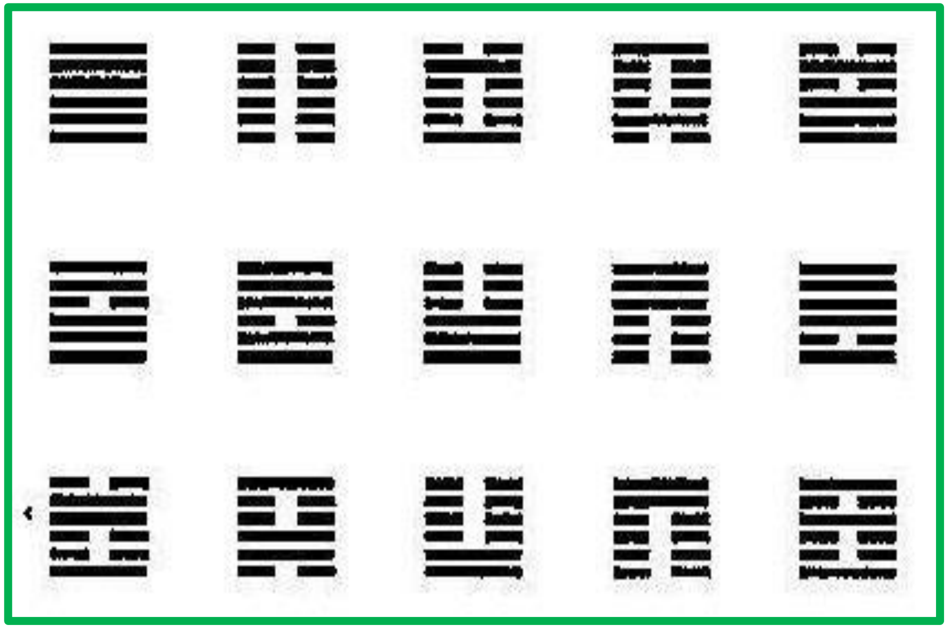
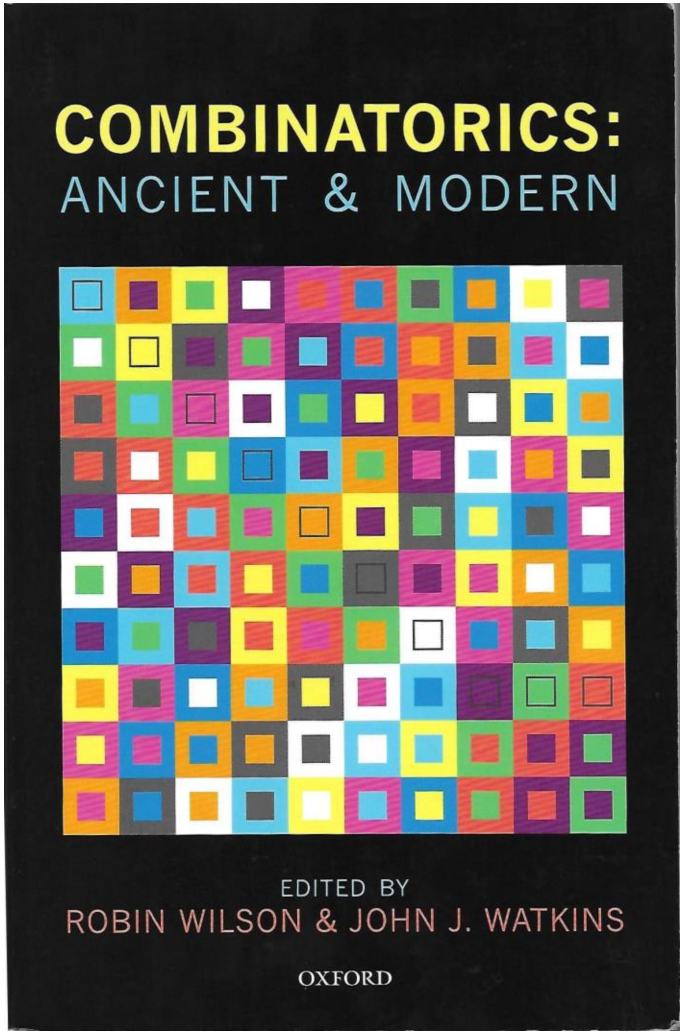
1. Early combinatorics

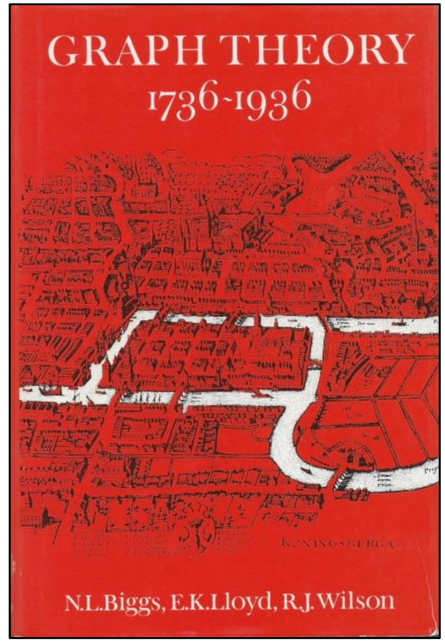
Robin Wilson

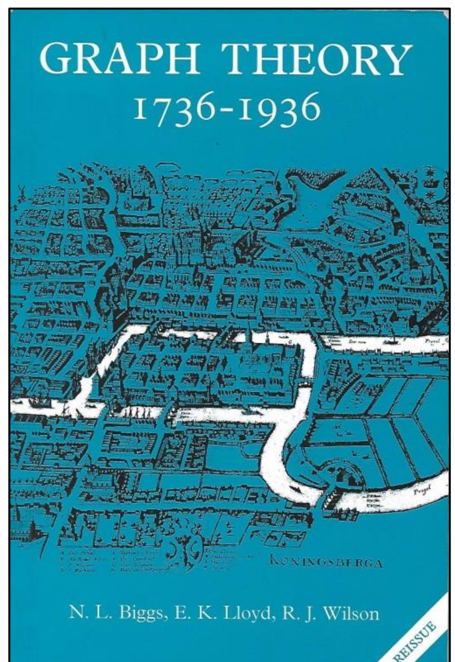


1. Early combinatorics
2. European combinatorics:
   Middle Ages to Renaissance
3. Euler’s combinatorics
4. Magic squares, Latin squares
   & triple systems
5. The 19th century
6. Colouring maps

7/8. A century of graph theory


COMBINATORICS:
ANCIENT & MODERN
EDITED BY
ROBIN WILSON & JOHN J. WATKINS
OXFORD


GRAPH THEORY
1736-1936
N.L.Biggs, E.K.Lloyd, R.J.Wilson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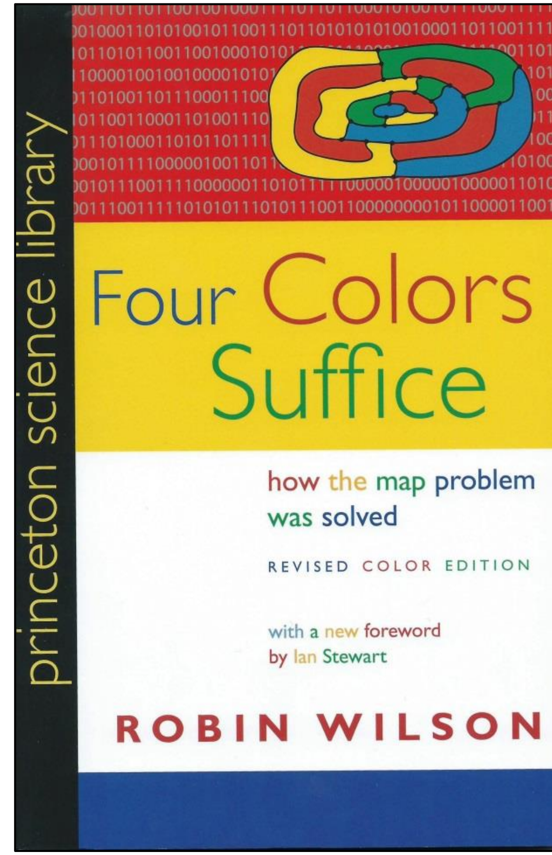
REISSUE


princeton science library
Four Colors
Suffice
how the map problem
was solved
REVISED COLOR EDITION
with a new foreword
by Ian Stewart
ROBIN WILSON

