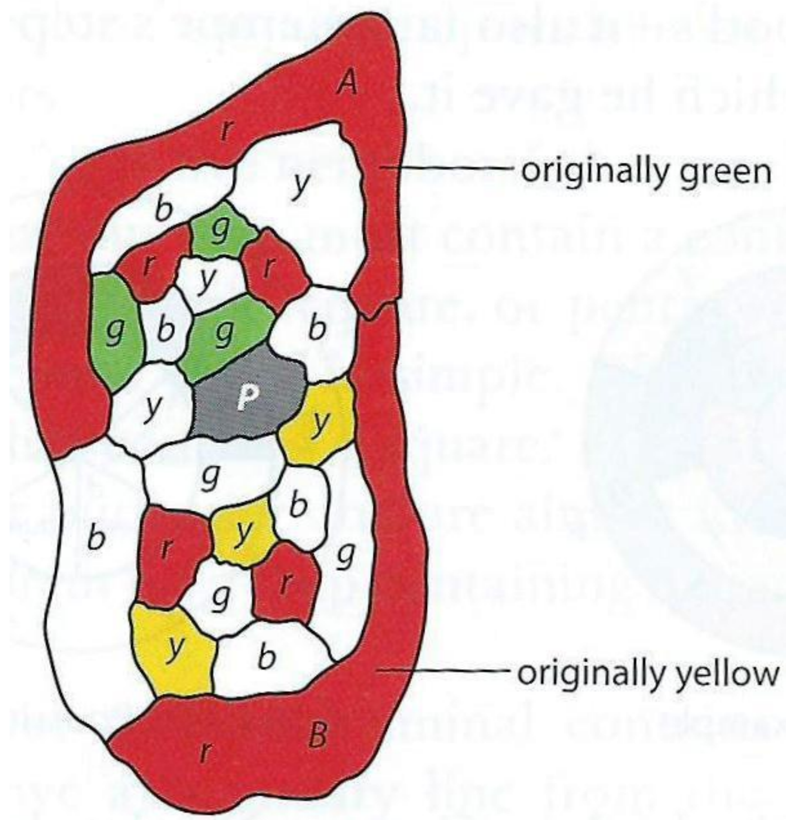
blue and green are connected . . .
so red and yellow are separated





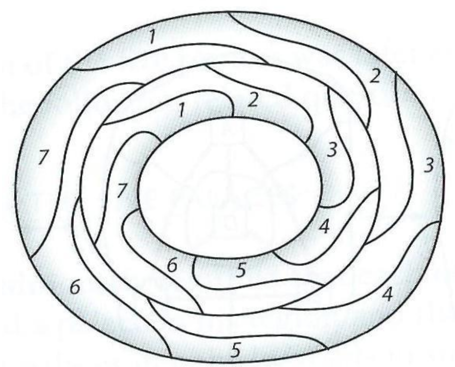
Heawood's example 4





Maps on other surfaces

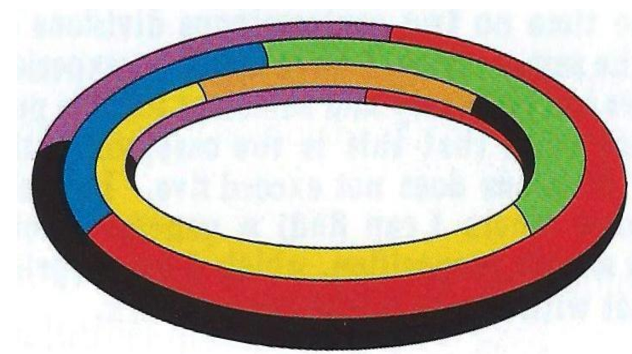
The four-colour problem concerns maps on a plane or sphere . . . but what about other surfaces?



Heawood: TORUS

7 colours suffice . . .

and may be necessary



HEAWOOD CONJECTURE

For a surface with h holes ($h \geq 1$)

$[{}^{1}/_{2}(7 + \sqrt{(1 + 48h)})]$ colours suffice:

$h = 1$: $[{}^{1}/_{2}(7 + \sqrt{49})] = 7$; $h = 2$: $[{}^{1}/_{2}(7 + \sqrt{97})] = 8$

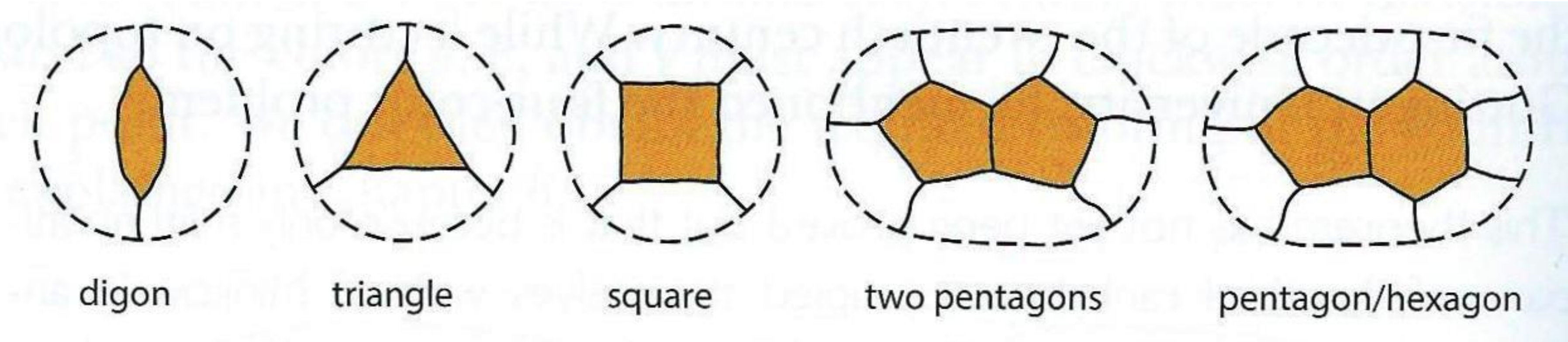
But do we need this number of colours?

