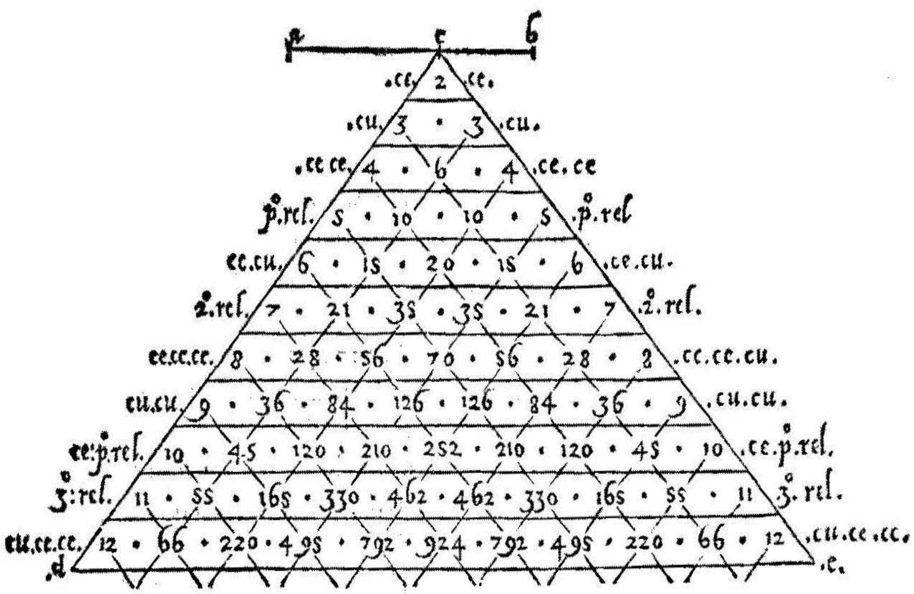
2. European Combinatorics

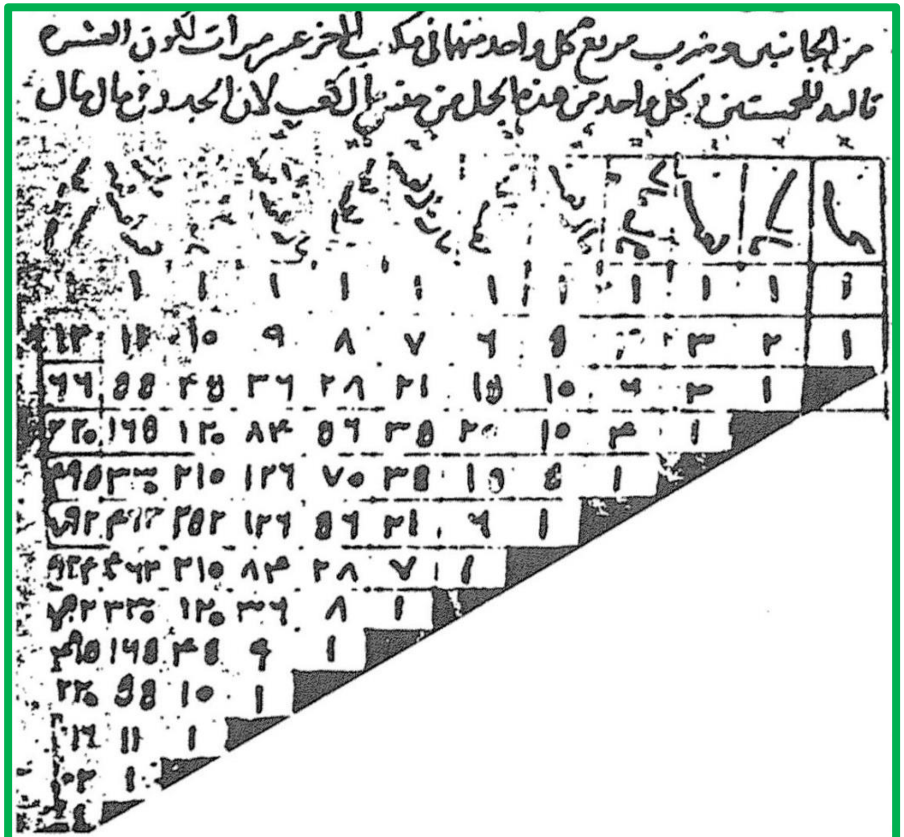
From Middle Ages to Renaissance

Robin Wilson



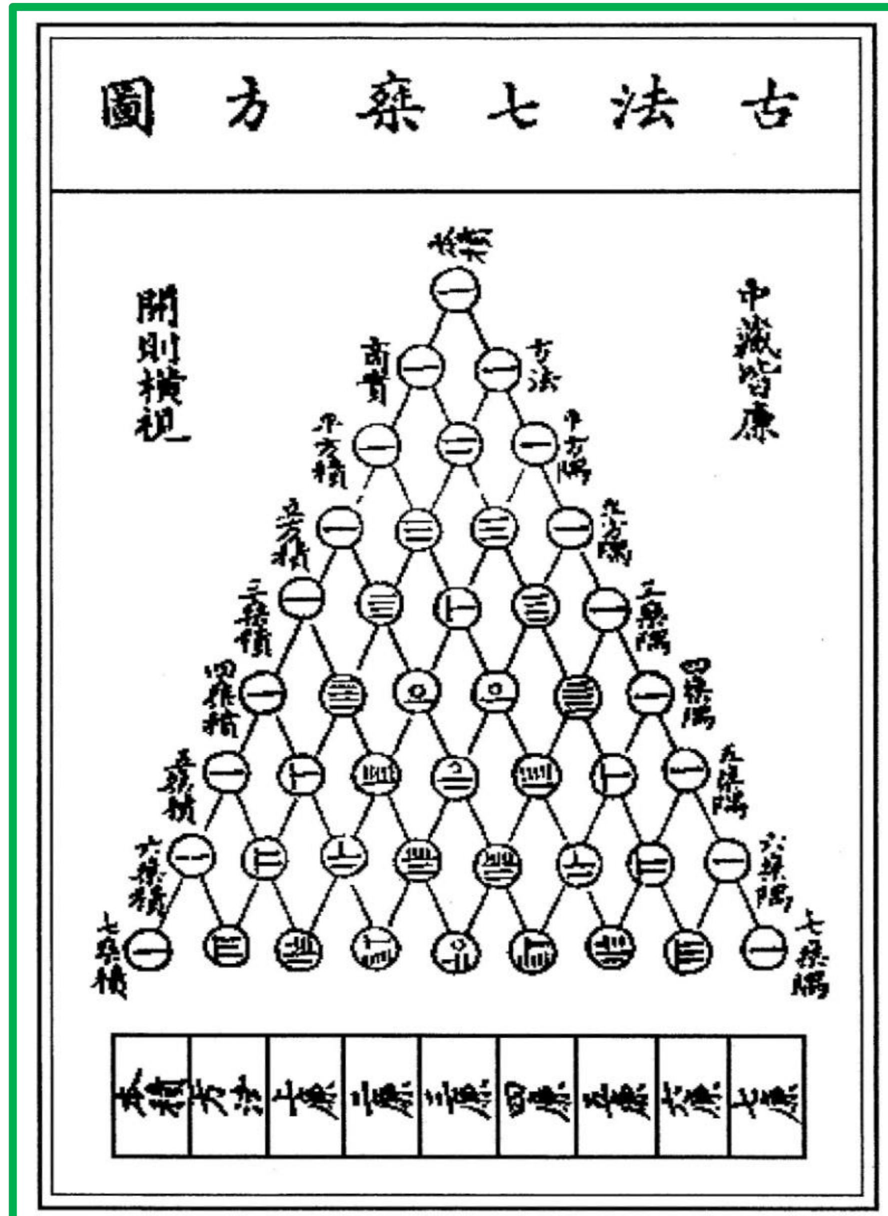
1. **The binomial theorem and the arithmetical triangle**
2. **Fibonacci**
3. **Ramon Llull and his followers**
4. **Pascal and Leibniz and their successors**
5. **Games of chance**

Arithmetical triangles of Al-Karaji (1007) and Nasir ad-Din at-Tusi(c.1240)



Zhu Shijie (1303): *Sijuan yujian*

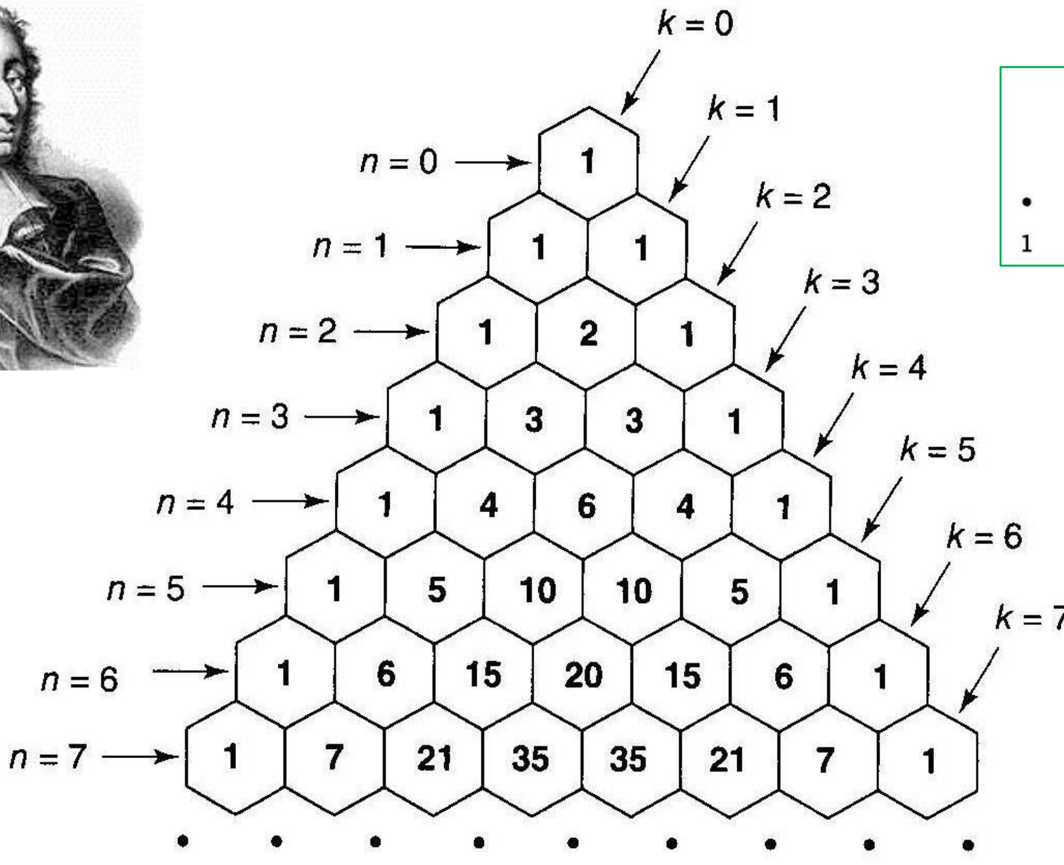
(Precious mirror of the four elements)

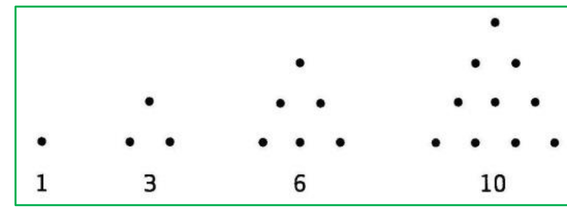


‘Pascal’s triangle’ of numbers C(n, k)

figurate numbers (such as triangular numbers)

‘binomial coefficients’ in the expansion of $(x + y)^n$





$$(1 + x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$$

The binomial theorem: $(a + b)^n$

$$(x+y)^0 = 1$$
$$(x+y)^1 = 1x + 1y$$
$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$
$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$
$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$
$$(x+y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1x^5$$
$$(x+y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1x^6$$
$$(x+y)^7 = 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7$$

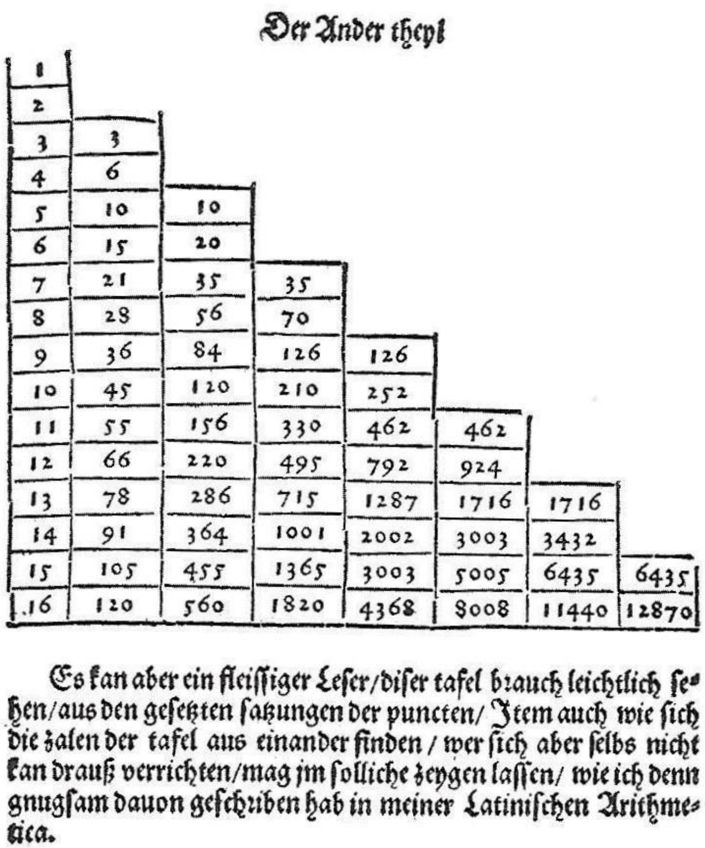
. . . . .

n = 2: Euclid's *Elements* (250 BC)

n = 3: Brahmagupta (AD 628)

n = 4, 5, 6: Omar Khayyam (c.1100)

Two 1545 German arithmetical triangles

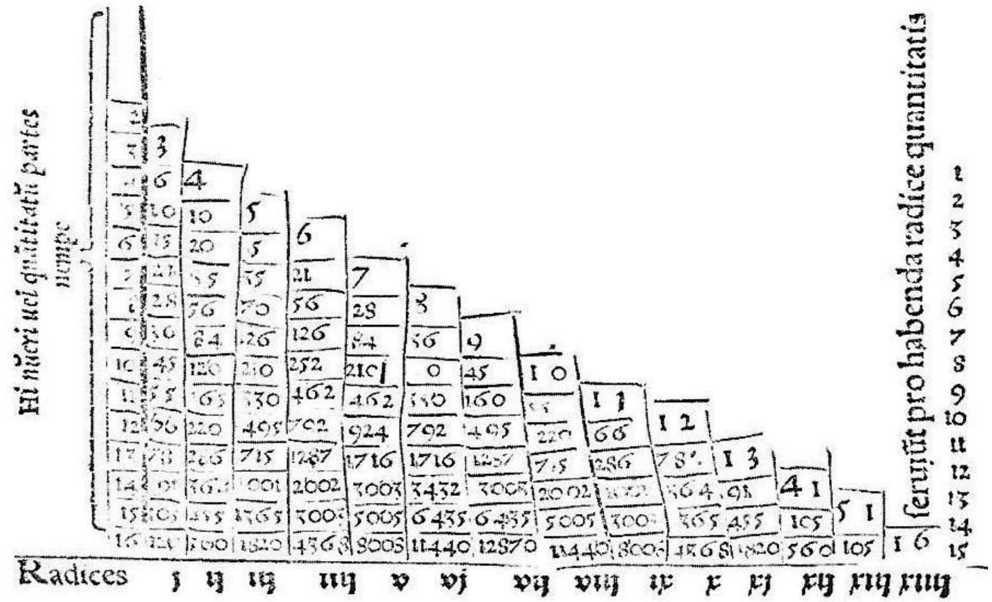


Der Ander theyl

1							
2							
3	3						
4	6						
5	10	10					
6	15	20					
7	21	35	35				
8	28	56	70				
9	36	84	126	126			
10	45	120	210	252			
11	55	156	330	462	462		
12	66	220	495	792	924		
13	78	286	715	1287	1716	1716	
14	91	364	1001	2002	3003	3432	
15	105	455	1365	3003	5005	6435	6435
16	120	560	1820	4368	8008	11440	12870