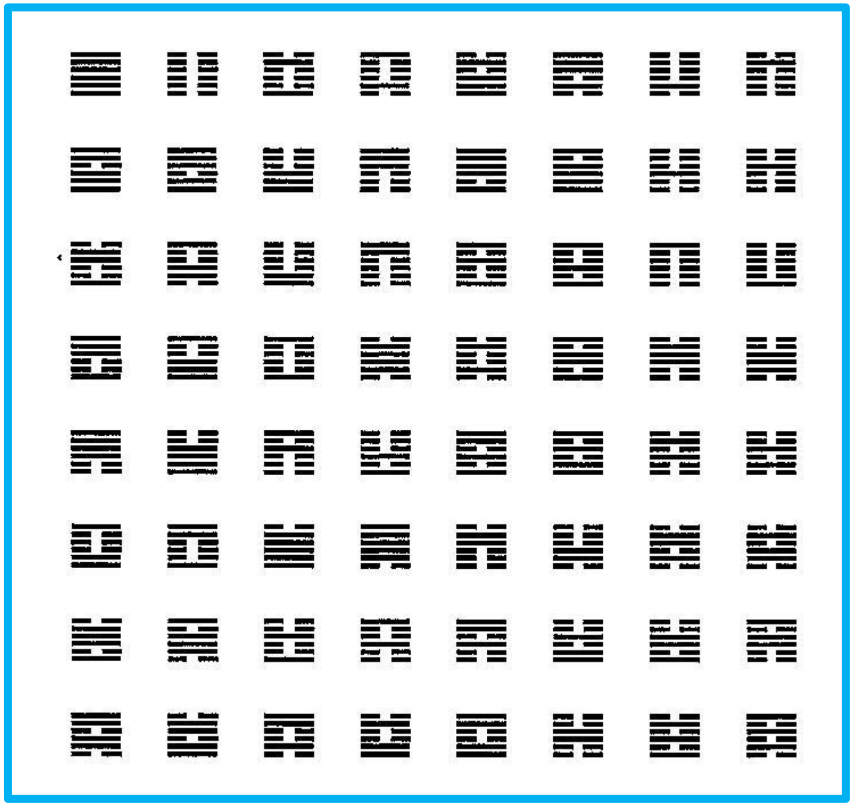
Early mathematics time-line

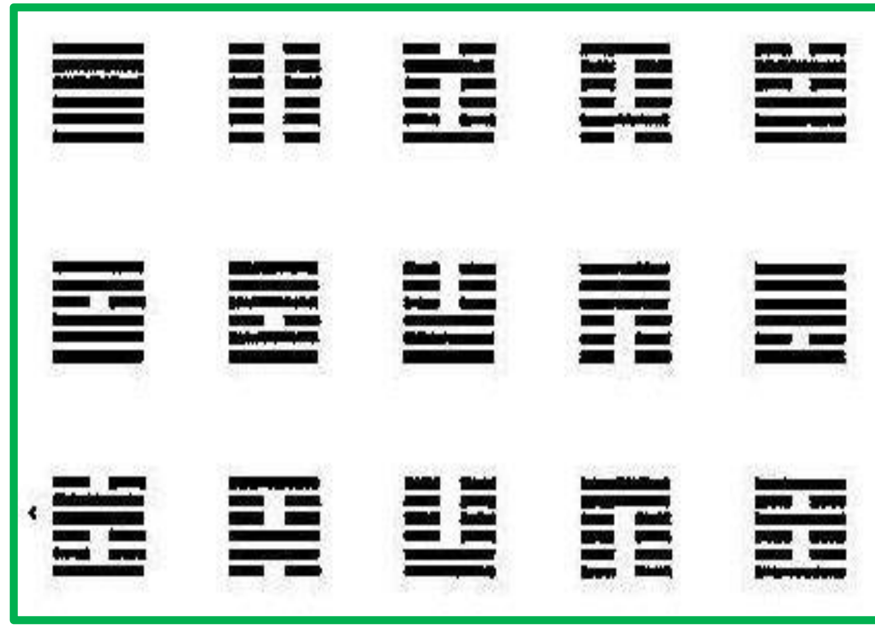
- ~~2700 – 1600 BC : Egypt~~
- ~~2000 – 1600 BC : Mesopotamia (‘Babylonian’)~~

- 1100 BC – AD 1400 : China
- 600 BC – AD 500 : Greece (three periods)
- 600 BC – AD 1200 : India
- ~~AD 500 – 1000 : Mayan~~

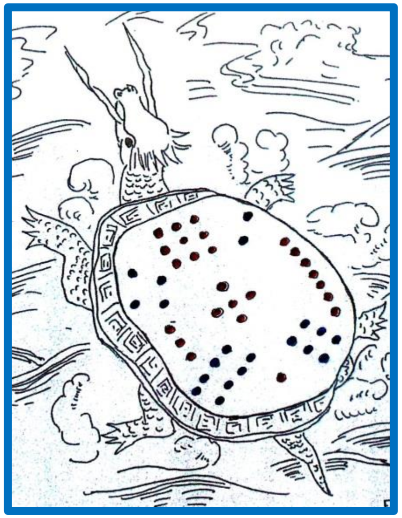
- AD 750 – 1400 : Islamic / Arabic
- AD 1000 – . . . : Europe

I Ching (yijing) (c.1100 BC)