1904: Paul Wernicke

Über den kartographischen Vierfarbensatz

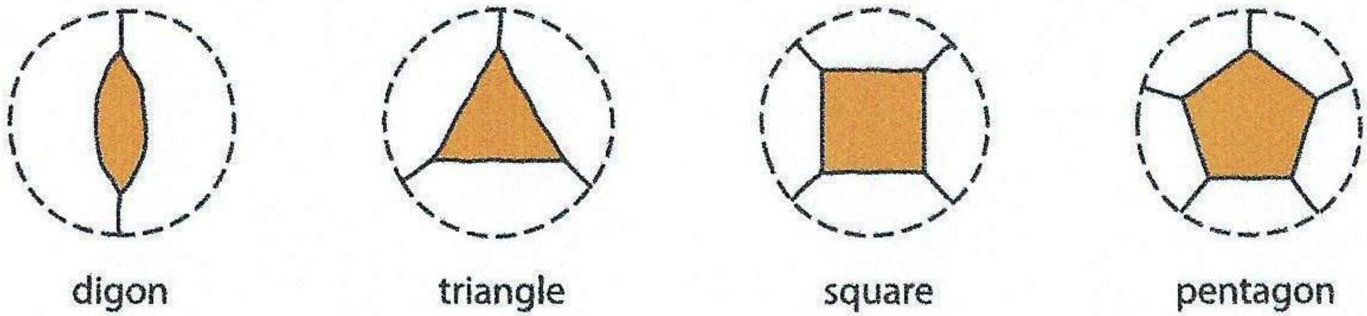
Kempe: Every map on the plane contains a digon, triangle, square or pentagon

Wernicke: Every map on the plane contains at least one of the following configurations



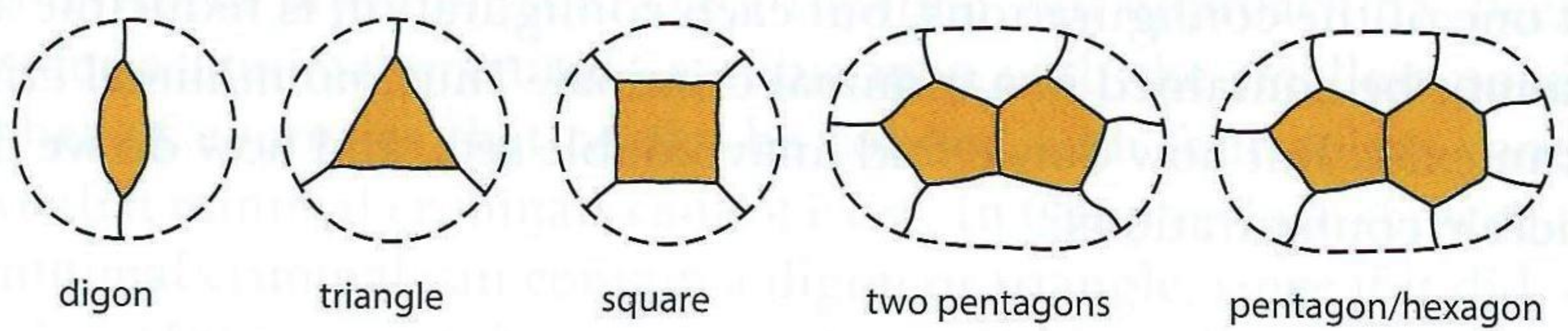
They form an 'unavoidable set': every map must contain at least one of them

Unavoidable sets

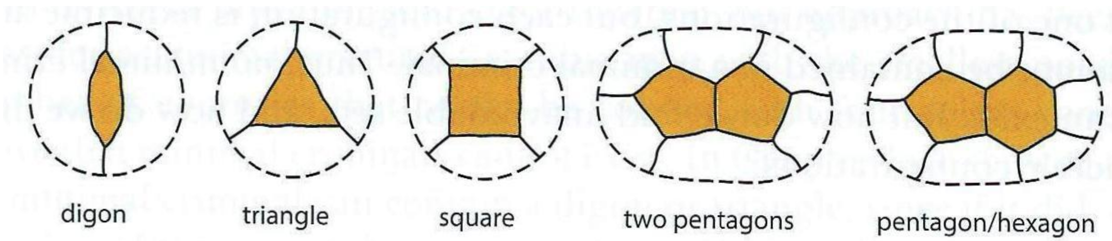


is an unavoidable set:
every map contains at least one of them

and so is the following set of Wernicke (1904):



An unavoidable set



If none of these appears, then each pentagon adjoins countries with at least 7 edges.

Now, if C_k is the number of k-sided countries, then

$$(4C_2 + 3C_3 + 2C_4) + C_5 - C_7 - 2C_8 - 3C_9 - \ldots = 12$$

Assign a ‘charge’ of 6 – k to each k-sided country:

pentagons 1, hexagons 0, heptagons – 1, . . .

Total charge = $C_5 - C_7 - 2C_8 - 3C_9 - \ldots = 12$

Now transfer charge of $^1/_5$ from each pentagon to each negatively-charged neighbour.

Total charge stays at 12, but pentagons become 0, hexagons stay at 0, and heptagons, octagons, . . . stay negative.