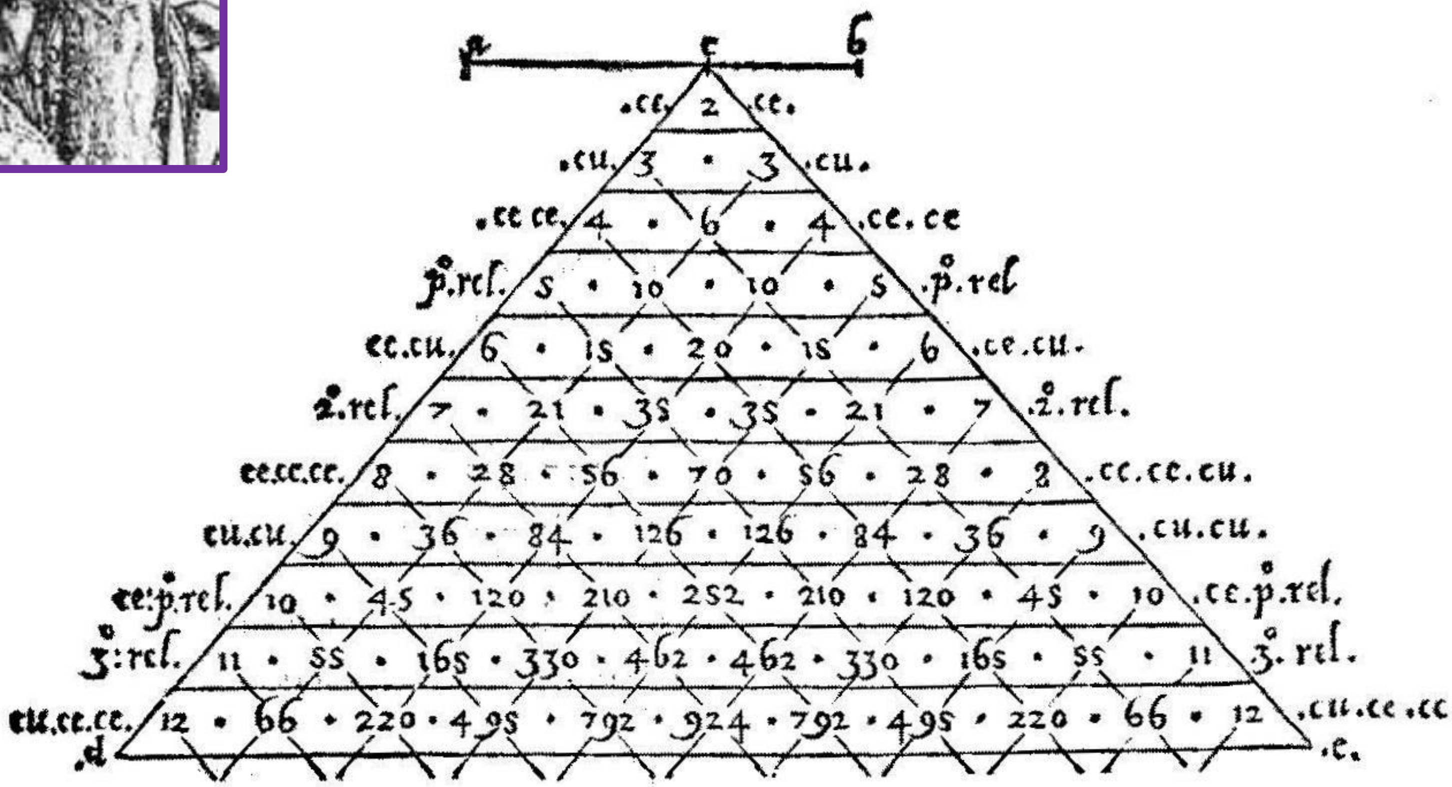
Es kan aber ein fleiſſiger Leſer/diſer tafel brauch leichtlich ſehen/aus den geſetzten ſatzungen der puncten/ Item auch wie ſich die zalen der tafel aus einander finden / wer ſich aber ſelbs nicht kan drauß verrichten/mag jm ſolliche zeygen laſſen/ wie ich denn gnugſam dauon geſchriben hab in meiner Latiniſchen Arithmetica.

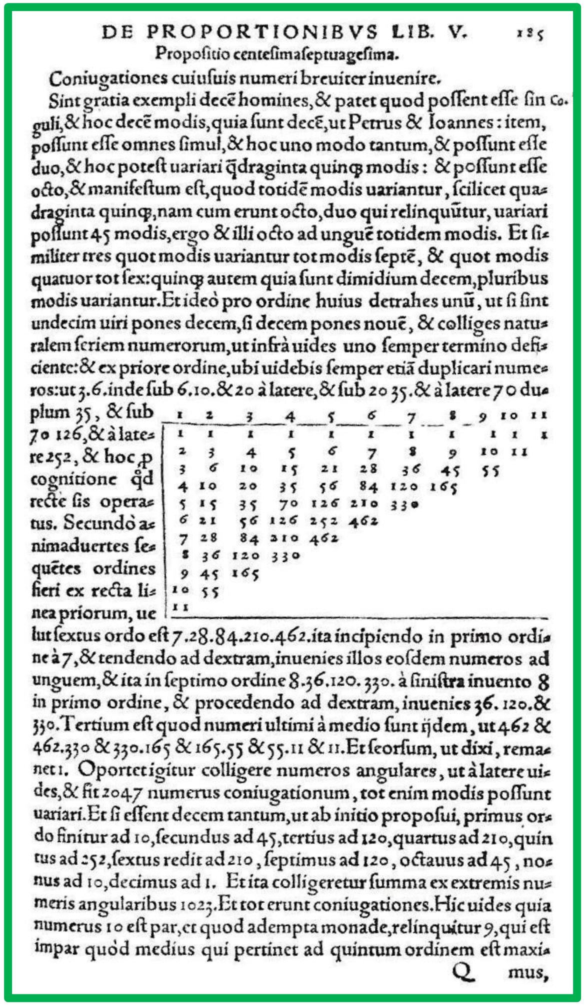
Michael Stifel
(figurate)



J. Scheubelius
(binomial)

Niccolò Tartaglia (1556)





Propositio centesimaseptuagesima.

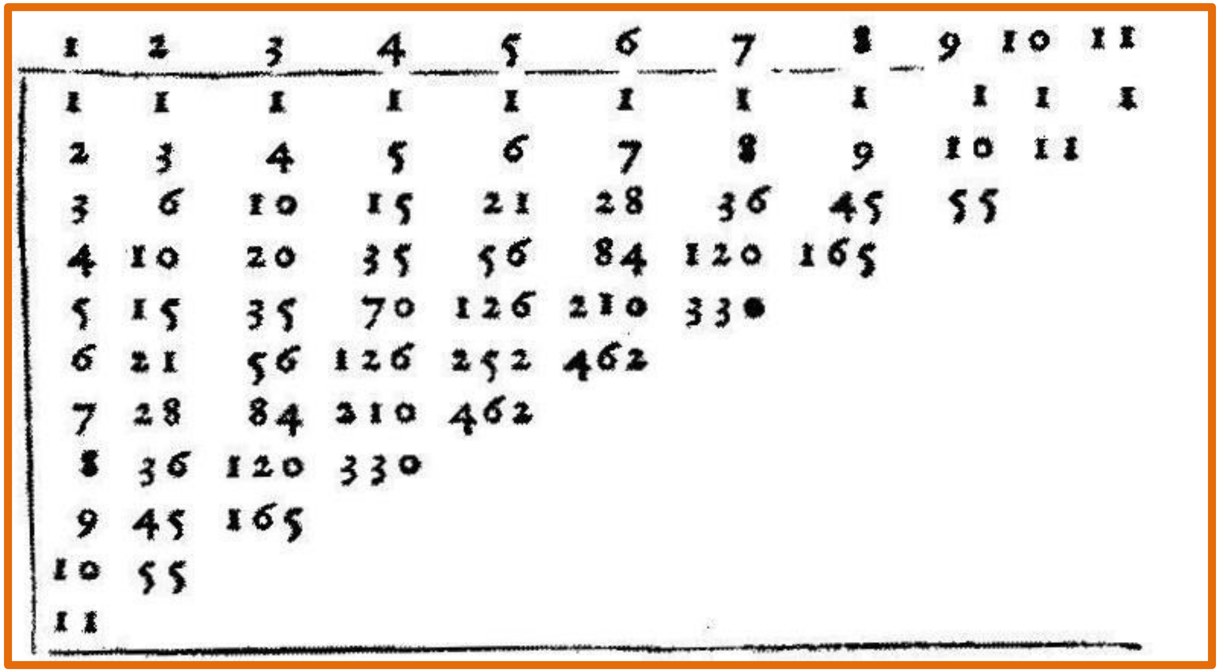
Coniugationes cuiusuis numeri breuiter inuenire.

Sint gratia exempli decē homines, & patet quod possent esse sin Co. guli, & hoc decē modis, quia sunt decē, ut Petrus & Ioannes: item, possunt esse omnes simul, & hoc uno modo tantum, & possunt esse duo, & hoc potest uariari q̄draginta quinq; modis: & possunt esse octo, & manifestum est, quod totidē modis uariantur, scilicet quadraginta quinq;, nam cum erunt octo, duo qui relinquūtur, uariari possunt 45 modis, ergo & illi octo ad unguē totidem modis. Et similiter tres quot modis uariantur tot modis septē, & quot modis quatuor tot sex: quinq; autem quia sunt dimidium decem, pluribus modis uariantur. Et ideò pro ordine huius detrahes unū, ut si sint undecim uiri pones decem, si decem pones nouē, & colliges naturalem seriem numerorum, ut infrà uides uno semper termino deficiente: & ex priore ordine, ubi uidebis semper etiā duplicari numeros: ut 3.6. inde sub 6.10. & 20 à latere, & sub 20 35. & à latere 70 duplum 35, & sub 70 126, & à latere 252, & hoc p cognitione q̊d recte sis operatus. Secundò animaduertes sequētes ordines fieri ex recta linea priorum, uelut sextus ordo est 7.28.84.210.462. ita incipiendo in primo ordine à 7, & tendendo ad dextram, inuenies illos eosdem numeros ad unguem, & ita in septimo ordine 8.36.120. 330. à sinistra inuento 8 in primo ordine, & procedendo ad dextram, inuenies 36. 120. & 330. Tertium est quod numeri ultimi à medio sunt ijdem, ut 462 & 462. 330 & 330. 165 & 165. 55 & 55. 11 & 11. Et seorsum, ut dixi, remanet 1. Oportet igitur colligere numeros angulares, ut à latere uides, & fit 2047 numerus coniugationum, tot enim modis possunt uariari. Et si essent decem tantum, ut ab initio proposui, primus ordo finitur ad 10, secundus ad 45, tertius ad 120, quartus ad 210, quintus ad 252, sextus redit ad 210, septimus ad 120, octauus ad 45, nonus ad 10, decimus ad 1. Et ita colligeretur summa ex extremis numeris angularibus 1023. Et tot erunt coniugationes. Hic uides quia numerus 10 est par, et quod adempta monade, relinquitur 9, qui est impar quòd medius qui pertinet ad quintum ordinem est maximus,

1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1
2	3	4	5	6	7	8	9	10	11	
3	6	10	15	21	28	36	45	55		
4	10	20	35	56	84	120	165			
5	15	35	70	126	210	330				
6	21	56	126	252	462					
7	28	84	210	462						
8	36	120	330							
9	45	165								
10	55									
11										

Q

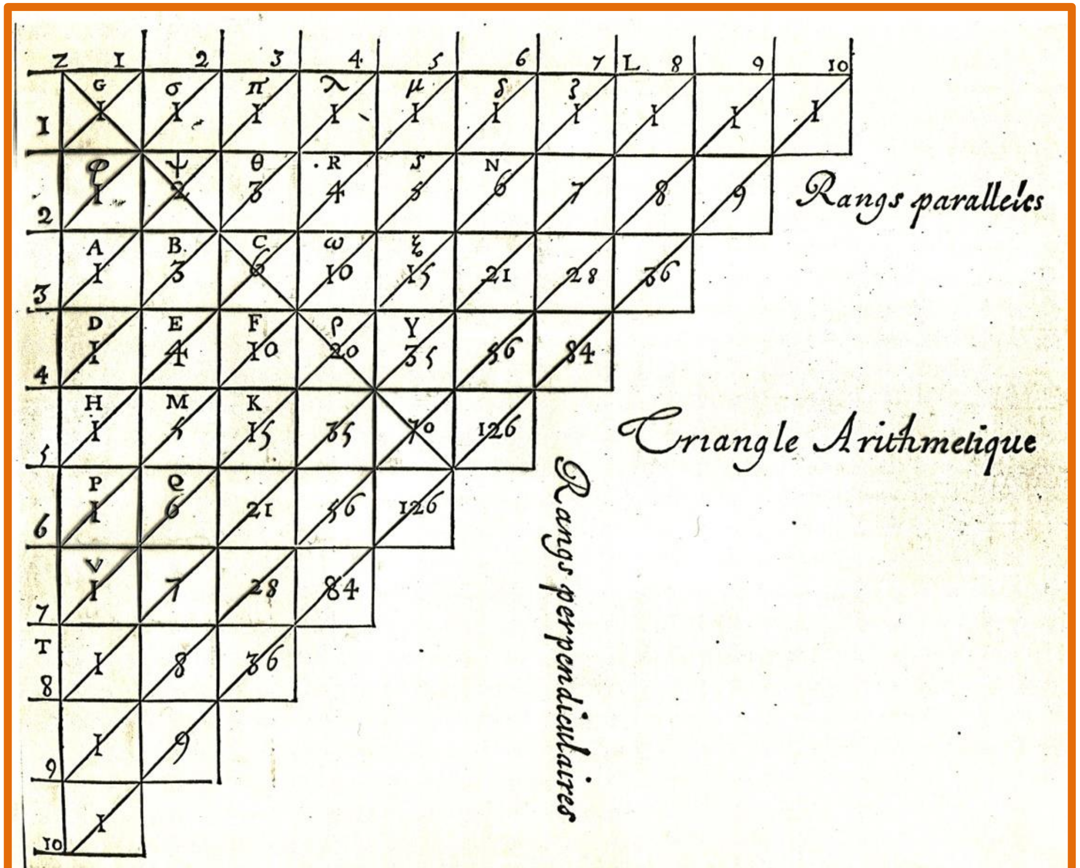
Girolamo Cardano (1570)

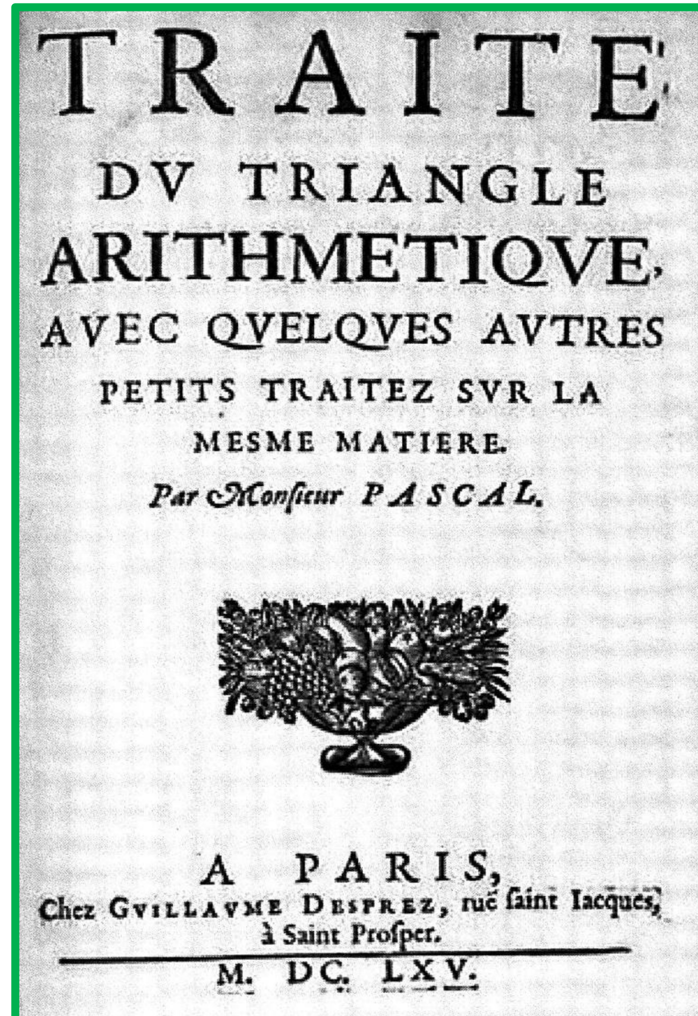


1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1
2	3	4	5	6	7	8	9	10	11	
3	6	10	15	21	28	36	45	55		
4	10	20	35	56	84	120	165			
5	15	35	70	126	210	330				
6	21	56	126	252	462					
7	28	84	210	462						
8	36	120	330							
9	45	165								
10	55									
11										