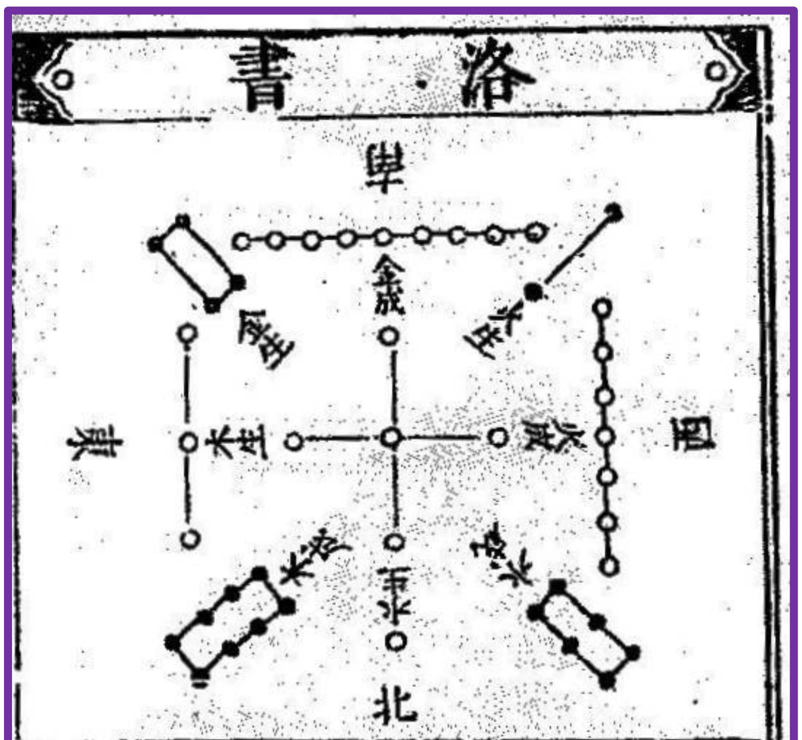


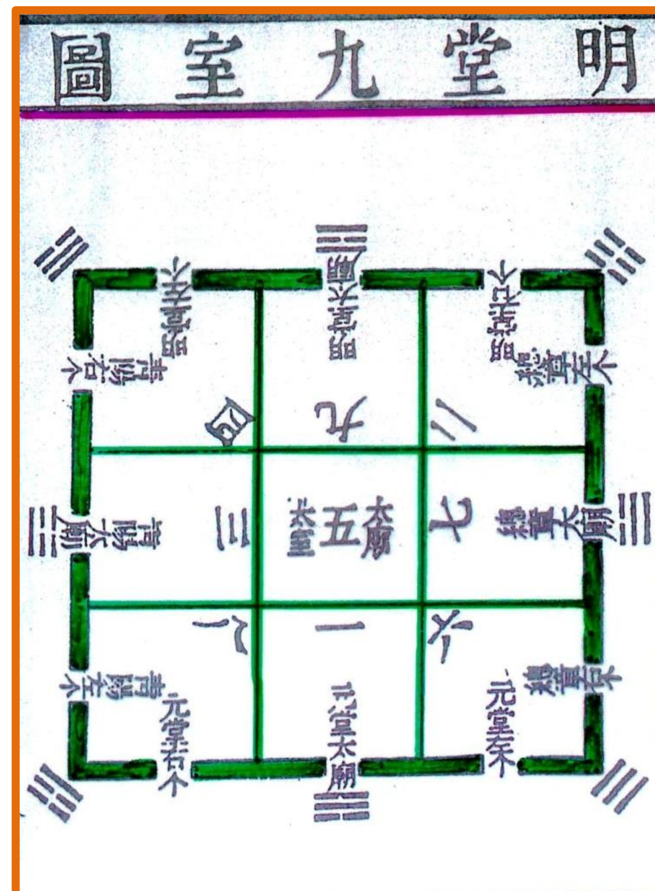
Number of yin-yang hexagrams (chapters) $= 2^6 = 64$



4	9	2
3	5	7
8	1	6

The Lo-shu diagram

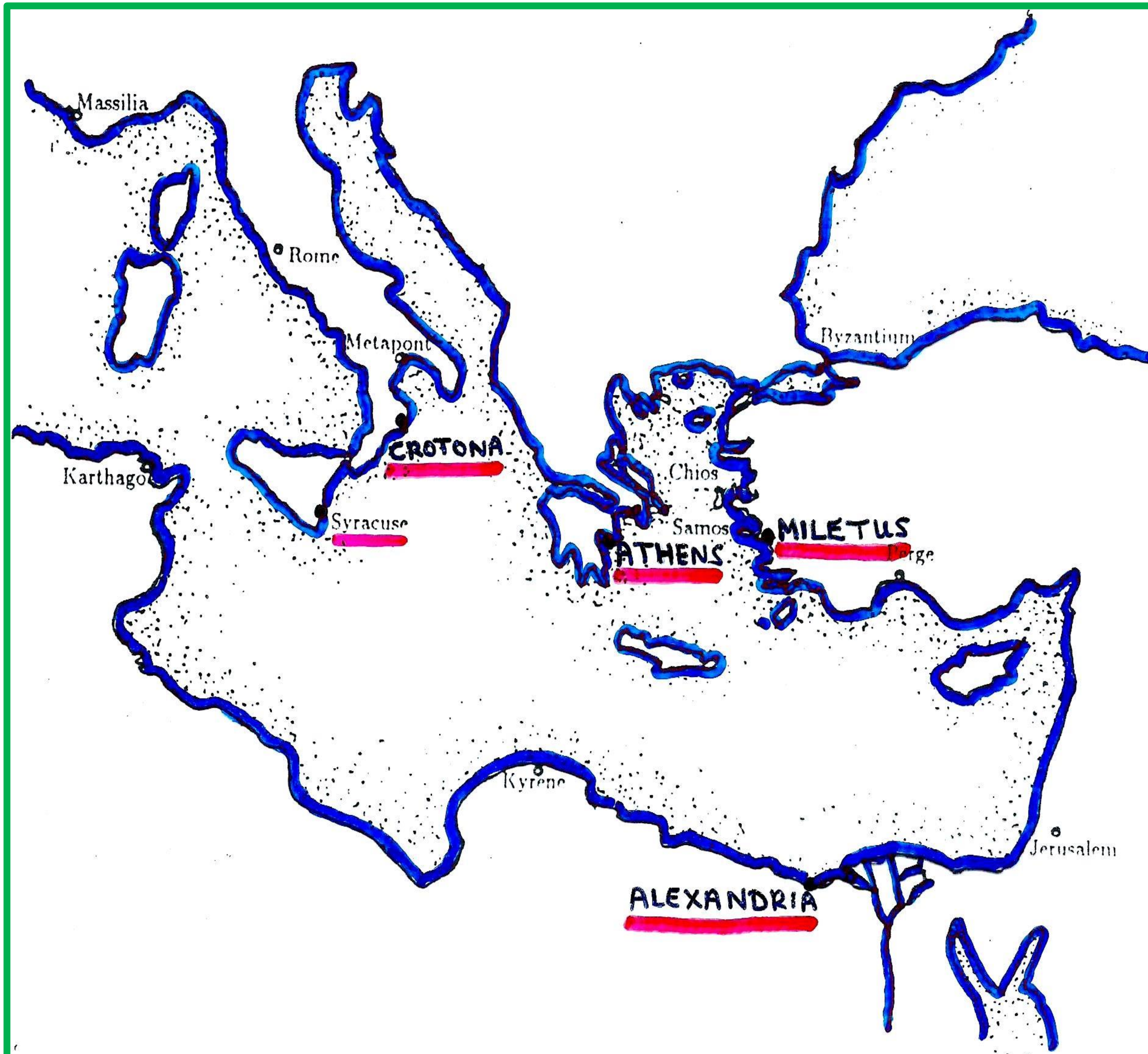




Greek mathematics: three periods

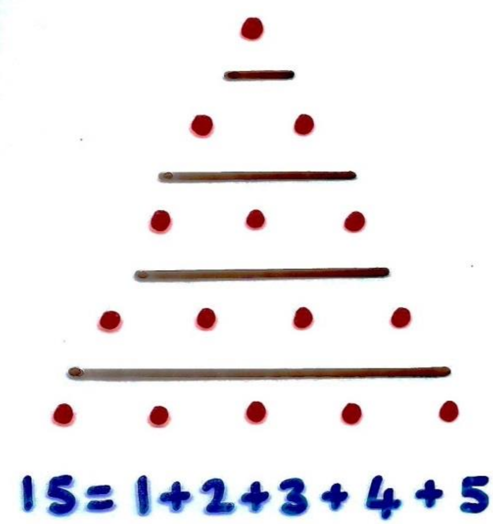
Early:	Thales	600 BC
	Pythagoras	520 BC
Athens:	Plato	387 BC
	Aristotle	350 BC
	Eudoxus	370 BC
Alexandria / Syracuse:		
	Euclid	250 BC?
	[Archimedes]	250 BC
	Apollonius	220 BC
	Ptolemy	AD 150
	Diophantus	AD 250?
	Pappus	AD 320
	Hypatia	AD 400

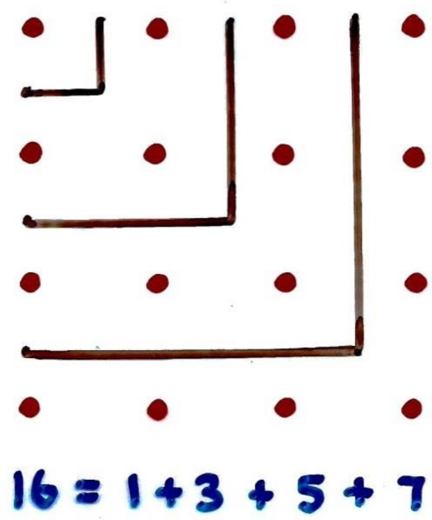
Map of Greece
(300 BC)



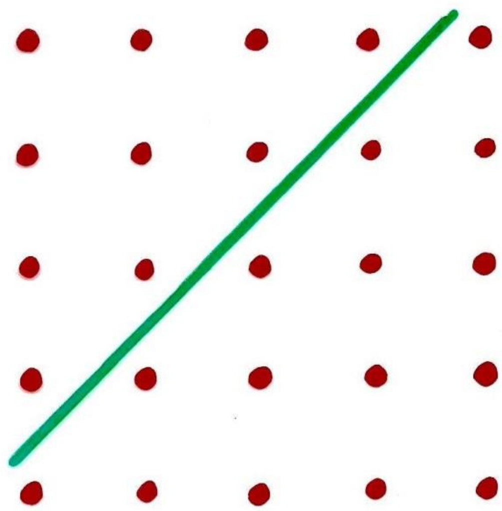
Pythagorean figurate numbers

triangular numbers
$n(n + 1)/2$
1, 3, 6, 10, 15, 21,
. . .