So total charge ≤ 0: CONTRADICTION

1913: G. D. Birkhoff

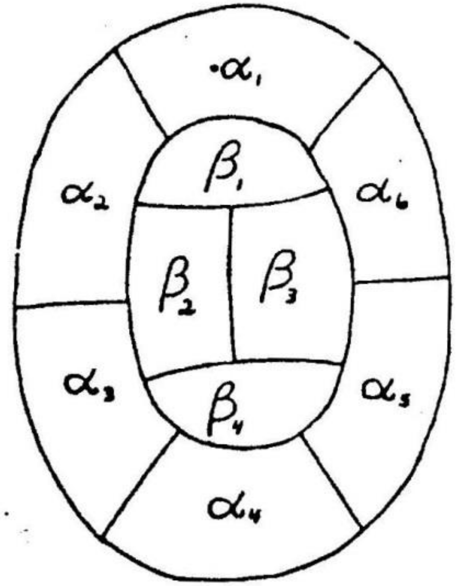
The reducibility of maps

A configuration of countries in a map is **reducible** if any 4-colouring of the rest of the map can be extended to the configuration.

Irreducible configurations cannot appear in counter-examples to the 4-colour theorem

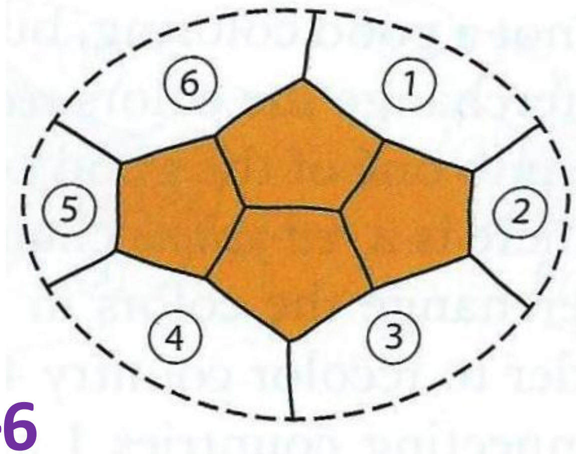
Kempe: digons, triangles and squares are reducible

Birkhoff: so is the Birkhoff diamond

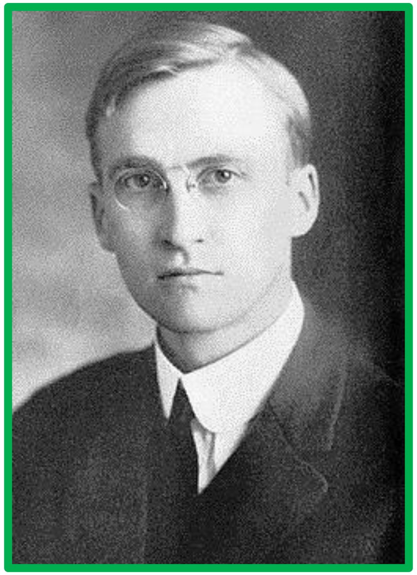


Testing for reducibility

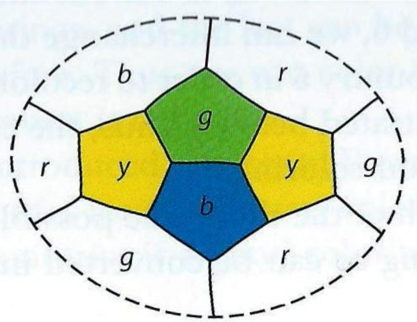
Colour the countries 1–6
in all 31 possible ways



Birkhoff diamond



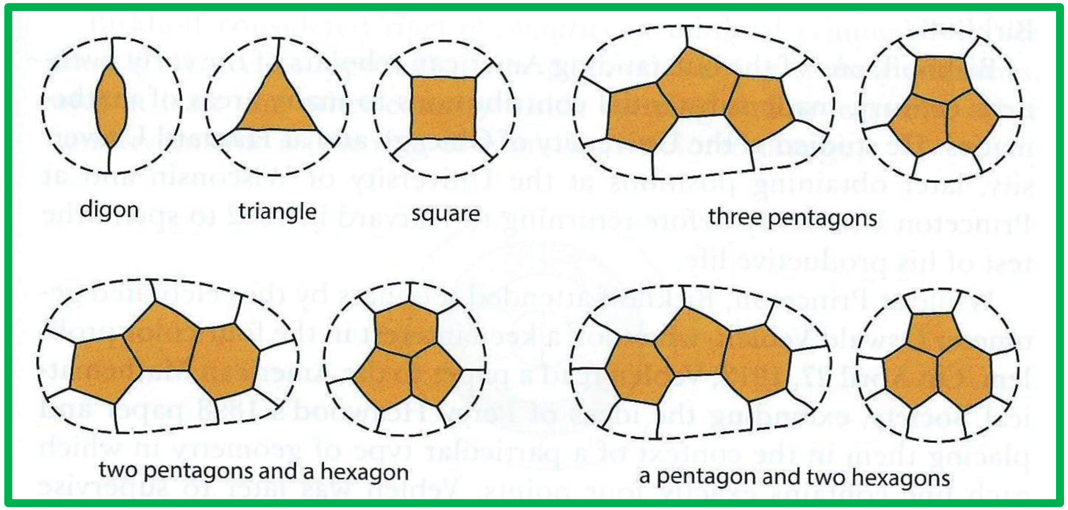
rgrgrg	*rgrbrg**	*rgrbgy**	*rgbrgy*	*rgbryb*	*rgbgbg**	*rgbyrg*	*rgbygy**
*rgrgrb**	*rgrbrb*	*rgrbyg**	*rgbrbg**	*rgbgrg**	*rgbgby*	*rgbyrb*	*rgbybg**
rgrgbg	*rgrbry*	*rgrbyb**	*rgbrby*	*rgbgrb**	*rgbgyg*	*rgbyry**	*rgbyby**
*rgrgby**	*rgrbgb**	*rgbrgb*	*rgbryg*	*rgbgry**	*rgbgyb*	*rgbygb*	



rgrgrb extends directly
and ALL can be done directly
or via Kempe interchanges of colour

Philip Franklin (1922)
The four color problem

Every cubic map containing no triangles or squares must have at least 12 pentagons.