Diagonal sums
$= 2^n - 1$

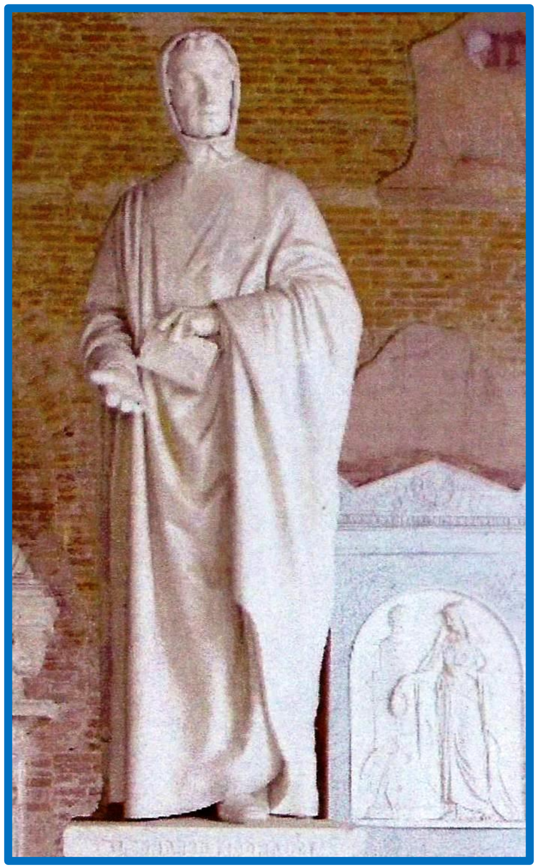
Blaise Pascal (1654/1665)



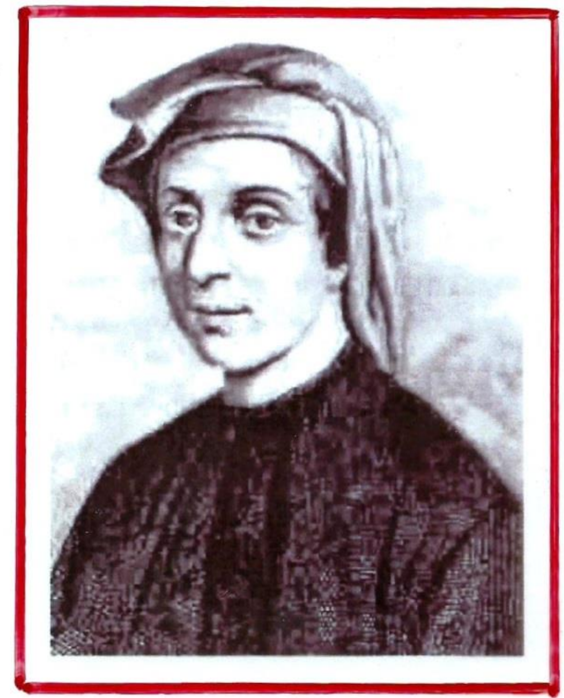

TRAITE
DV TRIANGLE
ARITHMETIQVE,
AVEC QVELQVES AVTRES
PETITS TRAITEZ SVR LA
MESME MATIERE.
Par Monsieur PASCAL.

A PARIS,
Chez GVILLAVME DESPREZ, ruë saint Iacques, à Saint Prosper.
M. DC. LXV.

Fibonacci (Leonardo of Pisa)
(c. 1170–1240)



Liber Abaci (1202)
Book of squares
Hindu-Arabic numerals
Many text problems
'Rabbits' problem
Fibonacci sequence

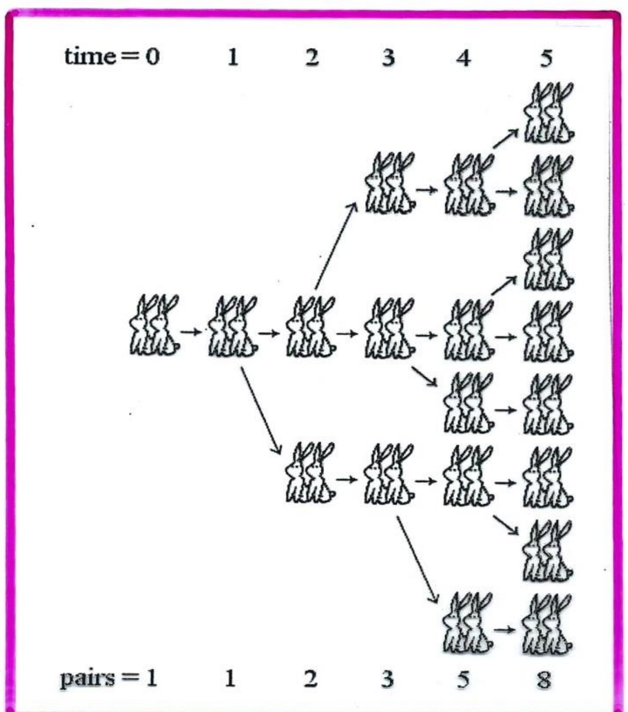


Fibonacci's 'rabbits' problem

Liber Abbaci (1202)

How many rabbits can be bred from one pair in a year?

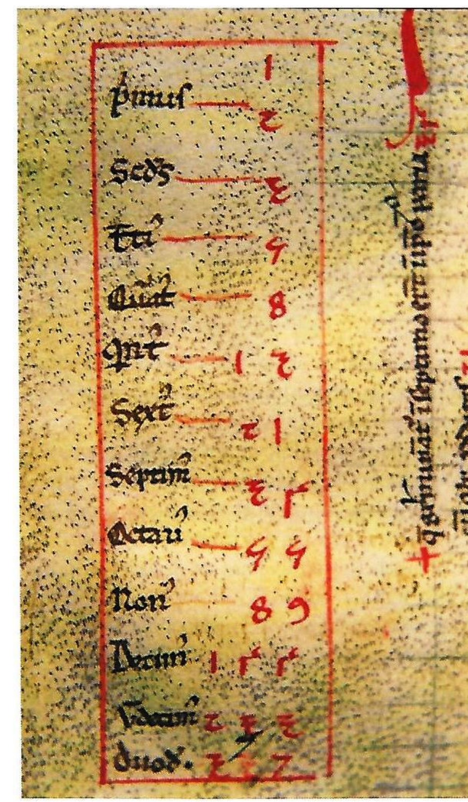
- **Each month they produce another pair**
- **From month two each new pair breeds**



Fibonacci sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, . . .

Each term is the sum of the previous two



‘Hemachandra numbers’ and pavings

A long syllable is two beats (—), and a short syllable is one beat (u).

How many rhythms are there with a given number of beats?

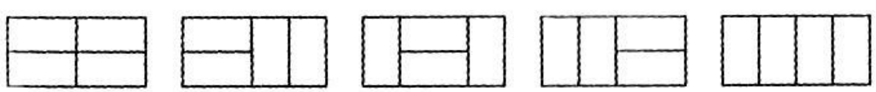
For example, there are 5 rhythms with four beats:

— — , — u u , u — u , u u — , and u u u u

In how many ways can we lay a path of length n with 2 x 1 rectangular paving stones that can be laid horizontally or vertically?

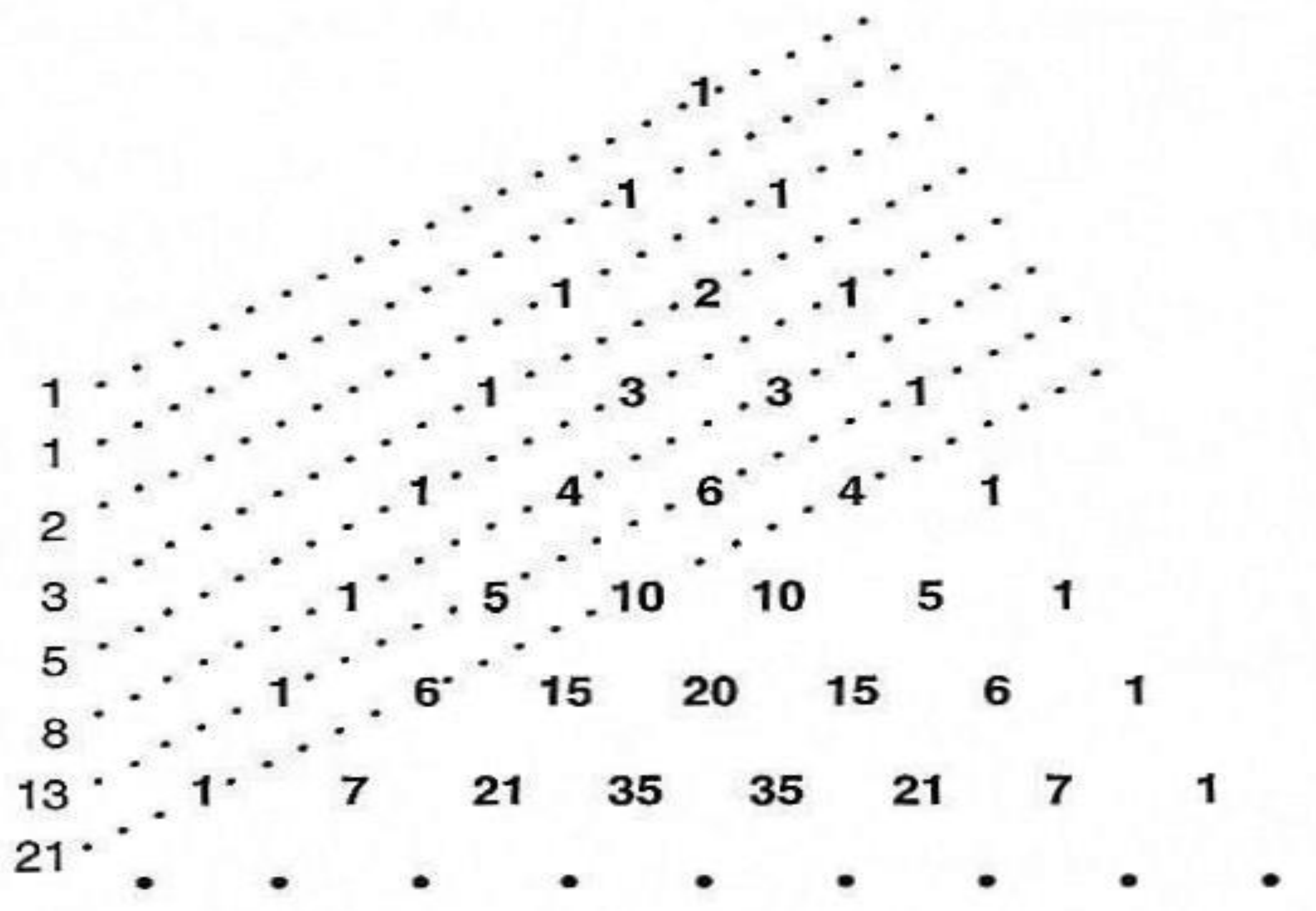
There are 5 pavings of length 4

and below are two of the 34 pavings of length 8



In the East, Fibonacci numbers are called *Hemachandra numbers*.

Pascal and the Fibonacci numbers



The influence of Ramon Llull

Aim: to unify all knowledge into a single system:
all knowledge derives from basic principles,
so we look at all combinations of these principles
to find all knowledge.

Ramon Llull (c.1235-1316): Catalan mystic and poet

Marin Mersenne (1588-1648): Minimite friar and communicator

Athanasius Kircher (1601/2-1680): Jesuit priest, and the last 'Renaissance man'

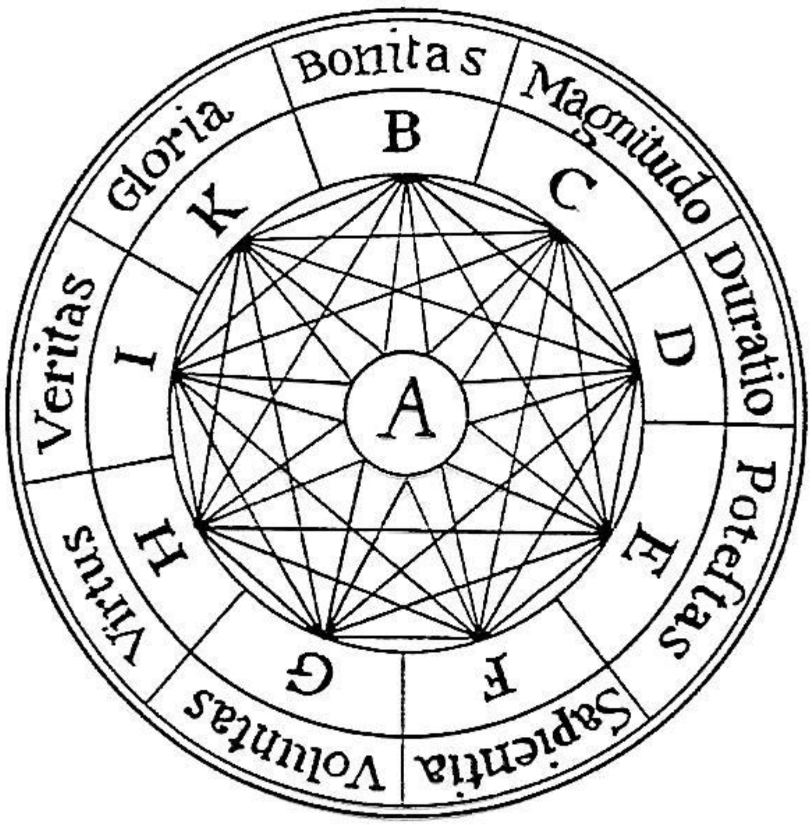
From Spain, Sebastian Izquierdo (1601-81) and Juan Caramuel de Lobkowitz (1606-82)

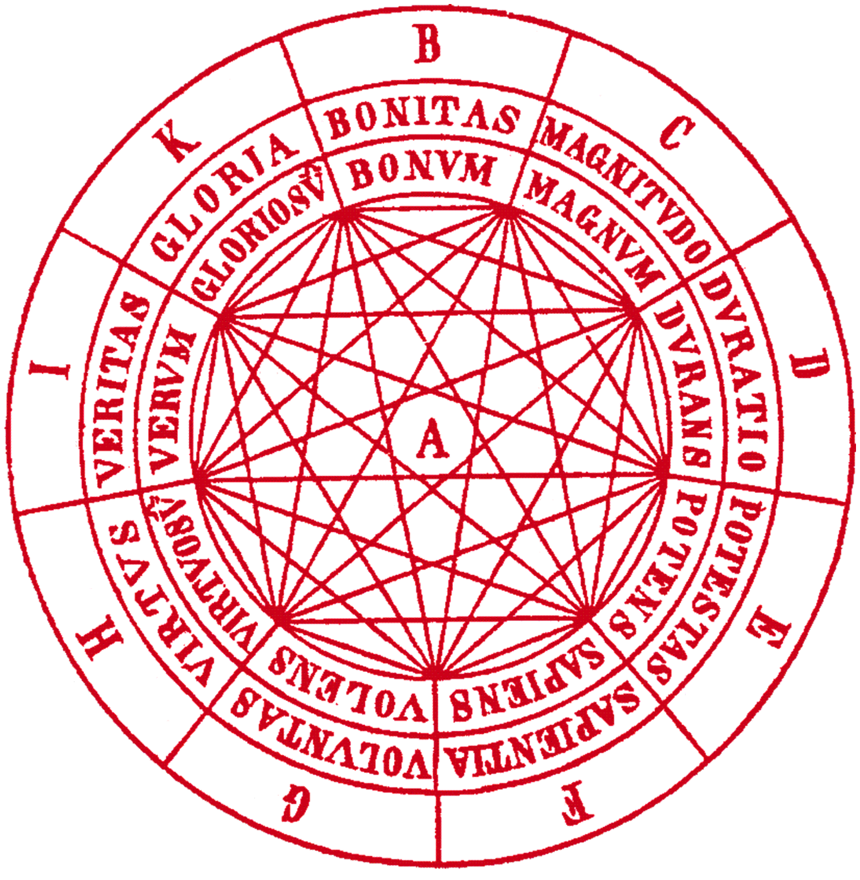
Gottfried Leibniz (1646-1716): 'Dissertatio de Arte Combinatoria'

Jakob Bernoulli (1615-1705): 'Ars Conjectandi'

Ramon Llull (c.1232–1315)

Combinations of divine attributes

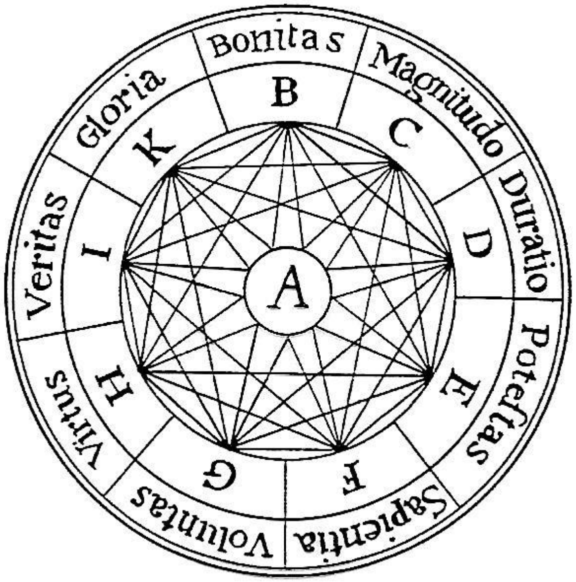


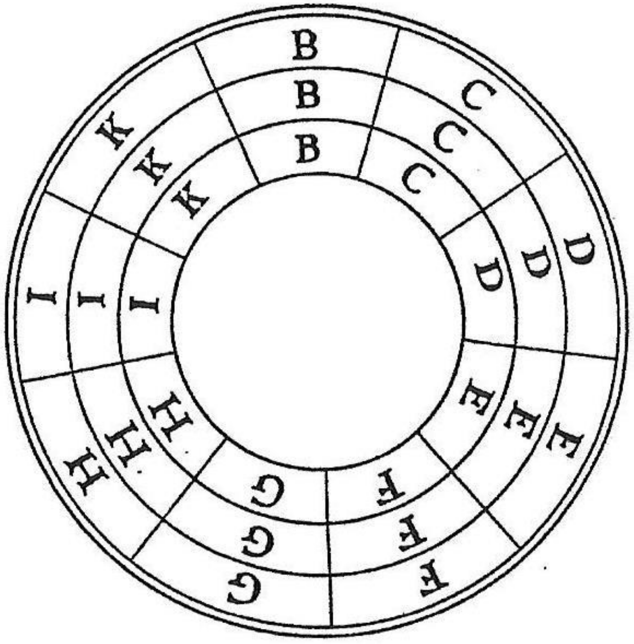


Llull's 'combinatory diagrams'

Llull's diagrams
give the relationships
among nine of the 'Dignities'
– the attributes of God, such as
B = Bonitas (goodness),
E = Potestas (power),
F = Sapientia (wisdom),
and I = Veritas (truth).