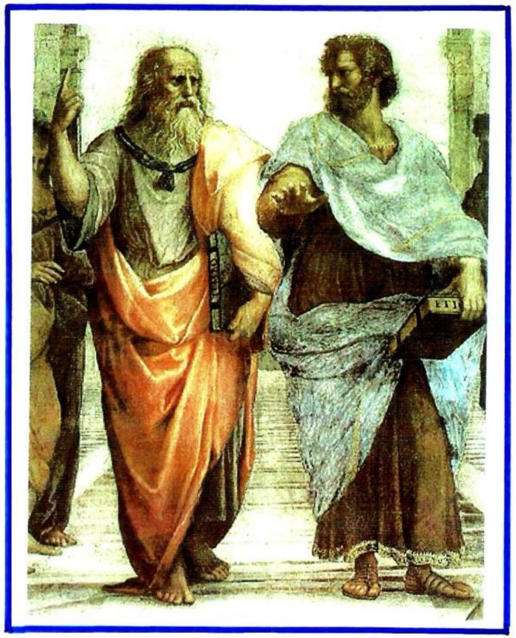




square numbers
n^2
1, 4, 9, 16, 25, 36
. . .



Any square number is the sum of two consecutive triangular numbers



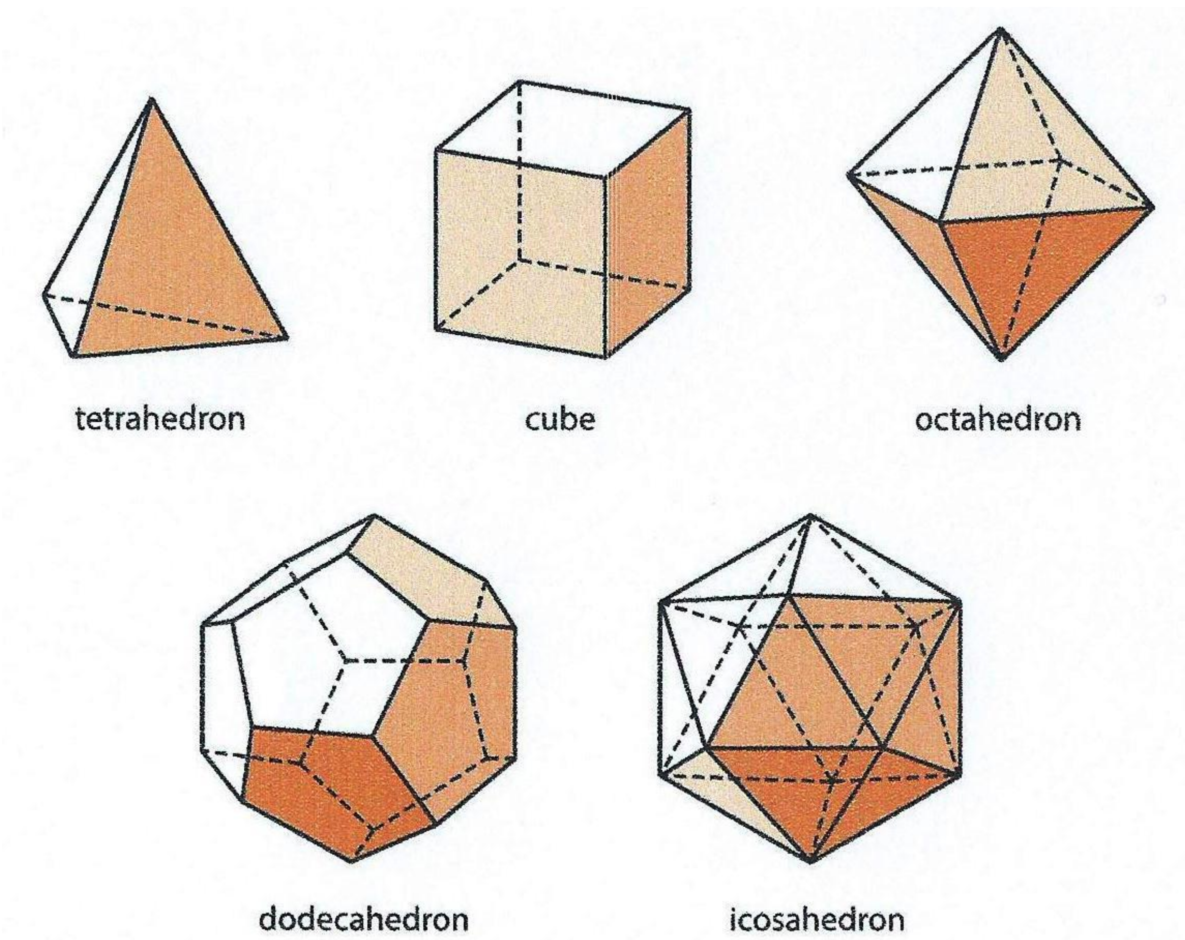
Plato’s Academy (387 BC)

Raphael’s ‘School of Athens’

Plato's *Timaeus*:

The five regular polyhedra

('Platonic solids')