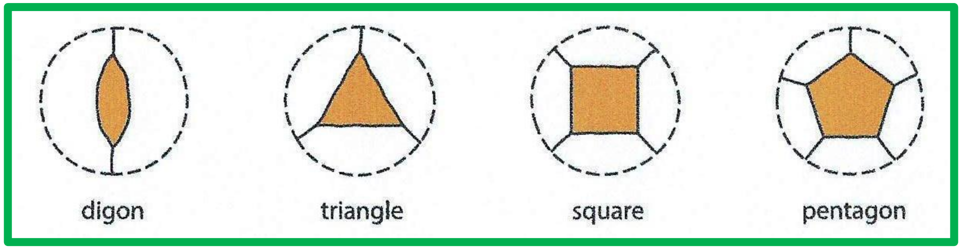


Any counter-example has at least 25 countries

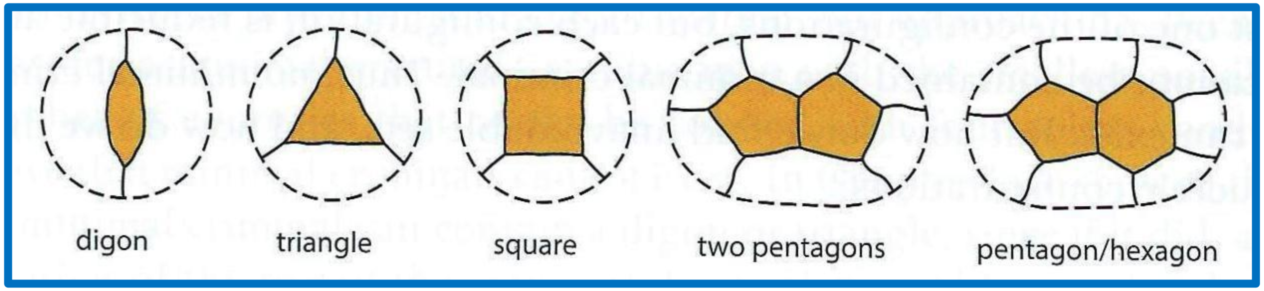
later extended by Reynolds (27), Winn (39) and others

Further unavoidable sets found by H. Lebesgue (1940)