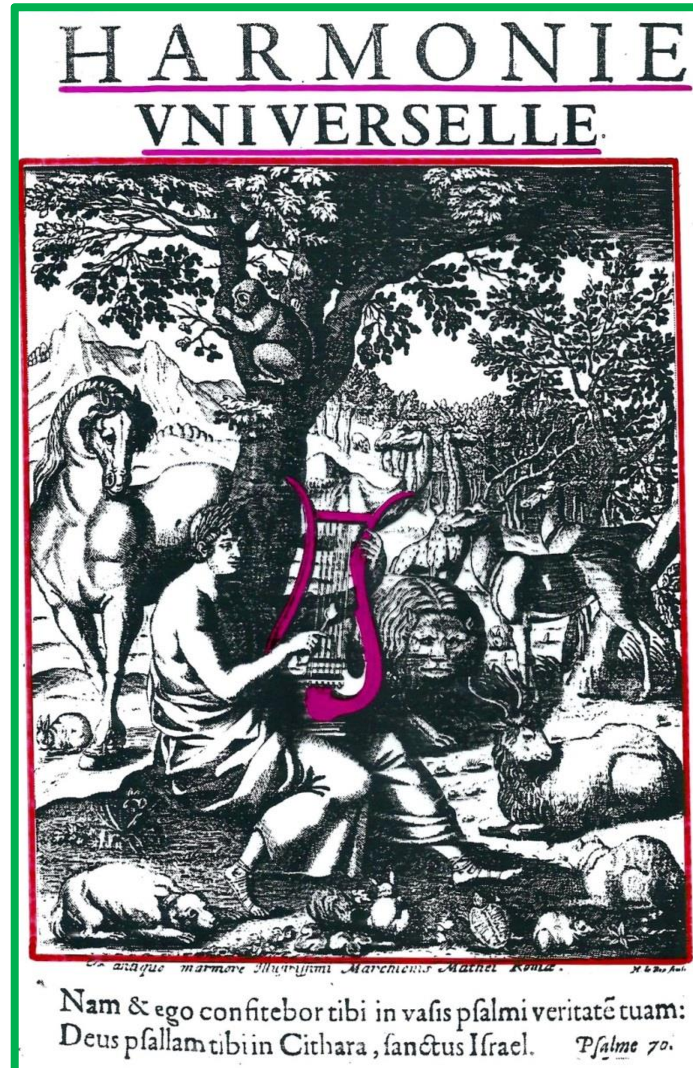
In the lower diagram, the inner wheels rotate, allowing all the combinations of three attributes.





Marin Mersenne

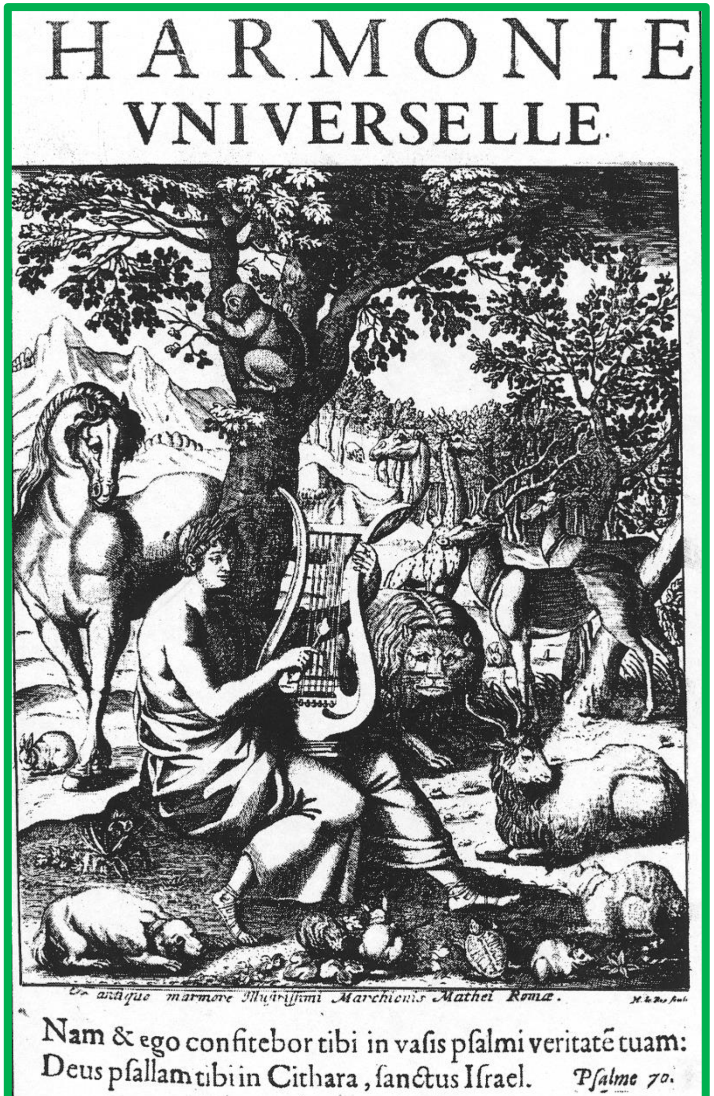
(1588-1648)



Musical composition consists of arrangements of consonances:
Harmonicorum Libri
= *Harmonie Universelle*:

- **table of values of factorials up to 64!**
- **exhibited the 720 'songs' with 6 notes**
- **extensive tables (in a musical setting) of permutations and combinations with and without repetition**
- **gave formula for n!/a!b! . . . where a + b + . . . = n**

Marin Mersenne (1636)



Number of arrangements of four notes is 4! = 24

For six notes it is 6! = 720

Mersenne's arithmetical triangle

Tabella pulcherrima & vtilißima Combinationis duodecim Cantilenarum.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.
1	1	1	1	1	1	1	1	1	1	1	1
2	3	4	5	6	7	8	9	10	11	12	13
3	6	10	15	21	28	36	45	55	66	78	91
4	10	20	35	56	84	120	165	220	286	364	455
5	15	35	70	126	210	330	495	715	1001	1365	1820
6	21	56	126	252	462	792	1287	2002	3003	4368	6188
7	28	84	210	462	924	1716	3003	5005	8008	12376	18564
8	36	120	330	792	1716	3432	6435	11440	19448	31824	50388
9	45	165	495	1287	3003	6435	12870	24310	43758	75582	125970
10	55	220	715	2002	5005	11440	24310	48620	92378	167960	293930
11	66	286	1001	3003	8008	19448	43758	92378	184756	352716	646646
12	78	364	1365	4368	12376	31824	75582	167960	352716	705432	1352078
13	91	455	1820	6188	18564	50388	125970	293930	646646	1352078	2704156
14	105	560	2380	8568	27132	77520	203490	497420	1144066	2496144	5200300
15	120	680	3060	11628	38760	116280	319770	817190	1961256	4457400	9657700
16	136	816	3876	15504	54264	170544	490314	1307504	3268760	7726160	17383860
17	153	969	4845	20349	74613	245157	735471	2042975	5311735	13037895	30421755
18	171	1140	5985	26334	100947	346104	1081575	3124550	8436285	21474180	51895935
19	190	1330	7315	33649	134596	480700	1562275	4686825	13123110	34597290	86493225
20	210	1540	8855	42504	177100	657800	2220075	6906900	20030010	54627300	141120525
21	231	1771	10626	53130	230230	888030	3108105	10015005	30045015	84672315	225792840
22	253	2024	12650	65780	296010	1184040	4292145	14307150	44352165	129024480	354817320
23	276	2300	14950	80730	376740	1560780	5852925	20160075	64512290	193536720	548354040
24	300	2600	17550	98280	475020	2035800	7888725	28048800	92561040	286097760	834451800
25	325	2925	20475	8755	593775	2629575	10518300	38567100	131128140	417225900	1251677700

Number of combinations = C(36, 12) = 1,251,677,700

Athanasius Kircher (1601/2-1680)

Wrote on
Noah's ark,
China,
Tower of Babel
Rome
Acoustics
The plague
Magnetism,
Optics, . . .

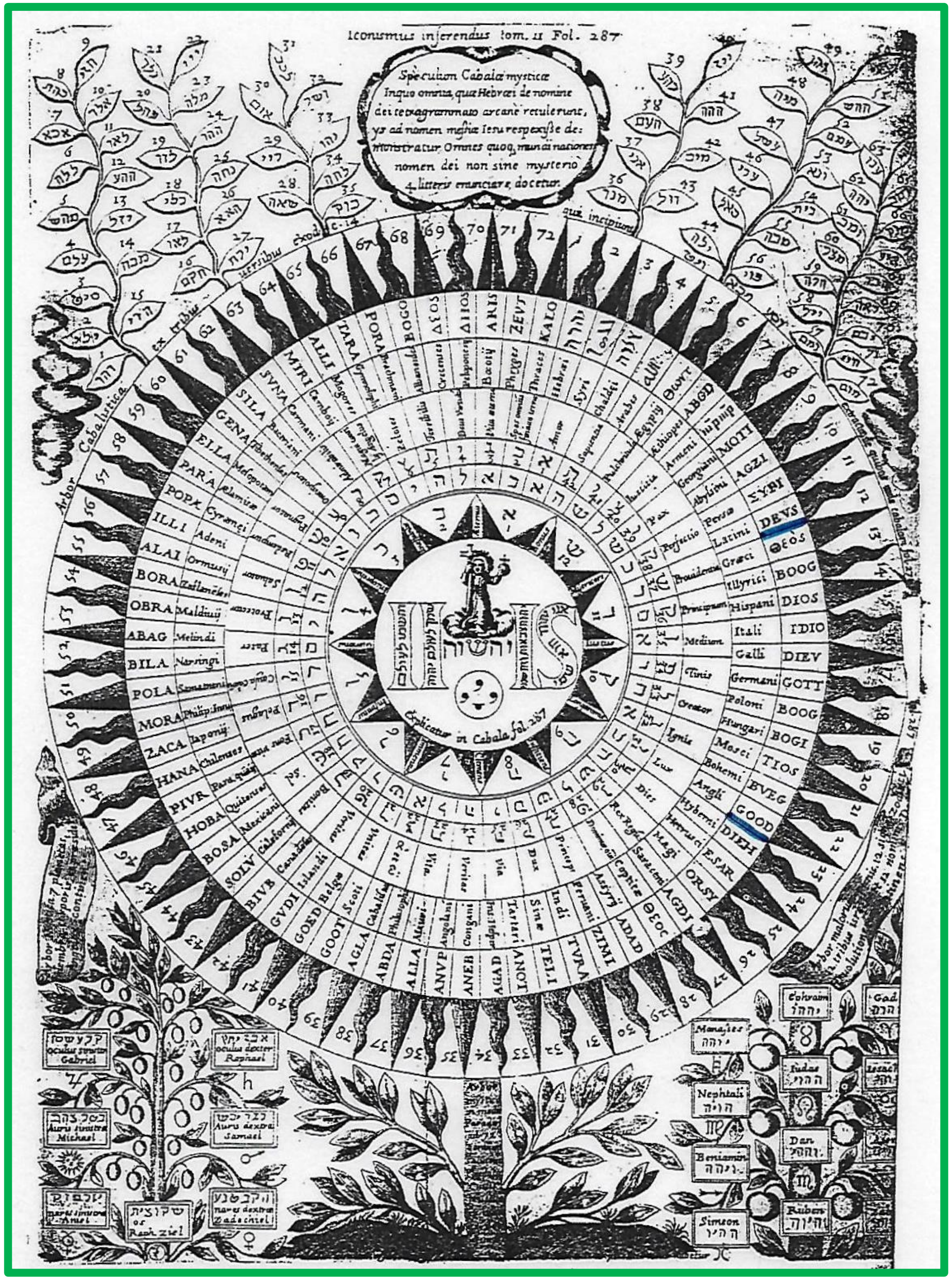
The last of the polymaths:

- great musical encyclopedists
- an early writer on germs
- translator of Egyptian hieroglyphs
- 'father of geology'
- designer of magic lanterns
- invented a system of logic
- designed magnetic toys for noblemen
- founded one of the earliest museums

. . . and much more

72 names of God

(God's name in 72 languages, all spelt with four letters)



The origins of writing

Mystic characters handed down by the angels, tracing writing and language back to divine revelation

Athanasius Kircher's *Ars Magna Sciendi* (1669)

Kircher's *Ars Magna Sciendi sive Combinatoria* (1669)

In XII Libros Digesta qua Nova & Universali Methodo per artificio sum Combinatorum contextum de omni re proposita plurimis & prope infinitis rationibus disputandi,

. . .

III: Methodus Lulliana

IV: Ars Combinatoria [pp. 153-201]

Quo Veluti proprio loco Artis Combinatoriae modus

Post 1 De Combinatione terminorum tam simplicium, quam compositorium . . .

Word arrangements

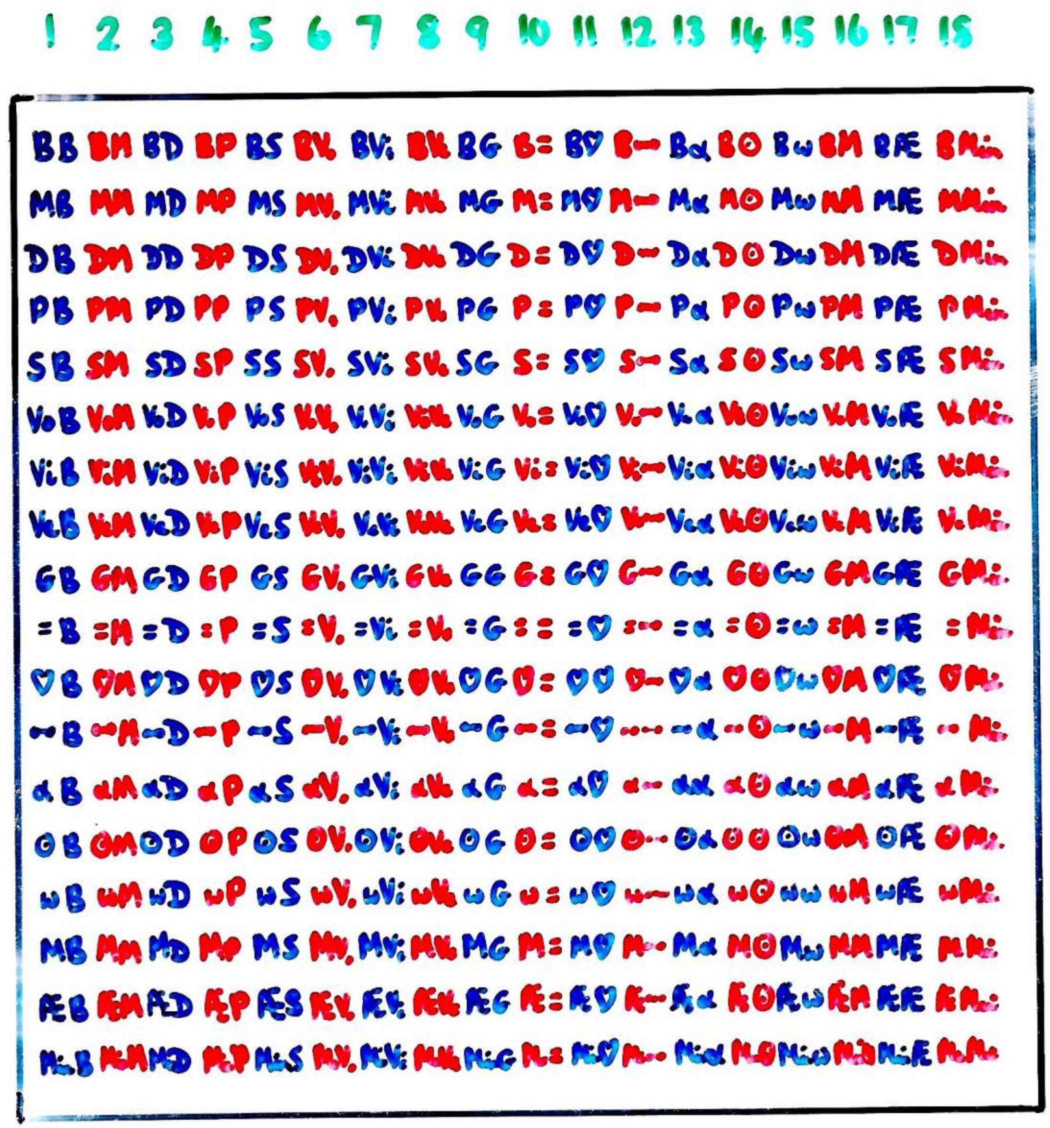
O	R	A
O	A	R
R	A	O
R	O	A
A	R	O
A	O	R

Tabula Generalis: Factorials from 1! to 50!