tetrahedron
= fire

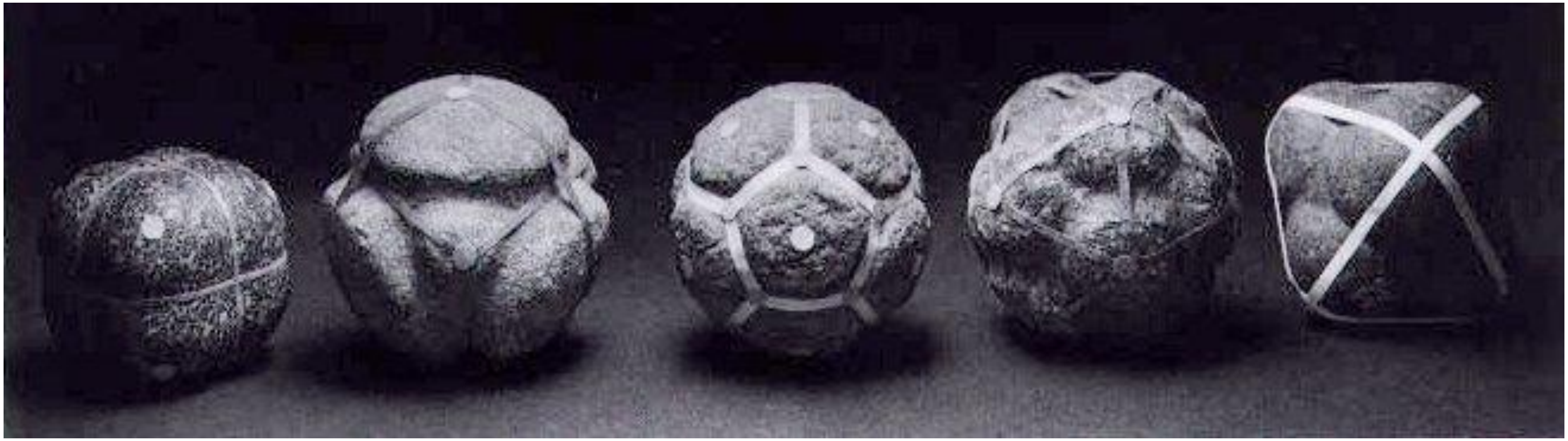
cube
= earth

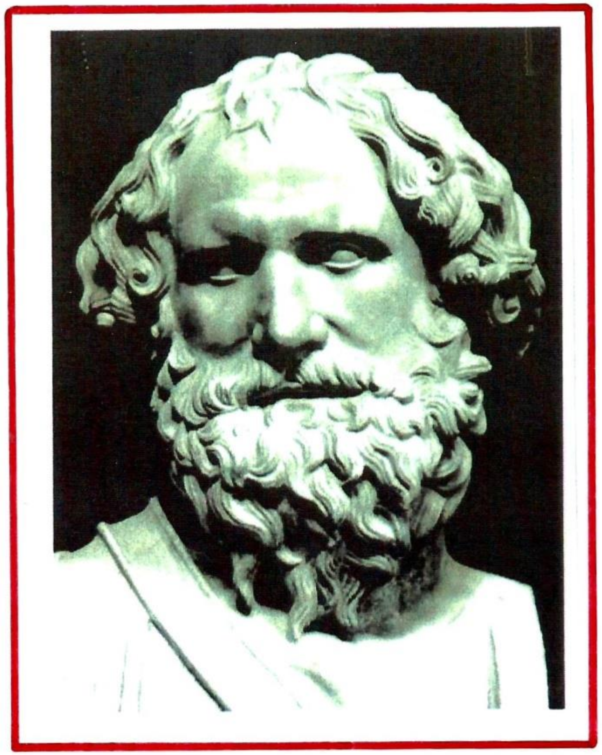
octahedron
= air

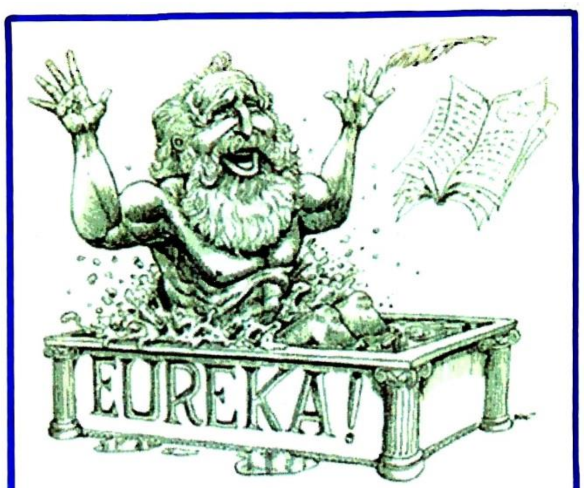
icosahedron
= water

dodecahedron
= cosmos

Polyhedral dice (*astragali*)





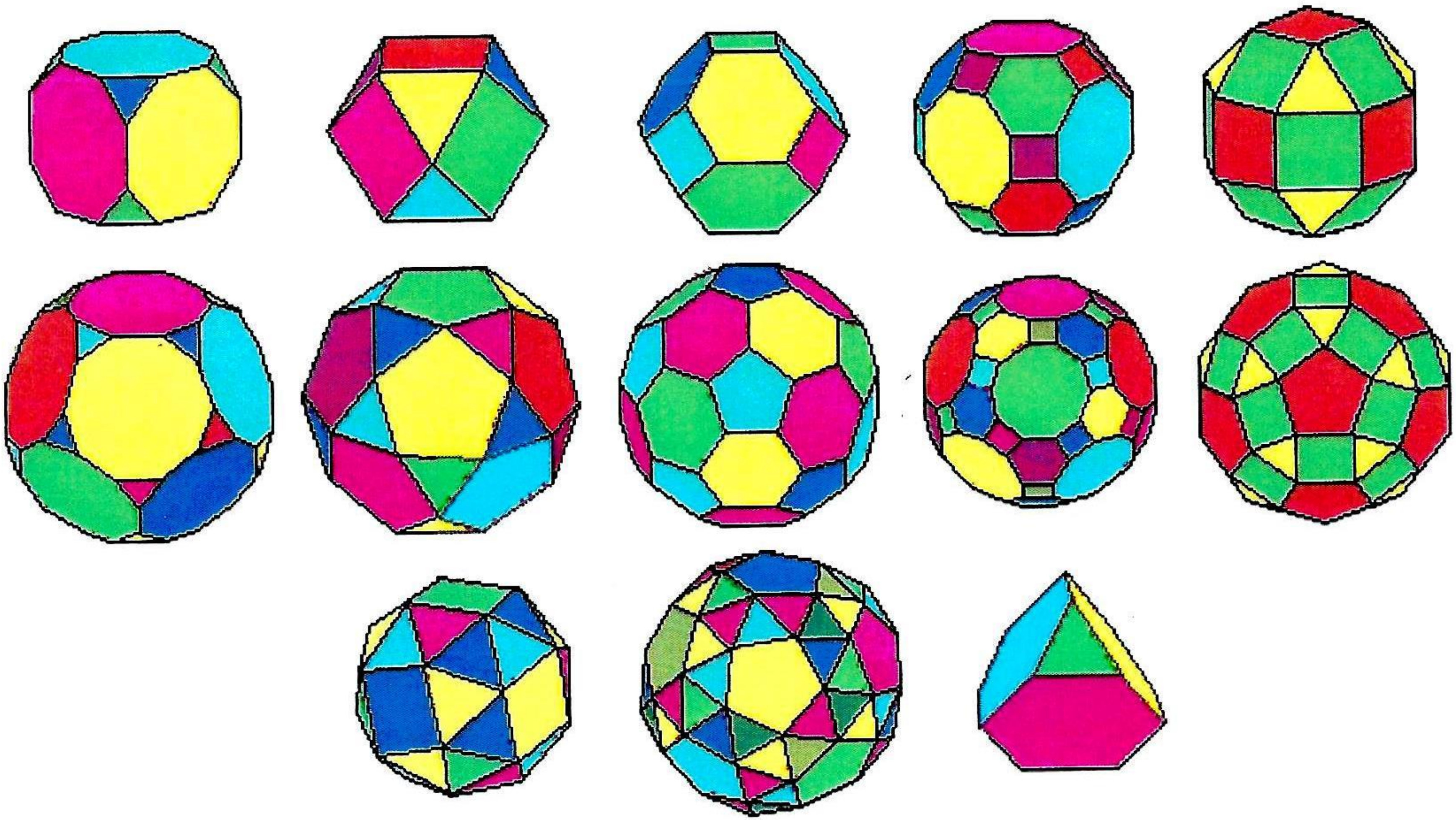


Archimedes

(c.287–212 BC)

- On floating bodies
- On the equilibrium of planes
- On the measurement of a circle
- *The Method*
- On Spirals
- On the sphere and cylinder I, II
- Quadrature of the parabola
- On conoids and spheroids
- The sand reckoner
- Semi-regular polyhedra

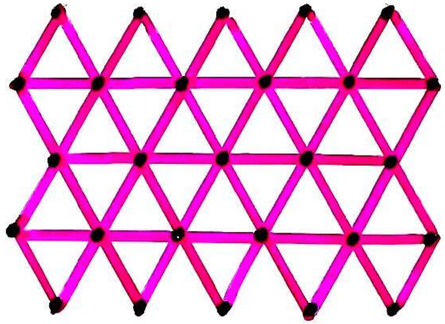
Archimedean (semi-regular) solids



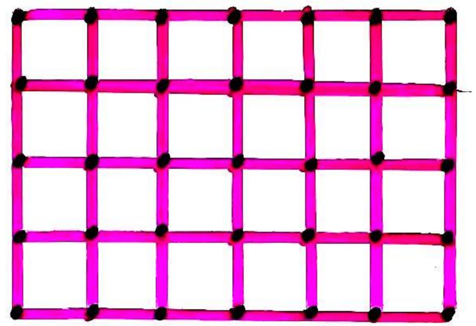
Pappus : ‘On the sagacity of bees’