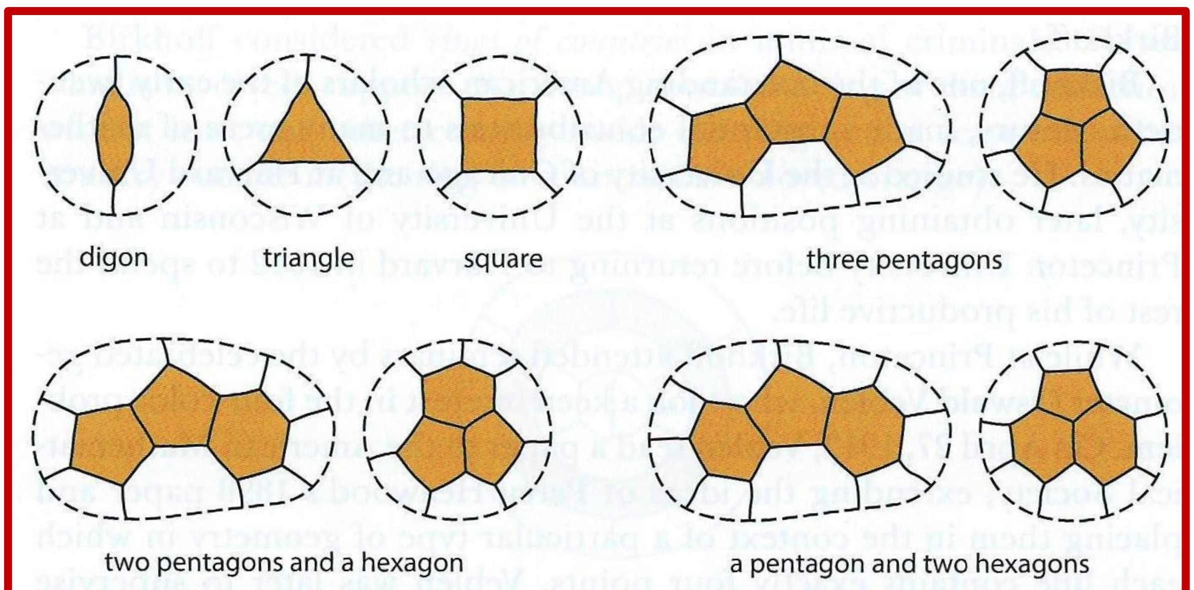
Unavoidable sets



Kempe
1879

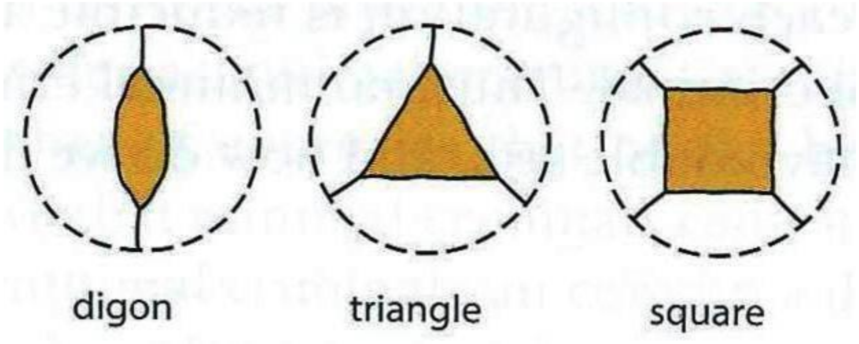


Wernicke
1904



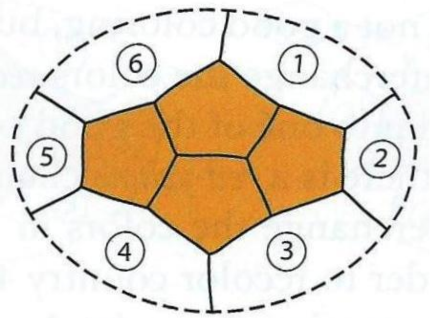
P. Franklin 1922: so the four-colour theorem is true for maps with up to 25 countries
[also H. Lebesgue]

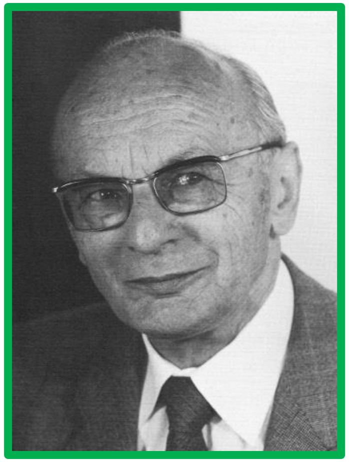
Reducible configurations



These configurations are reducible: any colouring of the rest of the map can be extended to include them

So are the 'Birkhoff diamond' (1913), and many hundreds of others





Aim (Heinrich Heesch):
To solve the four colour problem it is sufficient to find an unavoidable set of reducible configurations

1976
Kenneth Appel
& Wolfgang Haken
(Univ. of Illinois)
Every planar map is four colorable
(with John Koch)

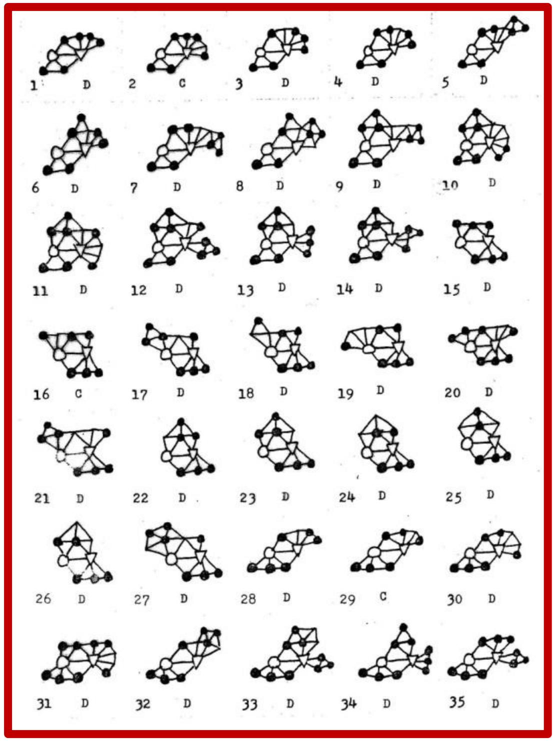
They solved the problem by finding an unavoidable set of 1936 (later 1482) reducible configurations

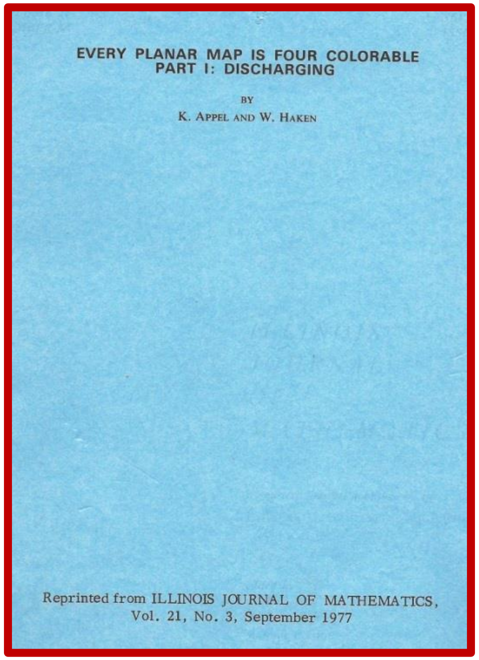
1976: K. Appel & W. Haken

Every planar map is four-colorable

H. Heesch: find an unavoidable set of reducible configurations

Using a computer Appel and Haken (and J. Koch) found an unavoidable set of 1936 reducible configurations (later 1482)