AMEN	MAEN	EAMN	NAME
AMNE	MANE	EANM	NAEM
AEMN	MEAN	EMAN	NMAE
AENM	MENA	EMNA	NMEA
ANEM	MNAE	ENAM	NEMA
ANME	MNEA	ENMA	NEAM

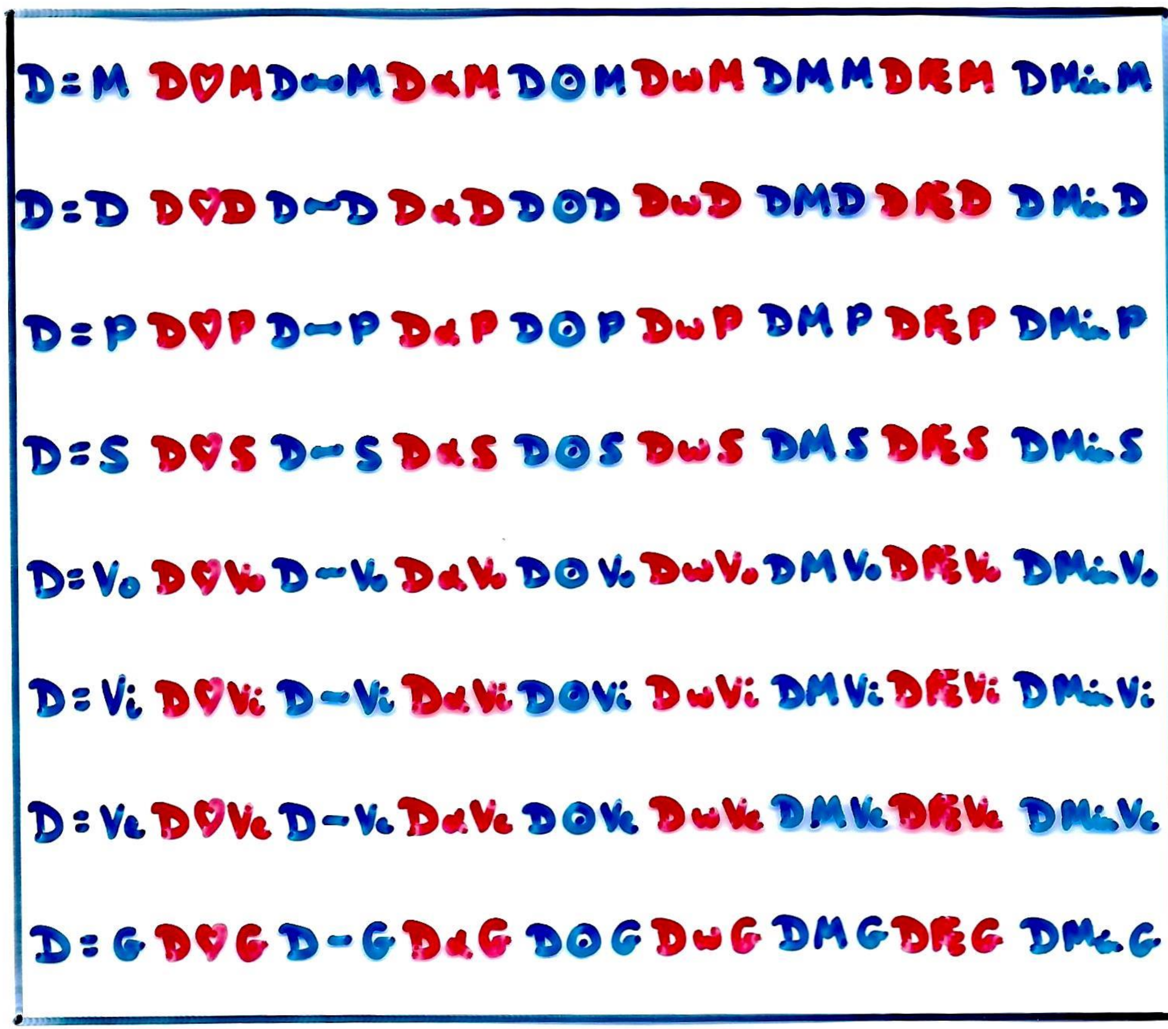
PATER (120 arrangements)

Kircher’s 2-combinations



Duratio cum principiis respectivis

(combinations involving Duratio)



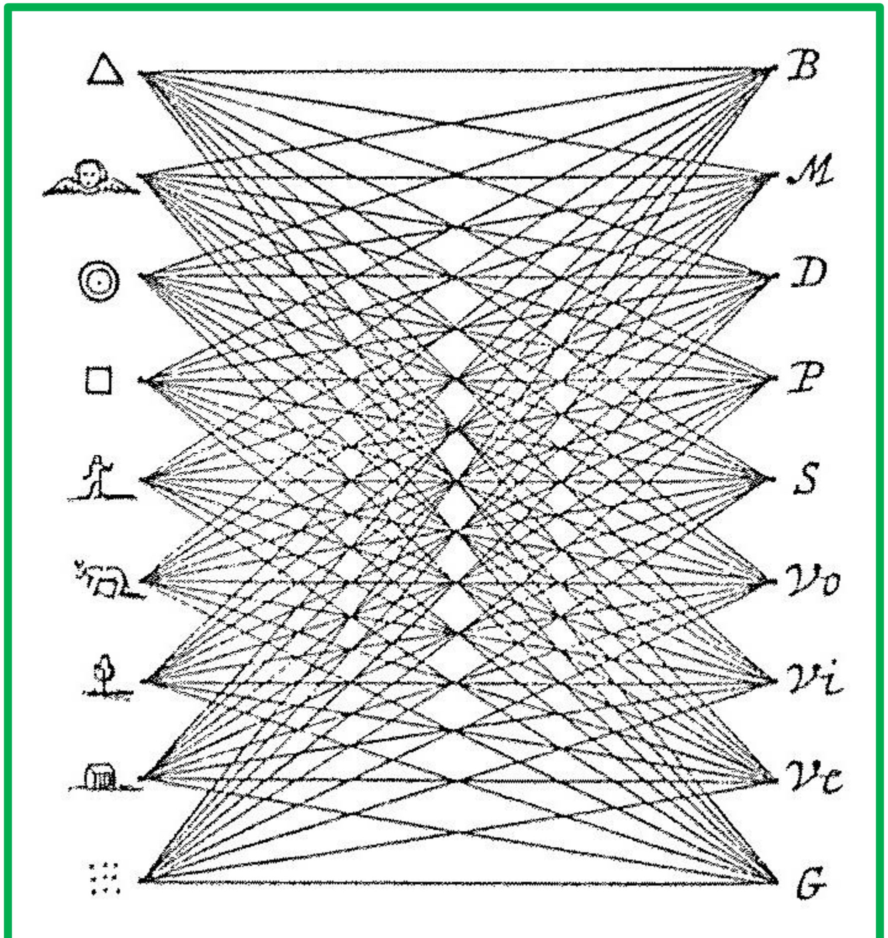
Ars Combinatoria

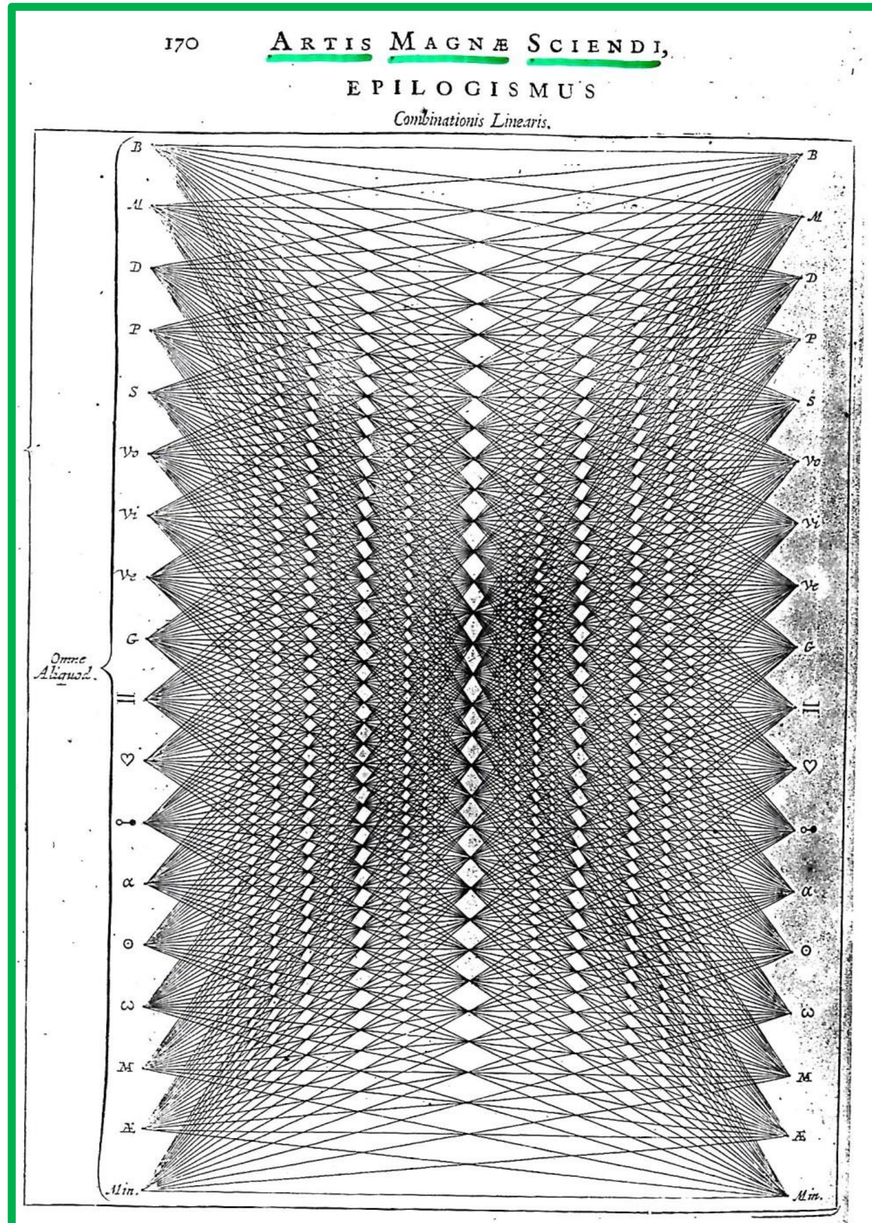
Another Kircher table

Combinatio rerum in qua 4 similes

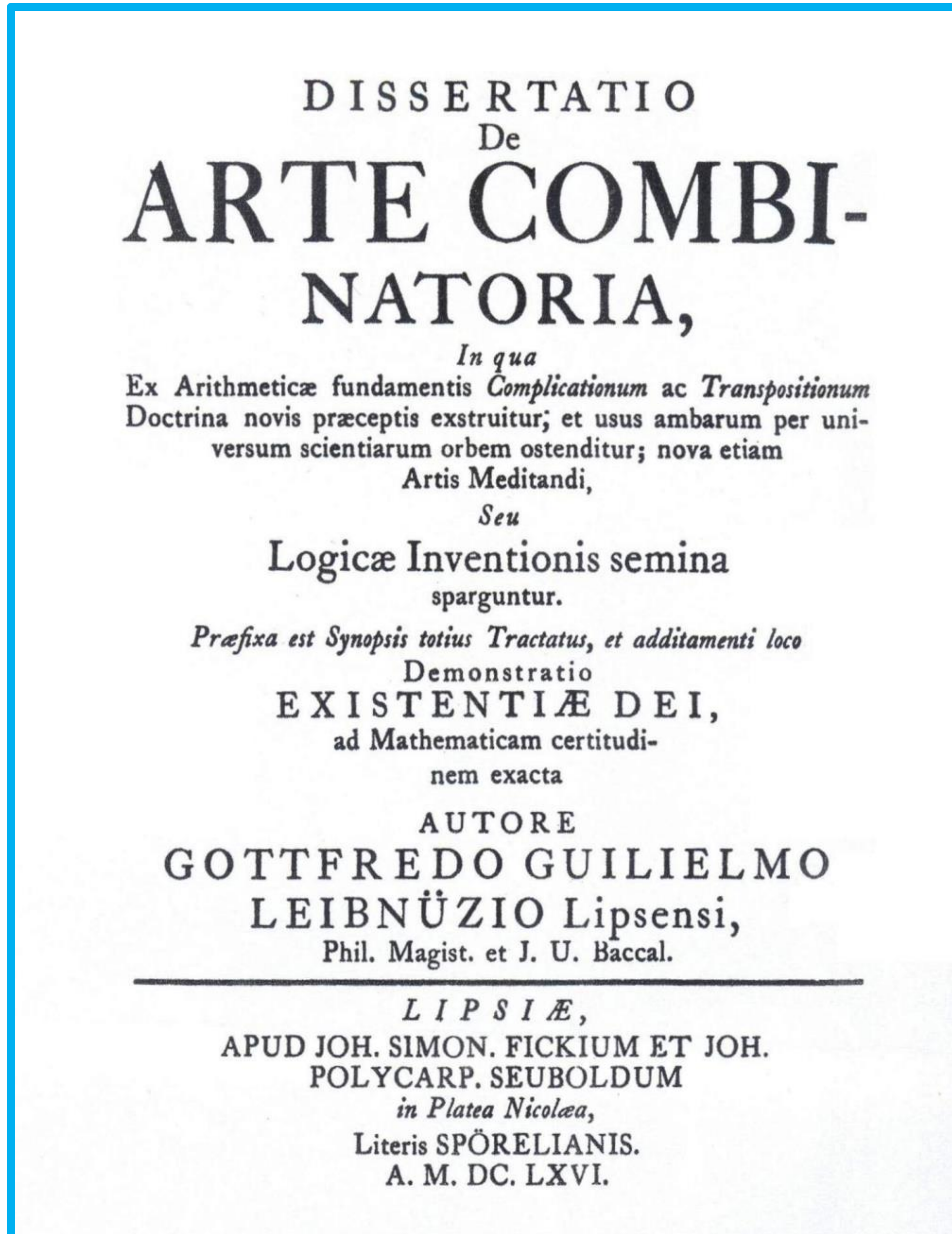
Series rerum diversarium	I	II	III	IV	V	VI	VII	VIII	IX
I	0								
II	2	0							
III	6	3	0						
IV	24	12	4	0					
V	120	60	20	5	0				
VI	720	360	120	30	6	0			
VII	5040	2520	840	210	42	7	0		
VIII	40320	20160	6720	1680	336	56	8	0	
IX	362880	181440	60480	15120	3024	504	72	9	0
X	3628800	1814400	604800	151200	30240	5040	720	90	10

Kircher's bipartite graphs





Leibniz's *De Arte Combinatoria* (1666)



DISSERTATIO
De
ARTE COMBI-
NATORIA,
In qua
Ex Arithmeticæ fundamentis *Complicationum* ac *Transpositionum* Doctrina novis præceptis exstruitur; et usus ambarum per universum scientiarum orbem ostenditur; nova etiam Artis Meditandi,
Seu
Logicæ Inventionis semina sparguntur.
Præfixa est Synopsis totius Tractatus, et additamenti loco
Demonstratio
EXISTENTIÆ DEI,
ad Mathematicam certitudinem exacta

AUTORE
GOTTFREDO GUILIELMO
LEIBNÜZIO Lipsensi,
Phil. Magist. et J. U. Baccal.