(regular tilings)

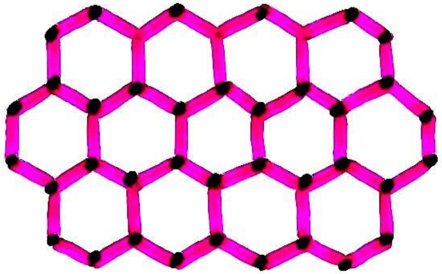
(early 4th century AD)



triangles

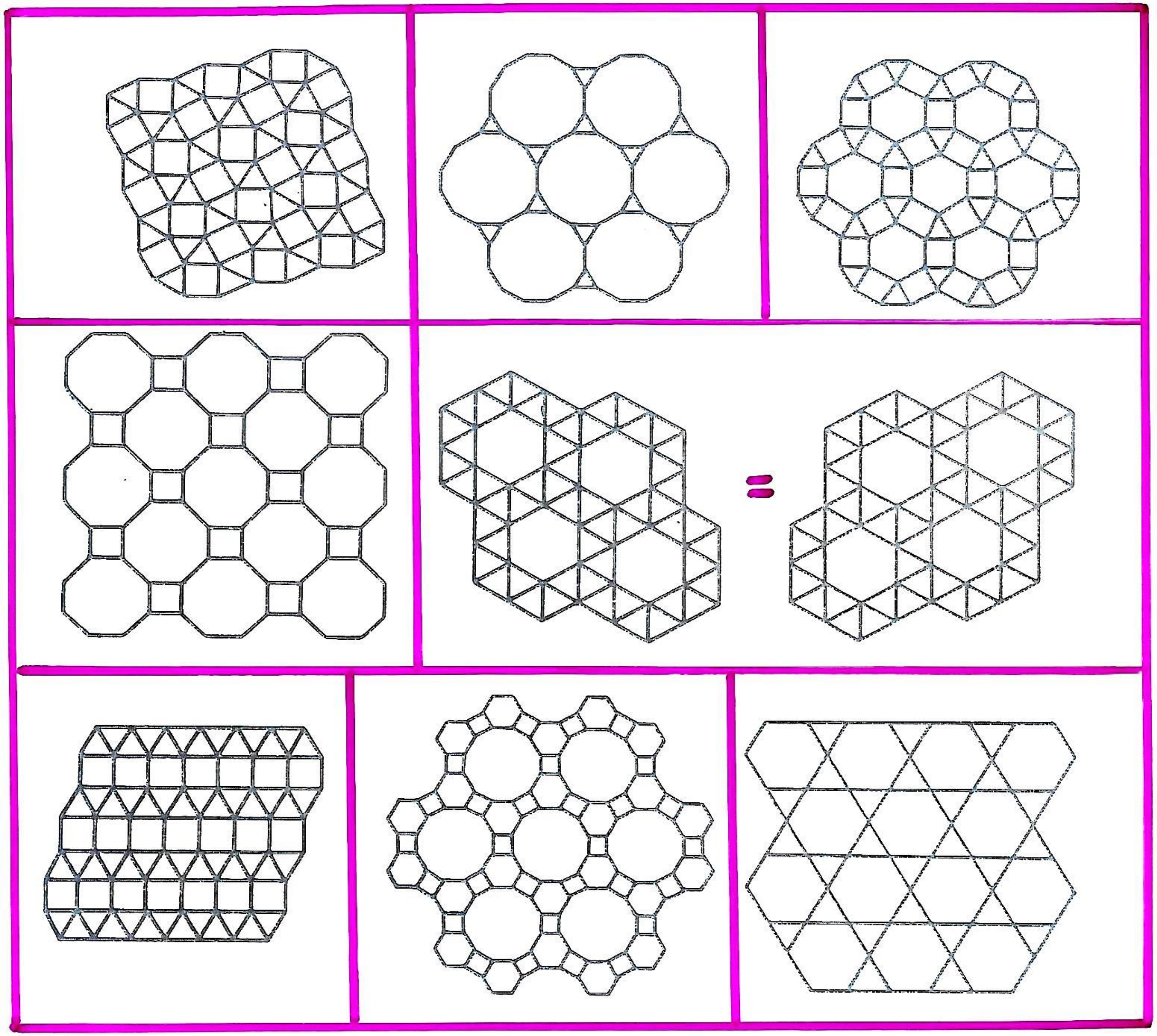


squares



hexagons

Semi-regular (Archimedean) tilings



Susruta's treatise (6th century BC)

Medicines can be sweet, sour, salty, pungent, bitter, or astringent.

Susruta listed:

6 combinations when taken 1 at a time

15 combinations when taken 2 at a time

20 combinations when taken 3 at a time

15 combinations when taken 4 at a time

6 combinations when taken 5 at a time

1 combination when taken 6 at a time

Thus: $C(6,1) = 6$; $C(6,2) = 15$; $C(6,3) = 20$;

$C(6,4) = 15$; $C(6,5) = 6$; $C(6,6) = 1$

Indian combinatorics

c. 300 BC: Jainas (Bhagabatisutra):
combinations of five senses, or of men,
women and eunuchs

c. 200 BC: Pingala (Chandrasutra):
combinations of short/long sounds
in a metrical poem (— u u — u, etc.)

c. AD 550: Varahamihira's *Brhatsamhita* on perfumes: choose 4 ingredients from 16

16	$C(16,1)$							
15	$C(15,1)$	120	$C(16,2)$					
14	$C(14,1)$	105	$C(15,2)$	560	$C(16,3)$			
13	$C(13,1)$	91	$C(14,2)$	455	$C(15,3)$	1820	$C(16,4)$	
12	$C(12,1)$	78	$C(13,2)$	364	$C(14,3)$	1365	$C(15,4)$	
11	$C(11,1)$	66	$C(12,2)$	286	$C(13,3)$	1001	$C(14,4)$	
10	$C(10,1)$	55	$C(11,2)$	220	$C(12,3)$	715	$C(13,4)$	
9	$C(9,1)$	45	$C(10,2)$	165	$C(11,3)$	495	$C(12,4)$	
8	$C(8,1)$	36	$C(9,2)$	120	$C(10,3)$	330	$C(11,4)$	
7	$C(7,1)$	28	$C(8,2)$	84	$C(9,3)$	210	$C(10,4)$	
6	$C(6,1)$	21	$C(7,2)$	56	$C(8,3)$	126	$C(9,4)$	
5	$C(5,1)$	15	$C(6,2)$	35	$C(7,3)$	70	$C(8,4)$	
4	$C(4,1)$	10	$C(5,2)$	20	$C(6,3)$	35	$C(7,4)$	
3	$C(3,1)$	6	$C(4,2)$	10	$C(5,3)$	15	$C(6,4)$	
2	$C(2,1)$	3	$C(3,2)$	4	$C(4,3)$	5	$C(5,4)$	
1	$C(1,1)$	1	$C(2,2)$	1	$C(3,3)$	1	$C(4,4)$	

Number of combinations = C(16, 4) = 1820

Arrangements: Vishnu

(from Bhaskara's *Lilavati*, AD 1150)

Vishnu holds in his four hands
a discus, a conch, a lotus,
and a mace:
the number of arrangements is
$4 \times 3 \times 2 \times 1 = 24 = 4!$

Bhaskara gave general rules for n!, C(n, k), etc:

The number of combinations of *k* objects selected from a set of *n* objects is

$$\frac{n \times (n-1) \times \cdots \times (n-k+1)}{k \times (k-1) \times \cdots \times 1}$$

Bhaskara's permutations: Sambhu

How many are the variations of form of the god Sambhu by the exchange of his ten attributes held in his ten hands: the rope, the elephant's hook, the serpent, the tabor, the skull, the trident, the bedstead, the dagger, the arrow and the bow?