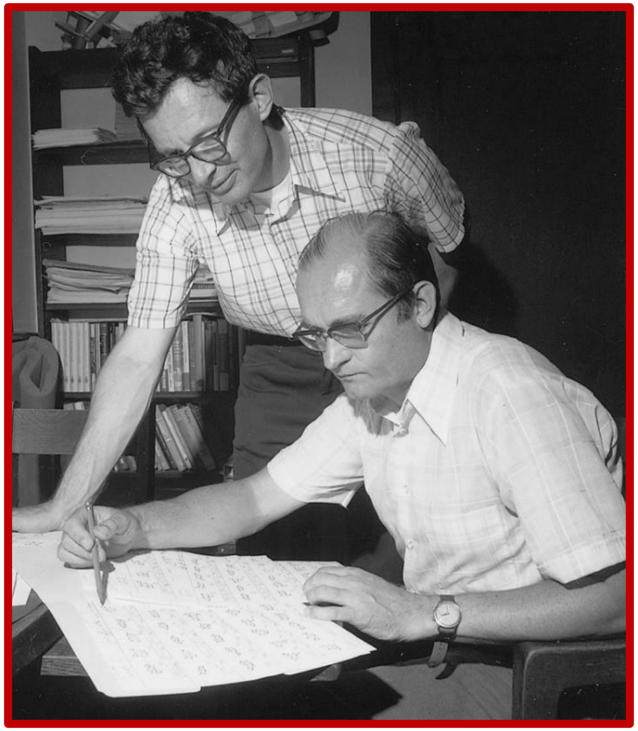


EVERY PLANAR MAP IS FOUR COLORABLE
PART I: DISCHARGING

BY

K. APPEL AND W. HAKEN

Reprinted from ILLINOIS JOURNAL OF MATHEMATICS,
Vol. 21, No. 3, September 1977

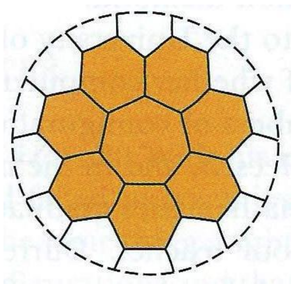


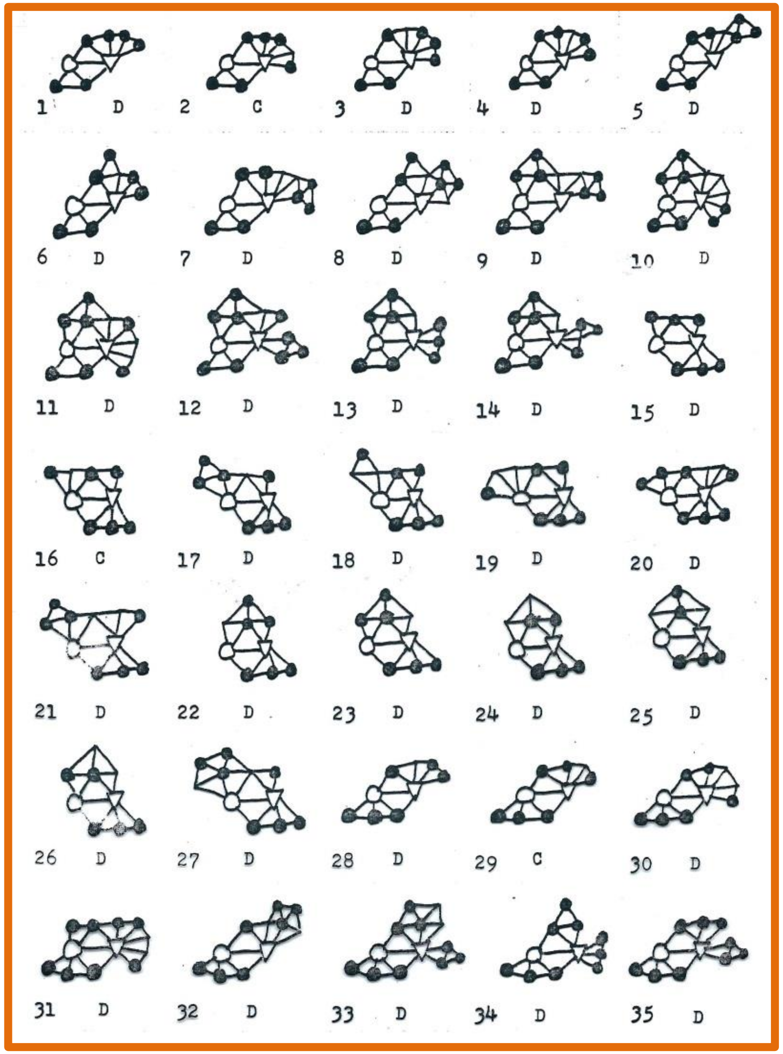
The Appel-Haken approach

Develop a 'discharging method' that yields an unavoidable set of 'likely-to-be-reducible' configurations.

Then use a computer to check whether the configurations are actually reducible: if not, modify the unavoidable set.

They had to go up to 'ring-size' 14 (199,291 colourings)





The proof is widely acclaimed

第3種郵便物認可

変わる証明の概念

電算機利用は不可欠に

「四色問題」解決の意義

どんな地図でも４色で塗り分けられる

四色問題の証明は、数学の専門家だけではなく、一般の数学ファンにとっても関心の深い問題だった。「平面または球面上のすべての地図は四色で塗り分けられる。これを証明せよ」という問題は、それほど難しい数式を使わなくても取り組みやすそうに見えるし、素人受けするテーマだ。百三十年前、数学者にこの問題を持ち込んで証明をたのんだのは、地図印刷業者が経験的に知っていたことを小耳にはさんだ大学生だった、といわれている。最初は、数学者たちには簡単に解けそうに思えたが、やってみると難しかった。

このように、ある定理の真偽が、肯定も否定もされずに何百年も未解決でいることは数学上の大問題であるだけでなく「数学者の精神衛生上の大問題」となる。なぜなら、一般の数学ファンがこれでもかこれでもかと〝証明成功〟の論文を数学者たちに持ち込んで点検を迫り、悩ますからだ。