LIPSIÆ,
APUD JOH. SIMON. FICKIUM ET JOH.
POLYCARP. SEUBOLDUM
in Platea Nicolæa,
Literis SPÖRELIANIS.
A. M. DC. LXVI.

Leibniz’s *De Arte Combinatoria*

Written under the influence of Llullism

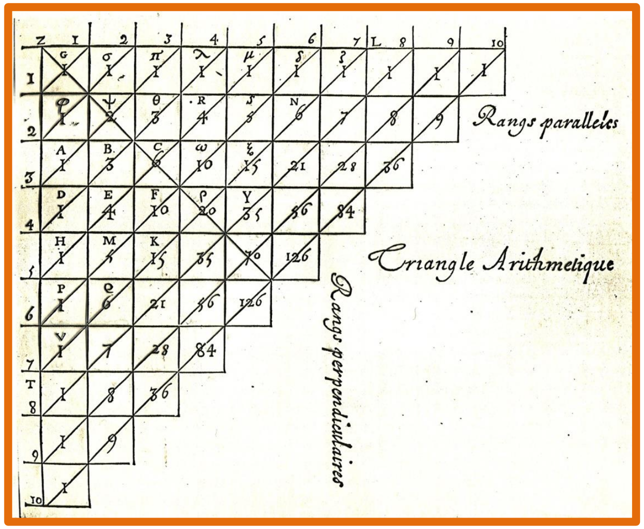
– wide view of ‘Ars combinatoria’

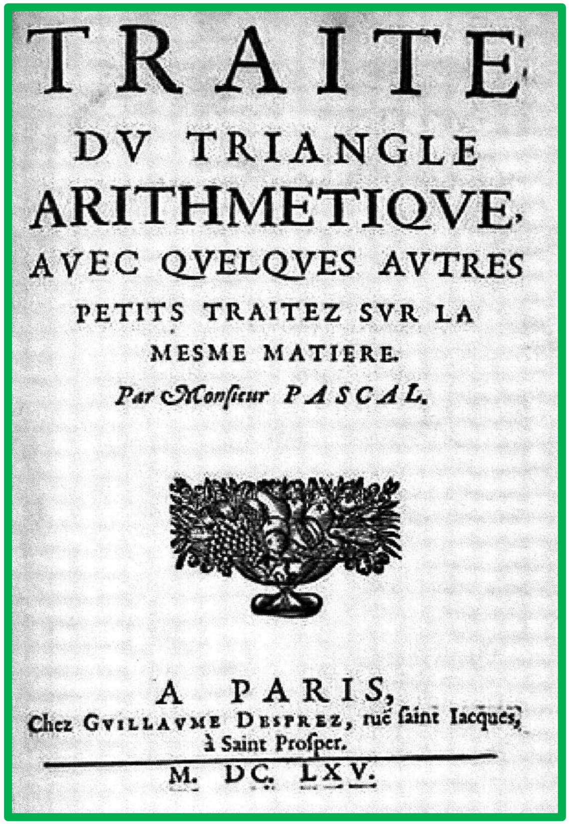
– many problems on permutations and combinations

– direct combinatorial proof of ‘Pascal’s triangle rule’

n prime → n divides C(n, r), for r = 1, 2, . . . , n – 1.

Blaise Pascal
(1654/1665)





TRAITE
DV TRIANGLE
ARITHMETIQVE,
AVEC QVELQVES AVTRES
PETITS TRAITEZ SVR LA
MESME MATIERE.
Par Monsieur PASCAL.

A PARIS,
Chez GVILLAVME DESPREZ, ruë saint Iacques,
à Saint Prosper.
M. DC. LXV.

Part I: A treatise on the arithmetical triangle

Part II: Uses of the arithmetical triangle:

- figurate numbers
- theory of combinations
- binomial expressions
- games of chance

Triangle ascribed to Pascal by de Montmort (1708) and De Moivre (1730)

Newton’s binomial theorem 1

...	-4	-3	-2	-1	0	1	2	3	4	5	6	...
...	1	1	1	1	1	1	1	1	1	1	1	...
...	-4	-3	-2	-1	0	1	2	3	4	5	6	...
...	10	6	3	1	0	0	1	3	6	10	15	...
...	-20	-10	-4	-1	0	0	0	1	4	10	20	...
...	35	15	5	1	0	0	0	0	1	5	15	...

$$(1+x)^{-4} = 1 - 4x + 10x^2 - 20x^3 + \cdots$$

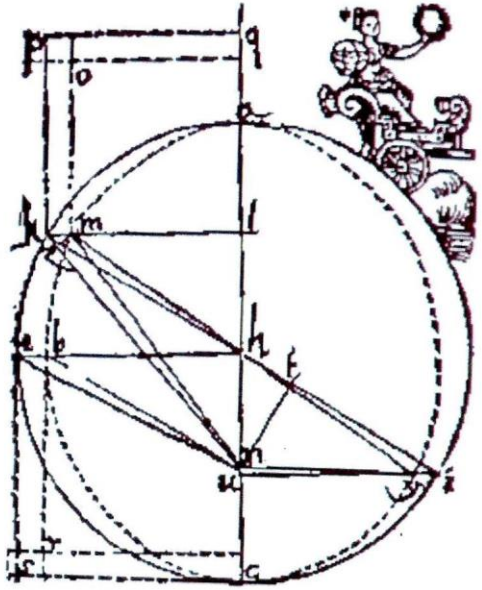
Newton's binomial theorem 2

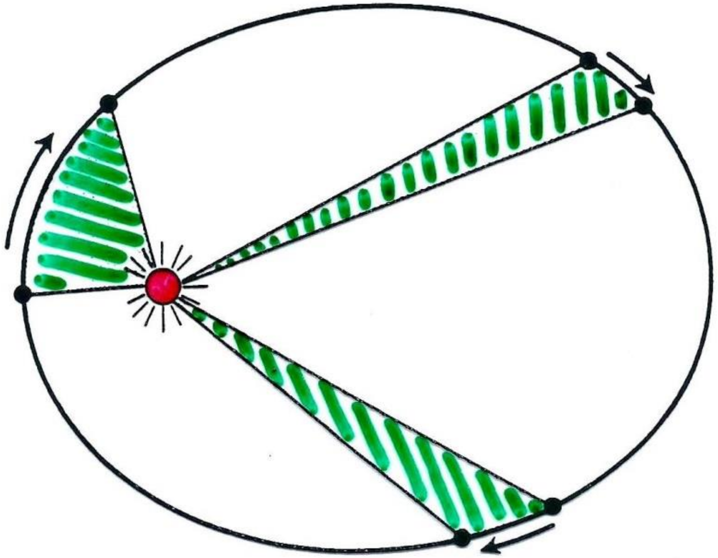
...	−1	−1/2	0	1/2	1	3/2	2	5/2	3	...
...	1	1	1	1	1	1	1	1	1	...
...	−1	−1/2	0	1/2	1	3/2	2	5/2	3	...
...	1	3/8	0	−1/8	0	3/8	1	15/8	3	...
...	−1	−5/16	0	1/16	0	−1/16	0	5/16	1	...
...	1	35/128	0	−5/128	0	3/128	0	−5/128	0	...