Statement: **number of places: 10**

The variations of form are found to be

(10! =) 3,628,800

Permutations and combinations

Four types of selection problem: choose r objects from n objects:

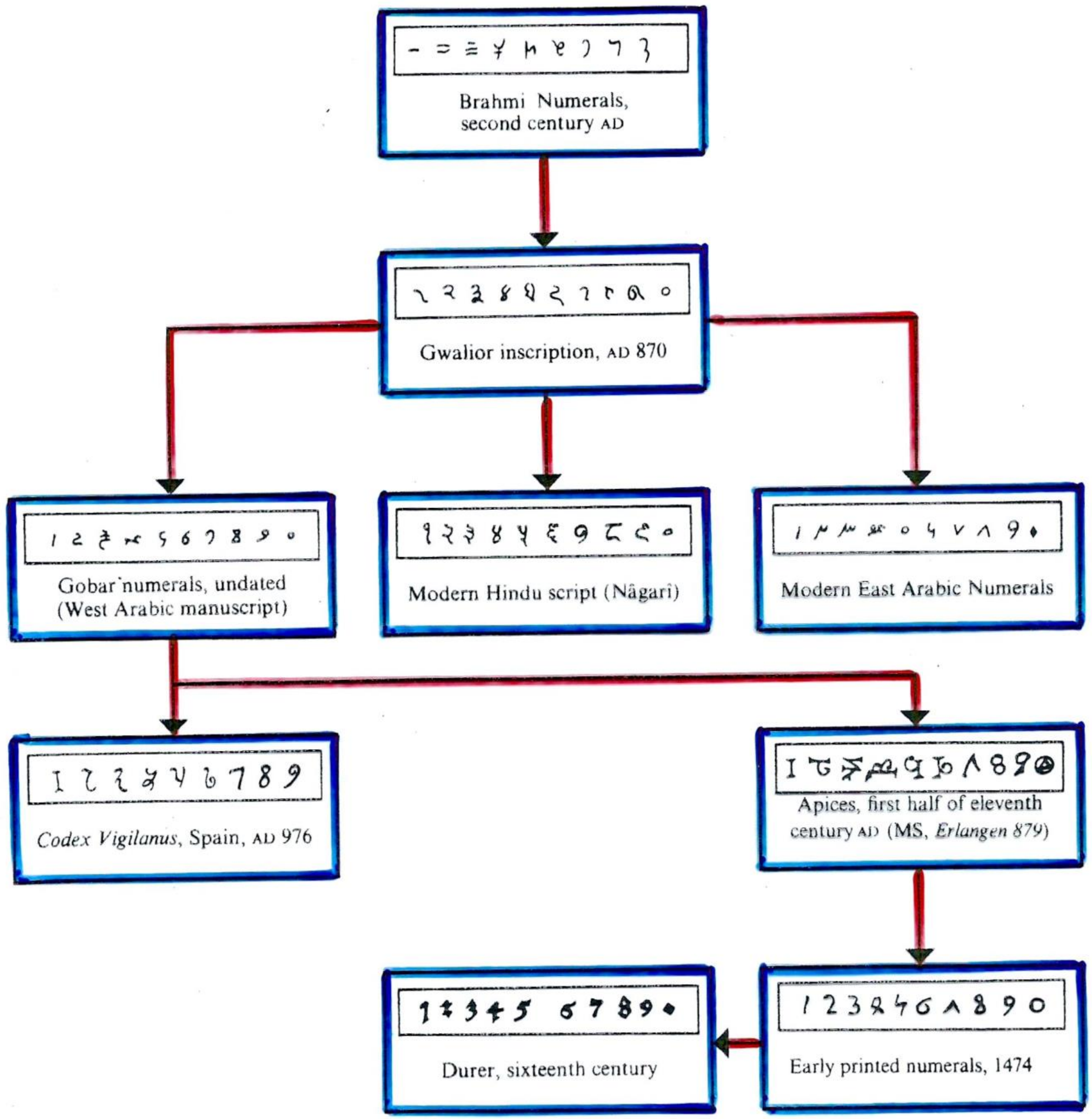
1. **Selections ordered, repetition allowed:**
   number of ways is $n \times n \times . . . \times n = n^r$

2. **Selections ordered, no repetition allowed (permutation):**
   number of ways is
   $P(n, r) = n \times (n - 1) \times . . . \times (n - r + 1) = n! / (n - r)!$

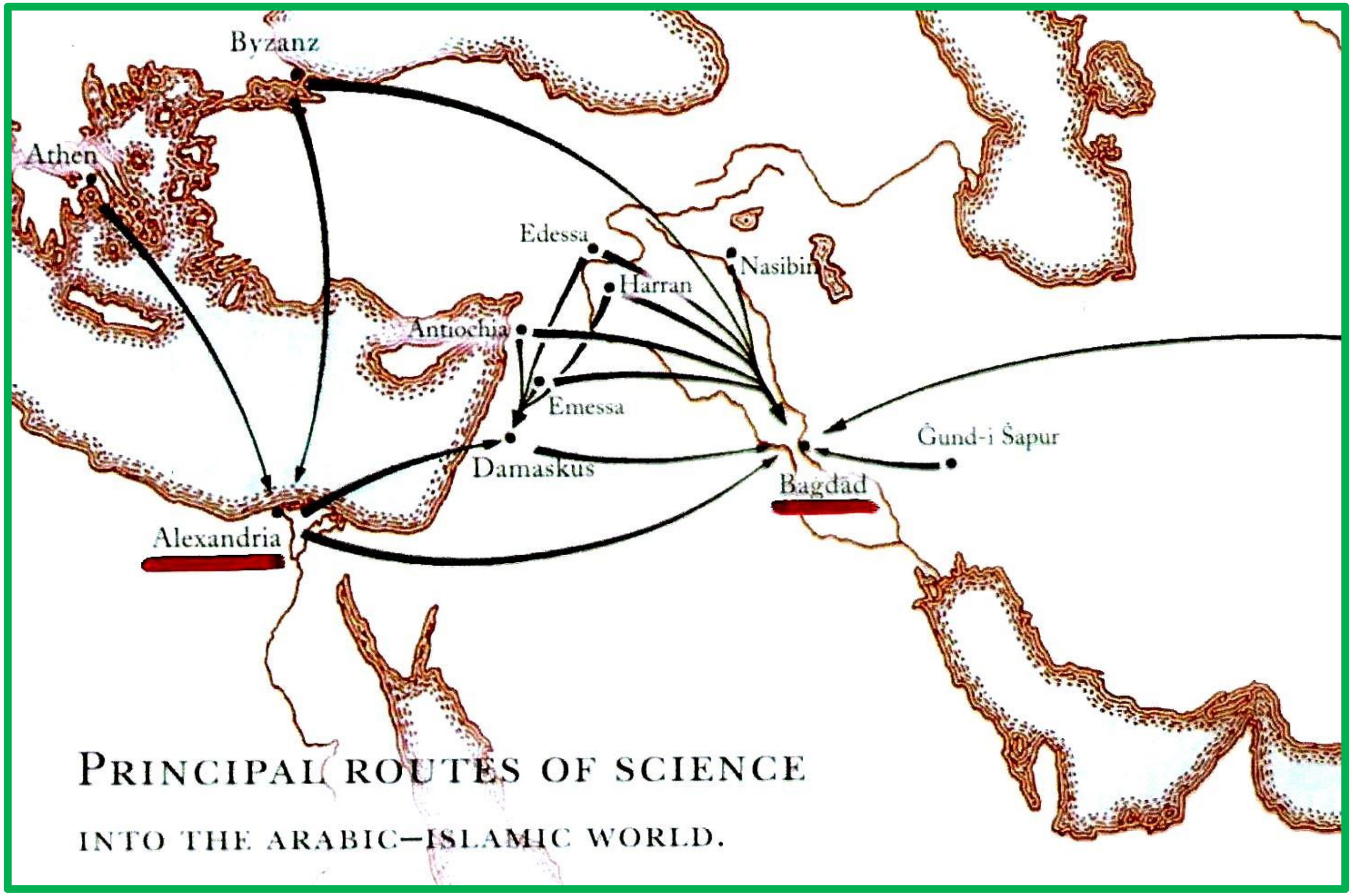
3. **Selections unordered, no repetition allowed (combination):**
   number of ways is $C(n, r) = P(n, r) / r! = n! / r! \, (n - r)!$