「たとえば、角の三等分の問題。これは定規とコンパスではできない。できないことを数学者が証明ずみだから、どんな珍証明を持ち込まれても、ただちにクズカゴへ捨ててしまう。しかし、四色問題は、そうはいかなかった。大多数の数学者は、四色問題を証明したという数学ファンの論文を、自賛の念を持ちつつ、クズカゴへ捨てていたのです」と矢野健太郎東工大名誉教授はいう。数学者の悩みの種は、まだかなり残されている。「中の島の両側にかかった七つの橋を、すべて一度ずつかならず渡る方法はあるか」という有名な「ケーニヒスベルクの七つの橋の問題」など、数学ファンの興味をそそるものだ。

イリノイ大学の二人の数学教授が証明した方法は、数学界に新しい論争を呼び起こしている。紙と鉛筆と人間の理性だけでスマートに解くのではなく、電子計算機を動員して、膨大な計算をした結果はじめて得られた証明が果たして、本当の証明といえるかどうかという議論だ。一松信京大数理解析研究所教授は、電子計算機の功罪を次のようにいう。

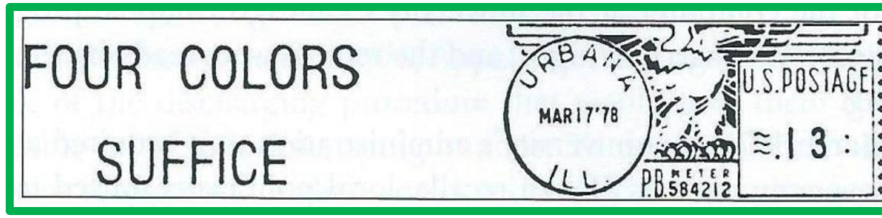
まず、難しそうな問題を電算機にやらせて、証明できたとする。証明できたとわかっていれば、数学者は、電子計算機なしでいずれはスマートな証明法が見つかる、とファイトを燃やせる。「こうして、最初は人間の理性ではとてもダメだと思われていた有限群をスマートに再証明した実例もかなりある」という。

しかし、数学的証明とは、誤りがまったくないものでなければならない。計算機は機械だから、どこかが故障して誤った計算をしているかも知れない。一つの計算機で、その命題を肯定する証明が完了しても、別の計算機は、それを否定する証明を出すかもしれない。このとき、どちらの機械が誤っているのか、第三の電算機が判定者として乗り出してきて、どちらか一方に軍配を上げたとしても、二対一になったに過ぎない。多数決で数学の定理が証明できるとは、伝統的な数学者は考えていない。四月にピッツバーグのカーネギーメロン大学で開かれたコンピューター数学のシンポジウムでも、この点が大議論になった。

にもかかわらず、数学の定理を電子計算機で証明するという新しい方法は、将来の数学にとって必要不可欠な方法となる、とみる数学者はふえている。数学者自身「証明」の定義と概念を変えざるを得ない事態に来ているのだ。なぜならスマートな証明ができるような問題は、ほとんどすべて証明ずみとなり、あとは電算機に頼るしかないような複雑な問題しか残っていないと予想されるからだ。

しかし、現在、数学者は電子計算機を十分には使えない状況だ。数学者以外の科学者の使用率が圧倒的に高いうえに、巨額の費用と時間がかかる。電算機の使用を最初から敬遠せざるを得ない数学者も多い。イリノイ大学の二教授の成功は、たまたま新設の電算機を千二百時間もぶっ続けに使える幸運に恵まれたからだ、という見方もある。

「定理の発見者、証明者、証明年月日のほかに、数学の論文にはもう一つのデータが不可欠となった。その定理を証明するのにいくらかかったかを明記する必要があるわけです」と佐藤大八郎カナダ・レジャイナ大学教授は、ため息をついている。（西村　幹夫記者）



Aftermath

The ‘computer proof’ was greeted with suspicion, derision and dismay – and raised philosophical issues: is a ‘proof’ really a proof if you can’t check it by hand?

Some minor errors were found in Appel and Haken’s proof and quickly corrected.

Using the same approach, N. Robertson, P. Seymour, D. Sanders and R. Thomas obtained a more systematic proof in 1994, involving about 600 configurations.

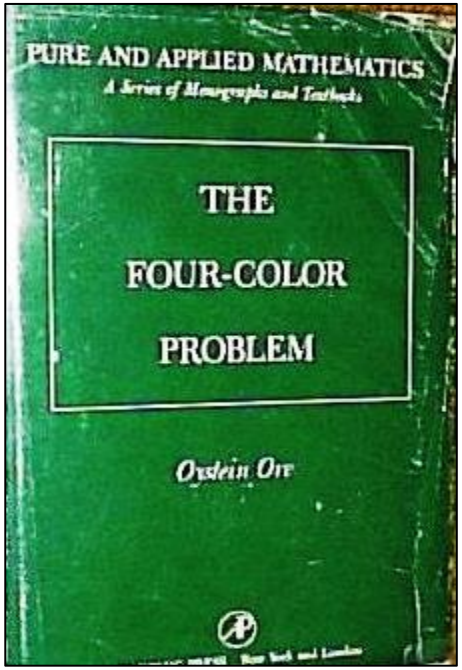
In 2004 G. Gonthier produced a fully machine-checked proof of the four-colour theorem (a formal machine verification of Robertson *et al.*’s proof).

The story is not finished . . .

Many new lines of research have been stimulated by the four-colour theorem, and for several conjectures it is but a special case.