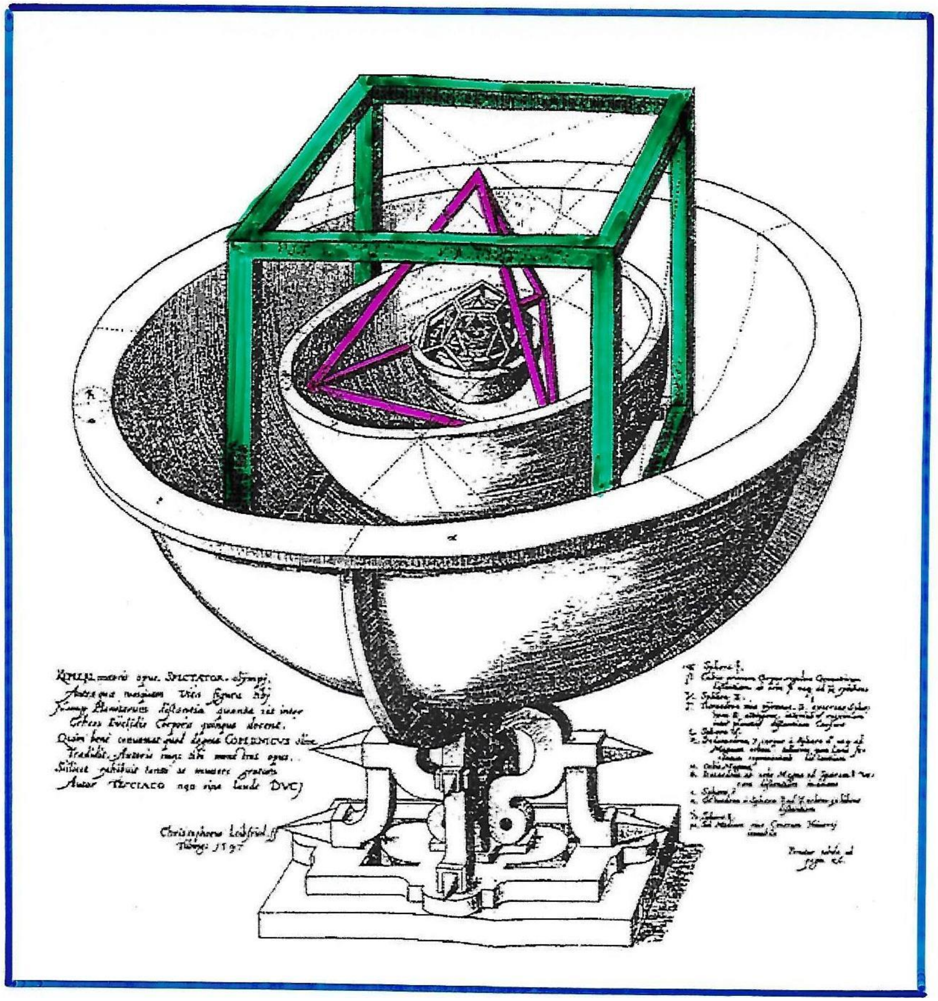
↑

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots$$

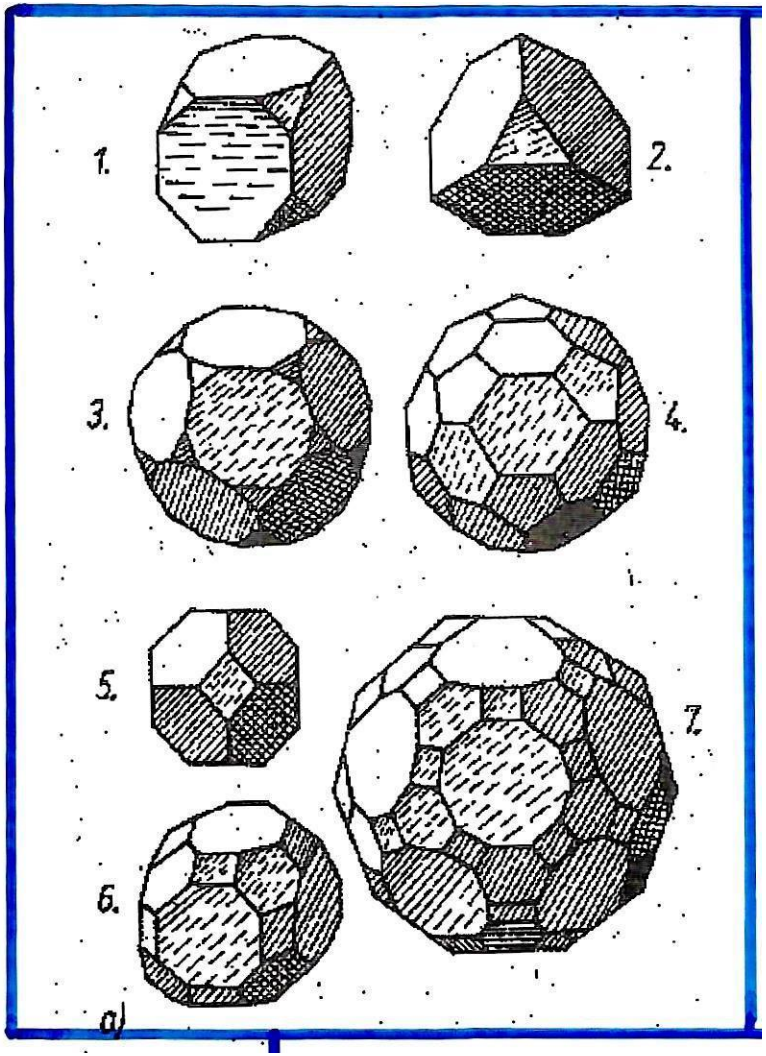
Astronomy: Kepler's polyhedral model

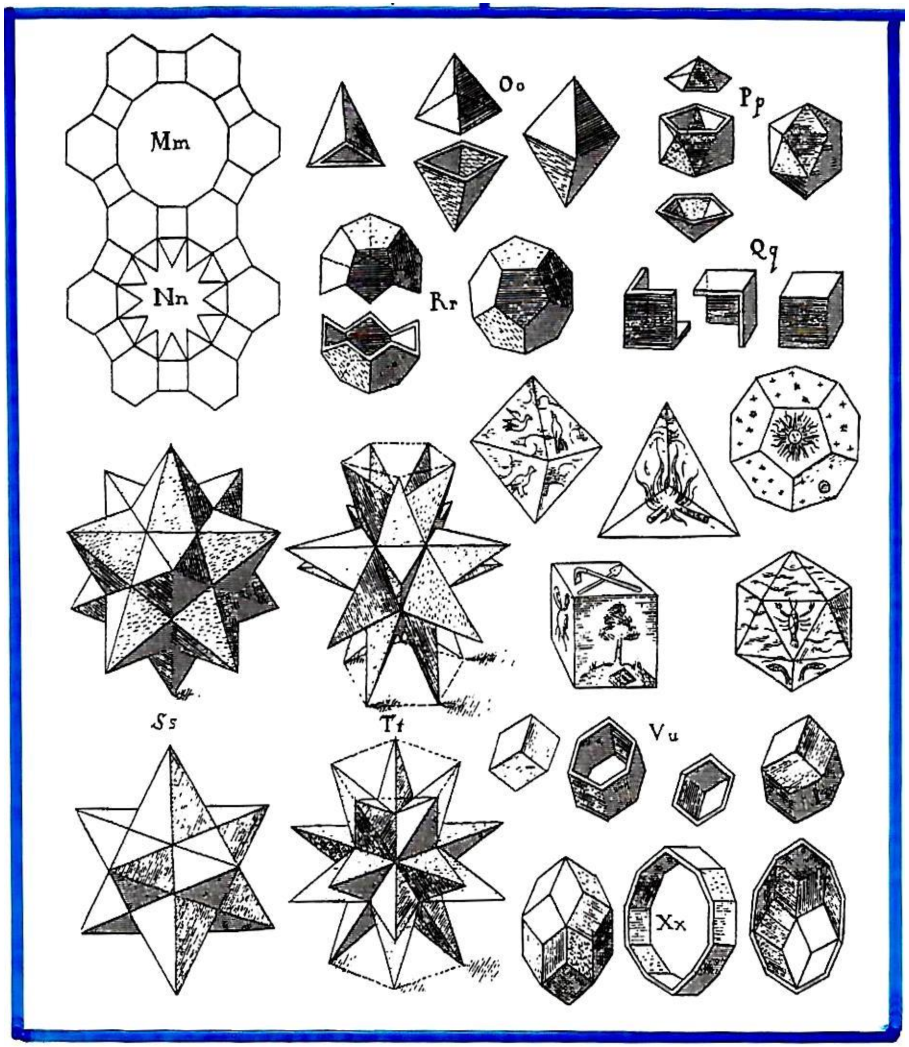




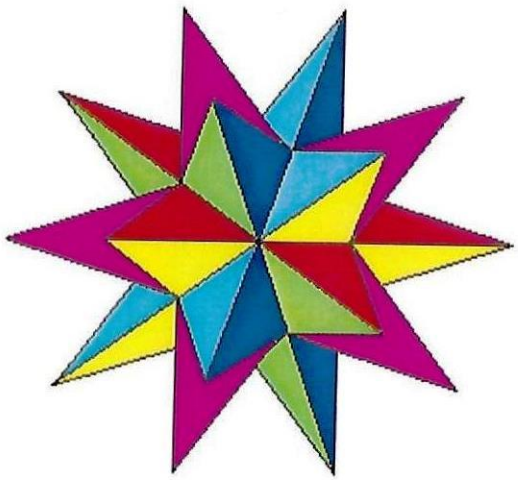


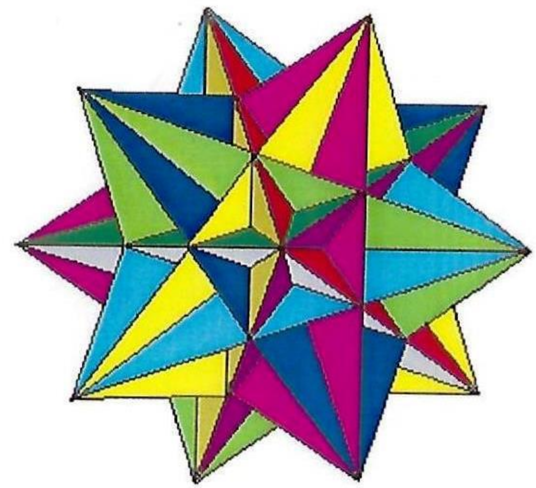
Kepler’s drawings of polyhedra

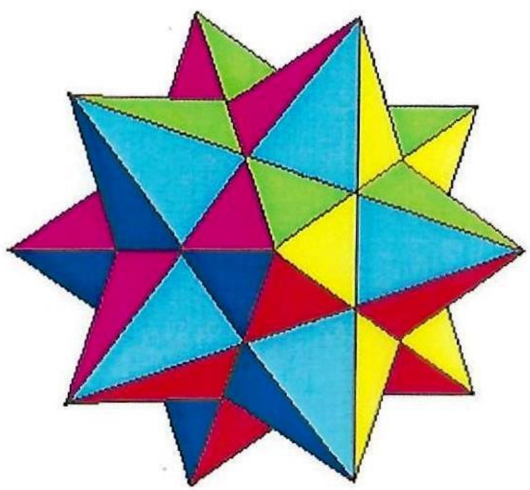


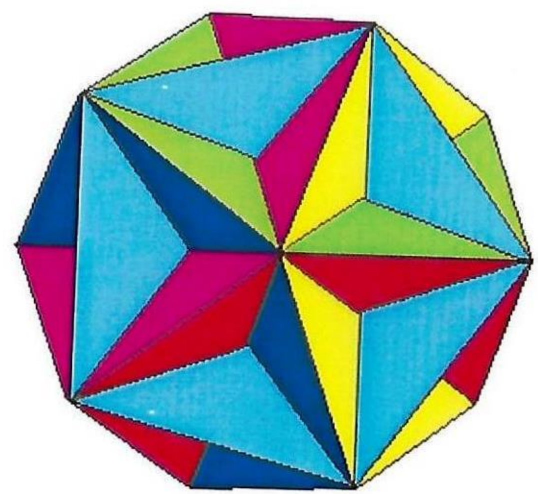


Non-convex polyhedra



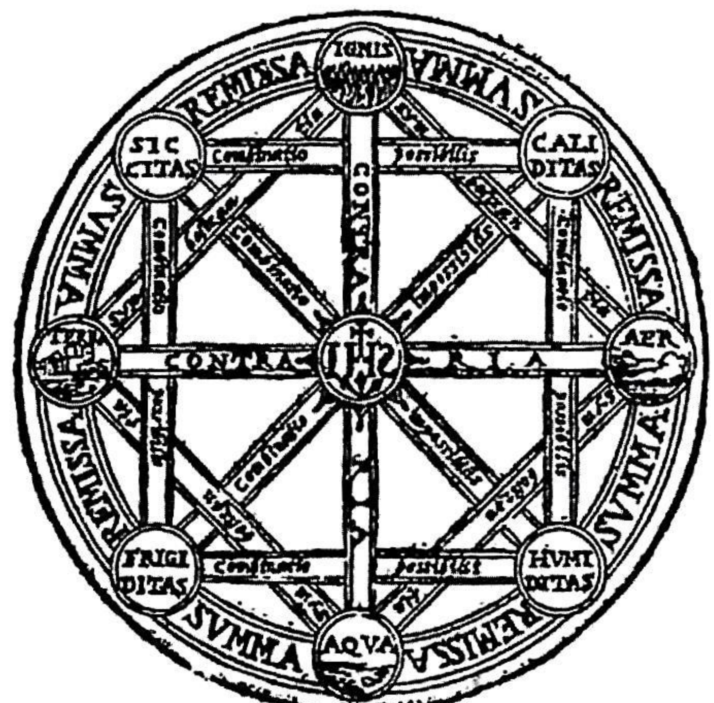






Christoph Clavius (1538-1612)

In one of his books Clavius gave ‘a most-wonderful digression on the combinations of things’, which influenced many later writers.



Permutable poetry (1617)

Tot tibi sunt dotes, Virgo, quot sidera caelo.

(Thou hast as many virtues, O Virgin, as there are stars in heaven.)

Tot dotes tibi, quot caelo sunt sidera, Virgo.
Dotes tot, caelo sunt sidera quot, tibi Virgo.
Dotes, caelo sunt quot sidera, Virgo tibi tot.
Sidera quot caelo, tot sunt Virgo tibi dotes.
Quot caelo sunt sidera, tot Virgo tibi dotes.
Sunt dotes Virgo, quot sidera, tot tibi caelo.
Sunt caelo tot Virgo tibi, quot sidera, dotes.

Dactyl – spondee – spondee – spondee – dactyl – spondee

(Dum-diddy dum-dum dum-dum dum-dum dum-diddy dum-dum)

Frans van Schooten (1657)

All the combinations of four letters *a, b, c, d*:

a

b. ab

c. ac. bc. abc

d. ad. bd. abd. cd. acd. bcd. abcd

All the divisors of 210:

2

3 6

5 10 15 30

7 14 21 42 35 70 105 210

Games of chance

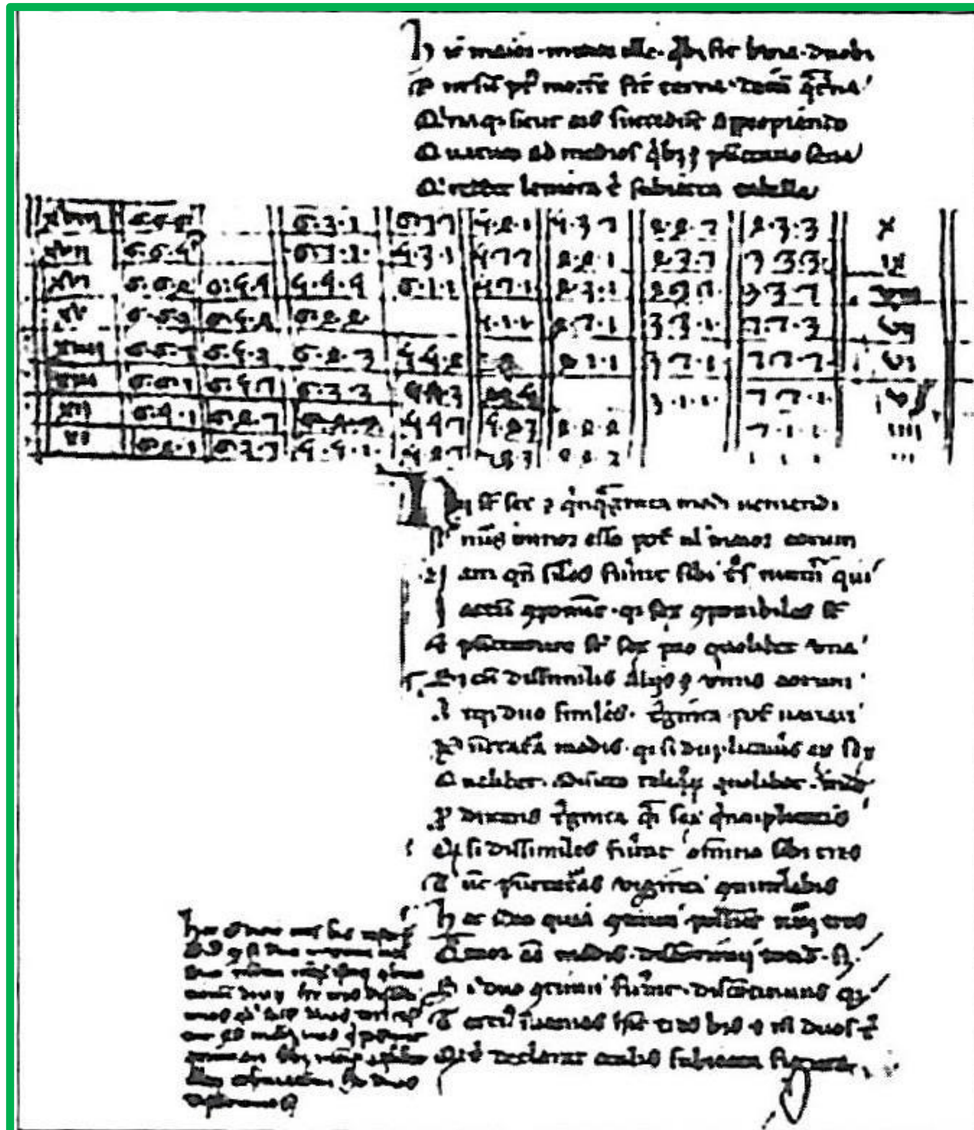
R. de Fournival?

De Vetula

(On a Little Old Woman)

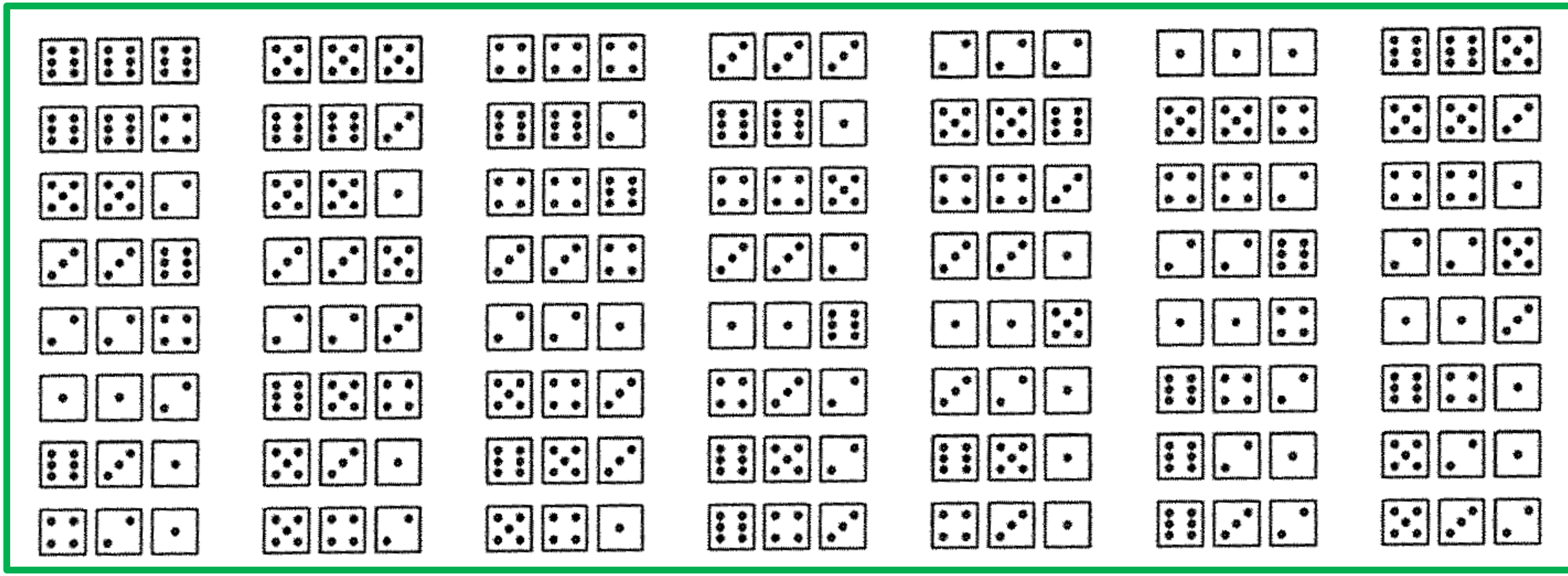
(c. 1240)

On dice games:
***De Vetula* enumerates the 56 different throws of three dice, etc.**