In 1978 W. T. Tutte wrote:

**The Four Colour Theorem
is the tip of the iceberg,
the thin end of the wedge
and the first cuckoo of Spring.**


PURE AND APPLIED MATHEMATICS
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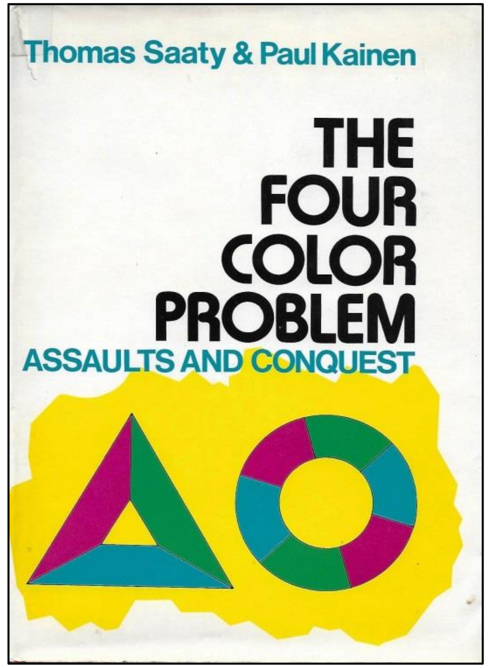
Thomas Saaty & Paul Kainen
THE
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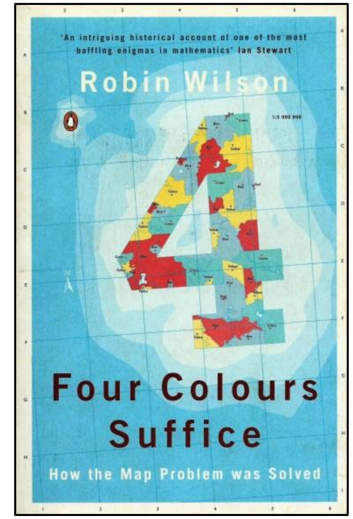
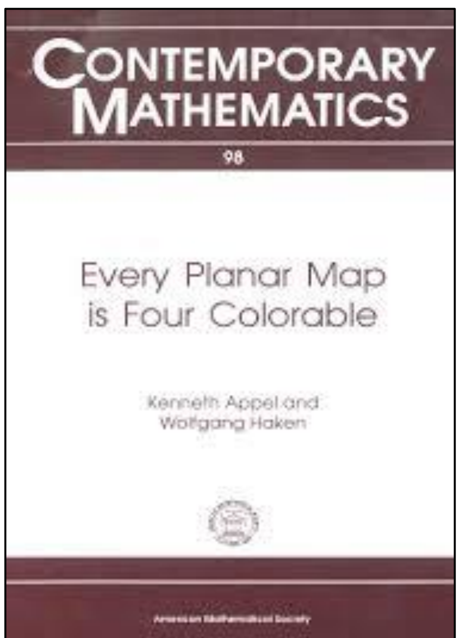
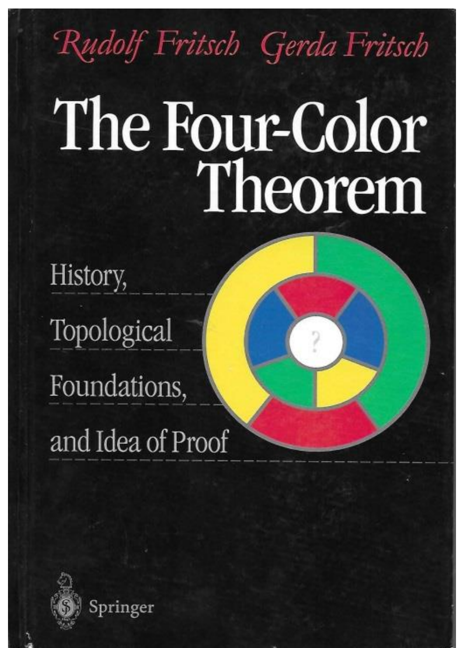

'An intriguing historical account of one of the most baffling enigmas in mathematics' Ian Stewart
Robin Wilson
4
Four Colours
Suffice
How the Map Problem was Solved


CONTEMPORARY
MATHEMATICS
98
Every Planar Map
is Four Colorable
Kenneth Appel and
Wolfgang Haken
American Mathematical Society


Rudolf Fritsch Gerda Fritsch
The Four-Color
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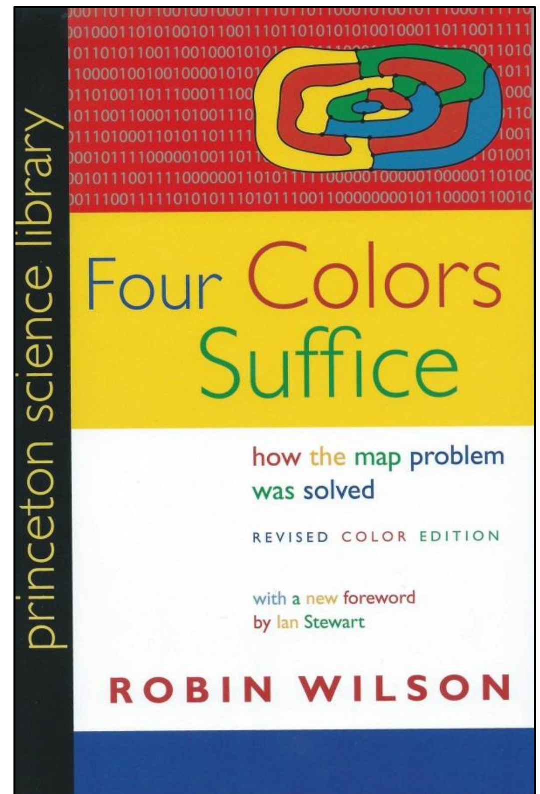
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