Combinatorial dice

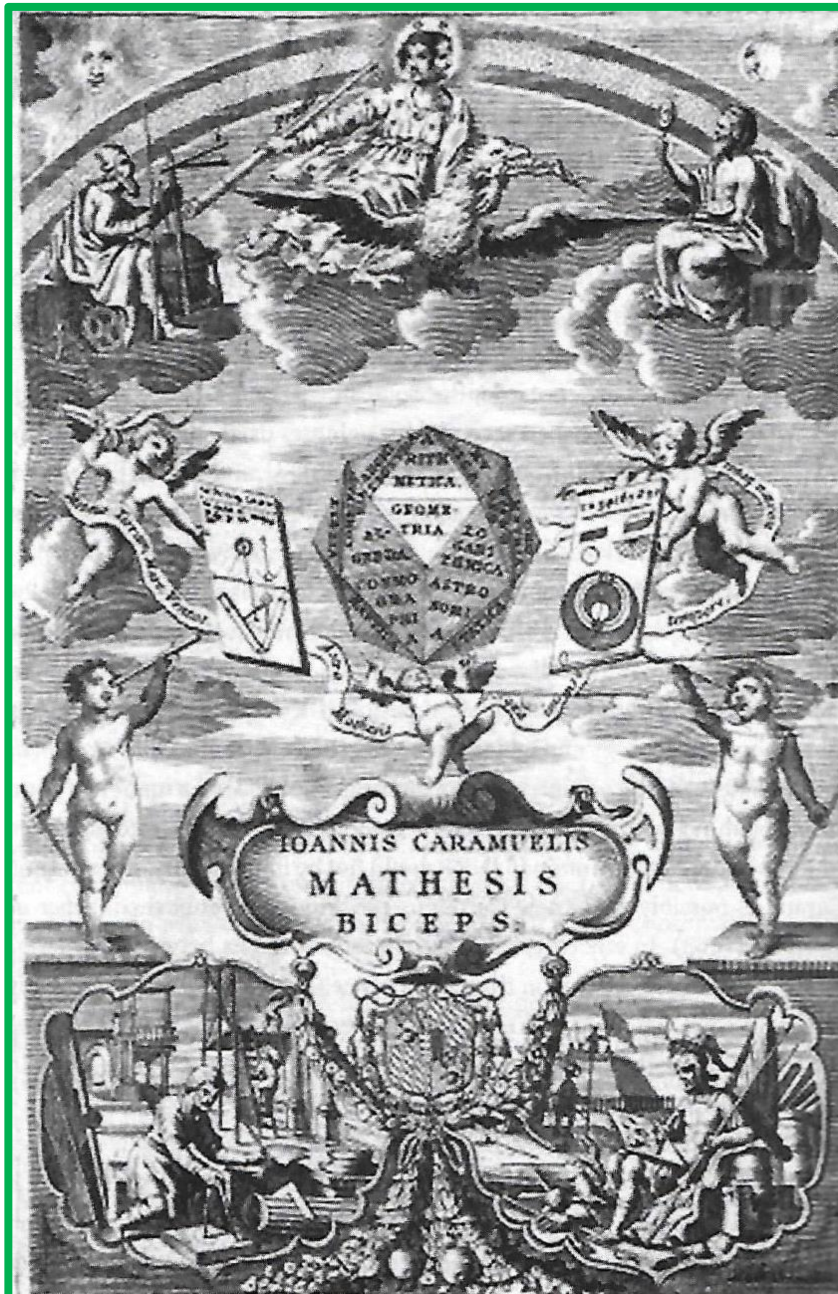
All unordered patterns of three dice



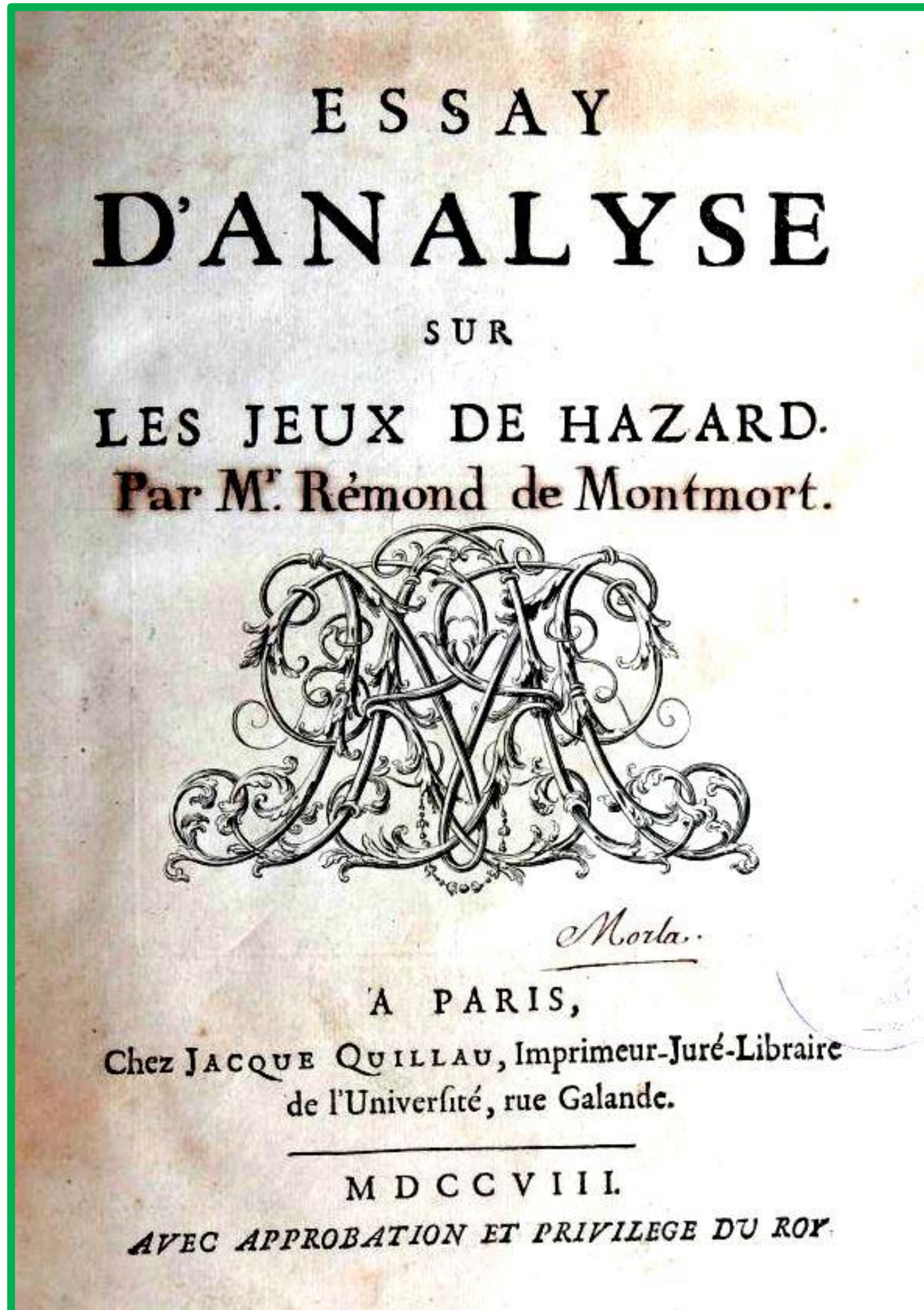
Caramuel's *Mathesis Biceps Vetus et Nova*

(Old and new two-headed mathematics)

(1670)



Pierre De Montmort's *Games of Chance* (1708) includes a 72-page *Treatise on Combinations* which discusses many problems, including derangements



ESSAY D'ANALYSE SUR LES JEUX DE HAZARD.

Par M^r. Rémond de Montmort.

A PARIS,

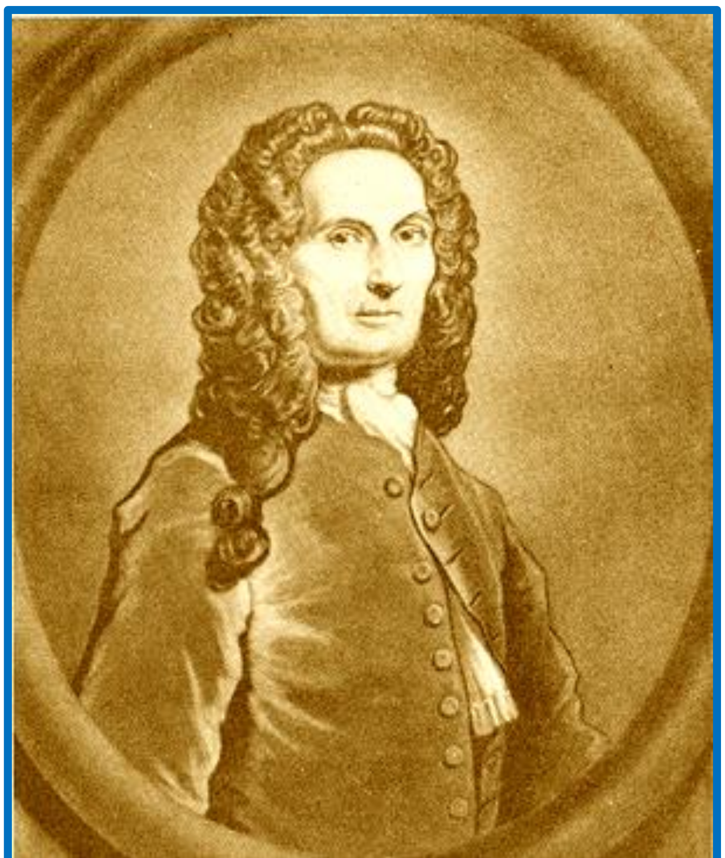
Chez JACQUE QUILLAU, Imprimeur-Juré-Libraire de l'Université, rue Galande.

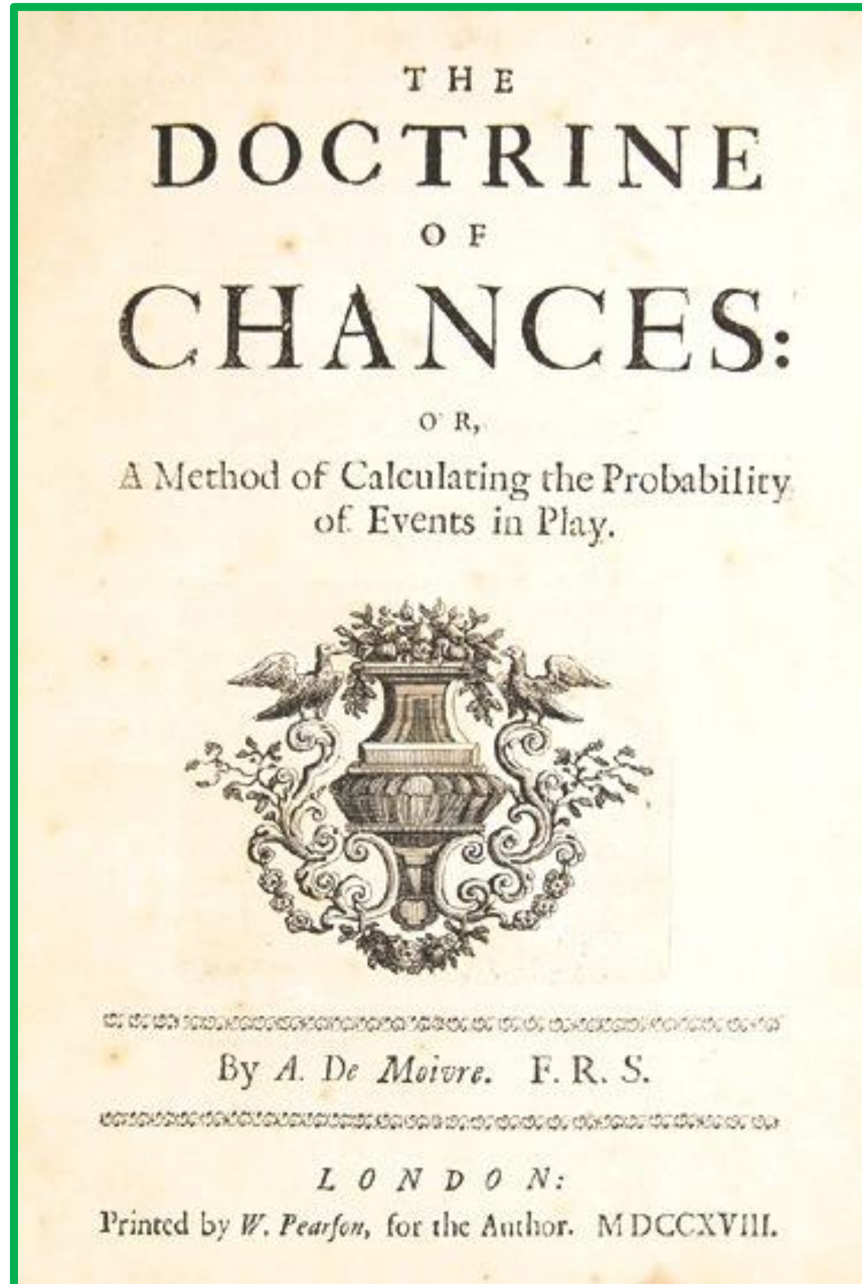
MDCCVIII.

AVEC APPROBATION ET PRIVILEGE DU ROY.

De Montmort's *Jeux de Hazard* (1708)

De Moivre's *Doctrine of Chances* (1718)





THE

DOCTRINE

OF

CHANCES:

OR,

A Method of Calculating the Probability of Events in Play.

By *A. De Moivre*. F. R. S.

LONDON:

Printed by *W. Pearſon*, for the Author. MDCCXVIII.

Principle of Inclusion–Exclusion

96 *The* DOCTRINE *of* CHANCES.

The Probability that a, b, c, d, e, f ſhall all be diſplaced is

$$1 - \frac{6}{6} + \frac{15}{6.5} - \frac{20}{6.5.4} + \frac{15}{6.5.4.3} - \frac{6}{6.5.4.3.2} + \frac{1}{6.5.4.3.2.1} \text{ or } 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} = \frac{265}{720} = \frac{53}{144}.$$

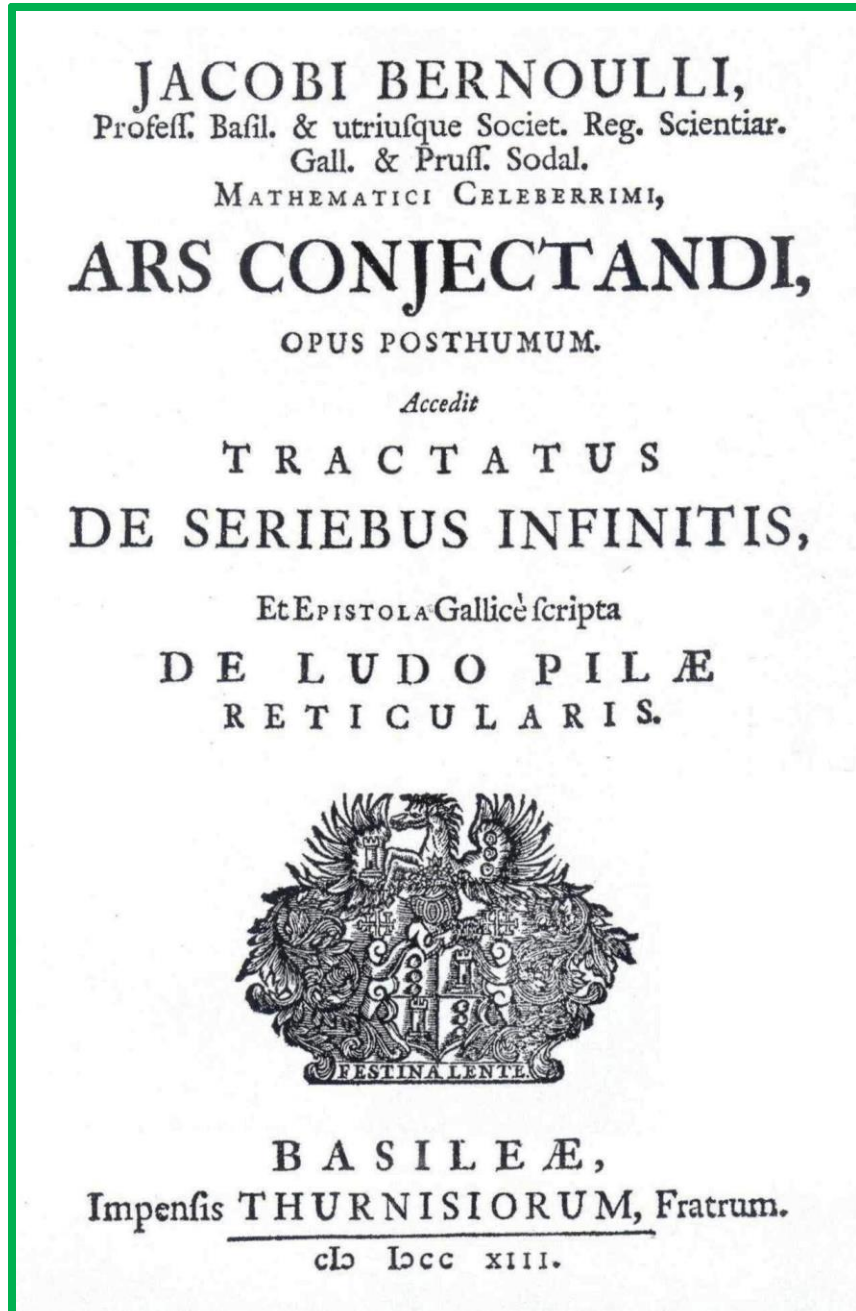
Hence it may be concluded, that the Probability that one of them at leaſt ſhall be in its place, is $1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} = \frac{91}{144}$, and that the Odds that one of them at leaſt ſhall be ſo found, are as 91 to 53.

It muſt be obſerved, that the foregoing Expreſſion may ſerve for any number of Letters, by continuing it to ſo many Terms as there are Letters: thus if the number of Letters had been ſeven, the Probability required would have been $\frac{177}{280}$.

For *n*= 6: the number of derangements is 265 (out of 6! = 720)

= 53 out of 144 (as stated above)

J. Bernoulli's *Ars Conjectandi* (1713)



JACOBI BERNOULLI,
Profess. Basil. & utriusque Societ. Reg. Scientiar. Gall. & Pruss. Sodal.
MATHEMATICI CELEBERRIMI,

ARS CONJECTANDI,

OPUS POSTHUMUM.

Accedit

TRACTATUS
DE SERIEBUS INFINITIS,

Et EPISTOLA Gallicè scripta
DE LUDO PILÆ
RETICULARIS.

BASILEÆ,
Impensis THURNISIORUM, Fratrum.
cIↃ IↃcc xiii.

Bernoulli's *Ars Conjectandi*

Opens with a hymn on the infinite variety of nature – this variety stems from the combinations and arrangements of its parts, and the combinatorial art helps us to understand these.

Also contains probability (the 'law of large numbers', limit theorems, and the binomial distribution), and the 'Bernoulli numbers' (actually discovered 100 years earlier by J. Faulhaber)

$$\int n = \frac{1}{2} nn + \frac{1}{2} n.$$
$$\int nn = \frac{1}{3} n^3 + \frac{1}{2} nn + \frac{1}{6} n.$$
$$\int n^3 = \frac{1}{4} n^4 + \frac{1}{2} n^3 + \frac{1}{4} nn.$$
$$\int n^4 = \frac{1}{5} n^5 + \frac{1}{2} n^4 + \frac{1}{3} n^3 \ast - \frac{1}{30} n.$$
$$\int n^5 = \frac{1}{6} n^6 + \frac{1}{2} n^5 + \frac{5}{12} n^4 \ast - \frac{1}{12} nn.$$
$$\int n^6 = \frac{1}{7} n^7 + \frac{1}{2} n^6 + \frac{1}{2} n^5 \ast - \frac{1}{6} n^3 \ast + \frac{1}{42} n.$$
$$\int n^7 = \frac{1}{8} n^8 + \frac{1}{2} n^7 + \frac{7}{12} n^6 \ast - \frac{7}{24} n^4 \ast + \frac{1}{12} nn.$$
$$\int n^8 = \frac{1}{9} n^9 + \frac{1}{2} n^8 + \frac{2}{3} n^7 \ast - \frac{7}{15} n^5 \ast + \frac{2}{9} n^3 \ast - \frac{1}{30} n.$$
$$\int n^9 = \frac{1}{10} n^{10} + \frac{1}{2} n^9 + \frac{3}{4} n^8 \ast - \frac{7}{10} n^6 \ast + \frac{1}{2} n^4 \ast - \frac{1}{12} nn.$$
$$\int n^{10} = \frac{1}{11} n^{11} + \frac{1}{2} n^{10} + \frac{5}{6} n^9 \ast - 1\, n^7 \ast + 1\, n^5 \ast - \frac{1}{2} n^3 \ast + \frac{5}{66} n.$$