4. **Selections unordered, repetition allowed:**
   number of ways is $C(n + r - 1, r) = (n + r - 1)! / (n - 1)! \, r!$

The Hindu – Arabic numerals



Influences on Baghdad



Baghdad was on the trade routes between the West (the Greek world) and the East (India)

Al-Khwarizmi (c.783–c.850)

- **Arithmetic text**
- **Algorithmi de numero Indorum**
  'Dixit Algorismi'
- **Algebra text**
  Kitab al-jabr w'al-muqabalah
  Ludus algebrae et almucgrabalaeque

المجتمع واحدا ثمّ ضرب الثلاثة فى الاثنين الباقيين [١] فاجتمع ستّة ،
و لكنّ ذلك لا يصحّ فى أكثر الصنوف وكأنّه وقع فى النسخة فساد
فأمّا الوضع فإنه إذًا كان هكذا : < < < ا
و هو أن يكون مزاج السطر الأيمن | < < ب
بالإغباب واحدا من آخر و مزاج < | < ج
السطر الأوسط اثنين من نوع و اثنين | | < د
من آخر و مزاج الأيسر أربعة من ذا < < | ه
و أربعة من ذاك بحسب أزواج الزوج | < | و
فى مزاجات الأسطر ثمّ زيد فى الحساب < | | ز
المذكور أنّ ابتداء الصفّ إن كان بحصّةِ | | | ح
ثقيل نُقص منها قبلَ الضرب واحدٌ و إن كان الضرب فى حصّة ثقيل
نُقص من المبلغ واحدٌ حصَلَ المطلوب من عدد رتبة الصفّ ؛ وكما أنّ
أبيات العربيّة تنقسم لنصفين بعروض و ضرب فإنّ أبيات أولئك تنقسم
لقسمين يسمّى كل واحد منها رِجلا [٢] و هكذا يسمّيها اليونانيّون ارجلا [٣] [٢]
ما يتركّب منه من الكلمات سلابى و الحروف بالصوت و عدمه و الطول
و القصر و التوسّط ؛ و ينقسم البيت لثلاث أرجل و لأربع و هو الأكثر
و ربّما زيد فى الوسط رجل خامسة و لا تكون مقفّاة و لكن إن كان
آخر الرجل الأولى و الثانية حرفا واحدا كالقافية و كذلك آخر الثالثة
و الرابعة أيضا حرفا واحدا سمّى هذا النوع " آرَلْ " و يجوز فى آخر.

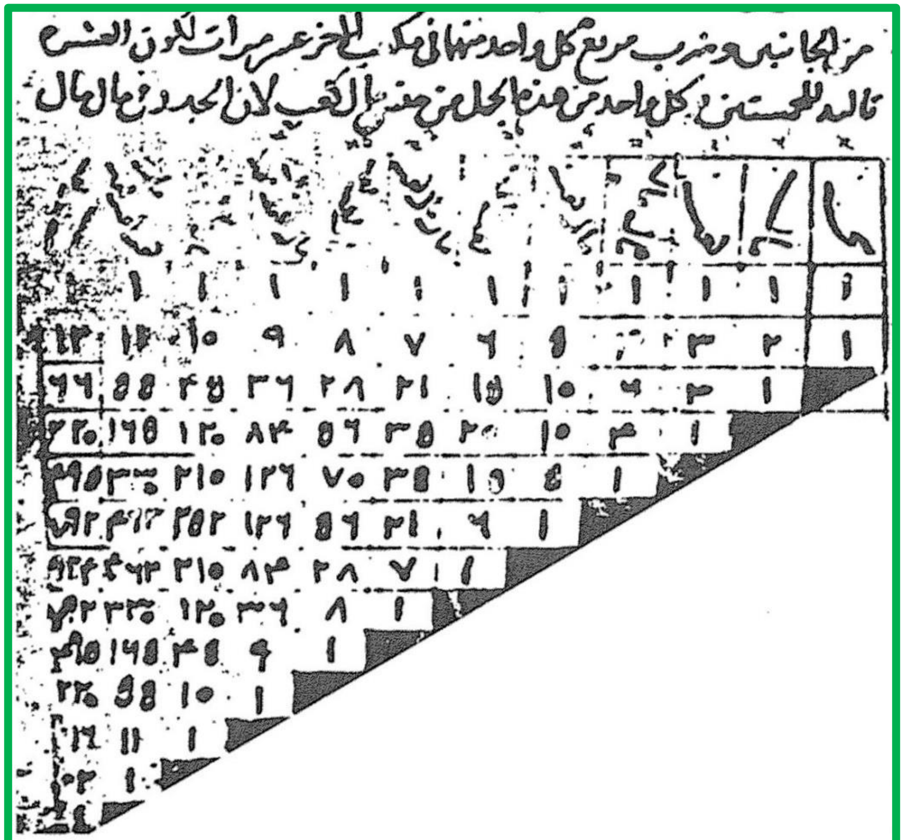
(١) فى ز، وش : الباقية (٢) من ز ، و فى ش : رجل (٣ – ٣) بياض فى ش
الرجل

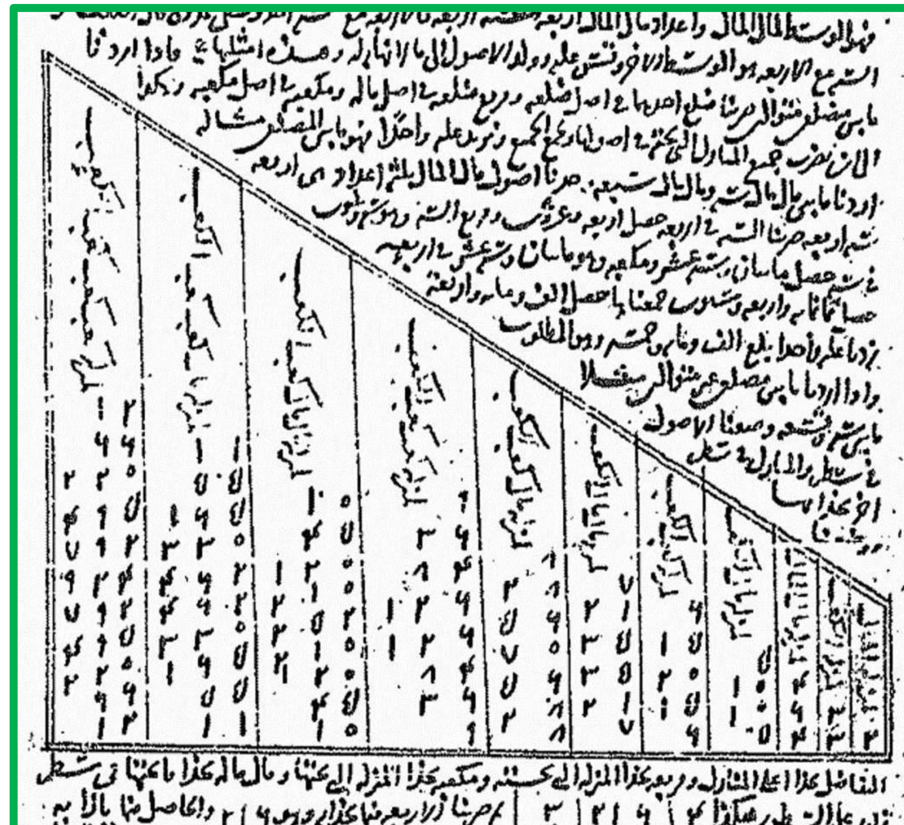
Al-Biruni on the combinatorics of Sanskrit metres

(11th century)

There are 8 possible three-syllable metres, where each symbol is either heavy (<) or light (|)

Arithmetical triangles of Al-Karaji (1007) and Nasir ad-Din at-Tusi (c.1240)





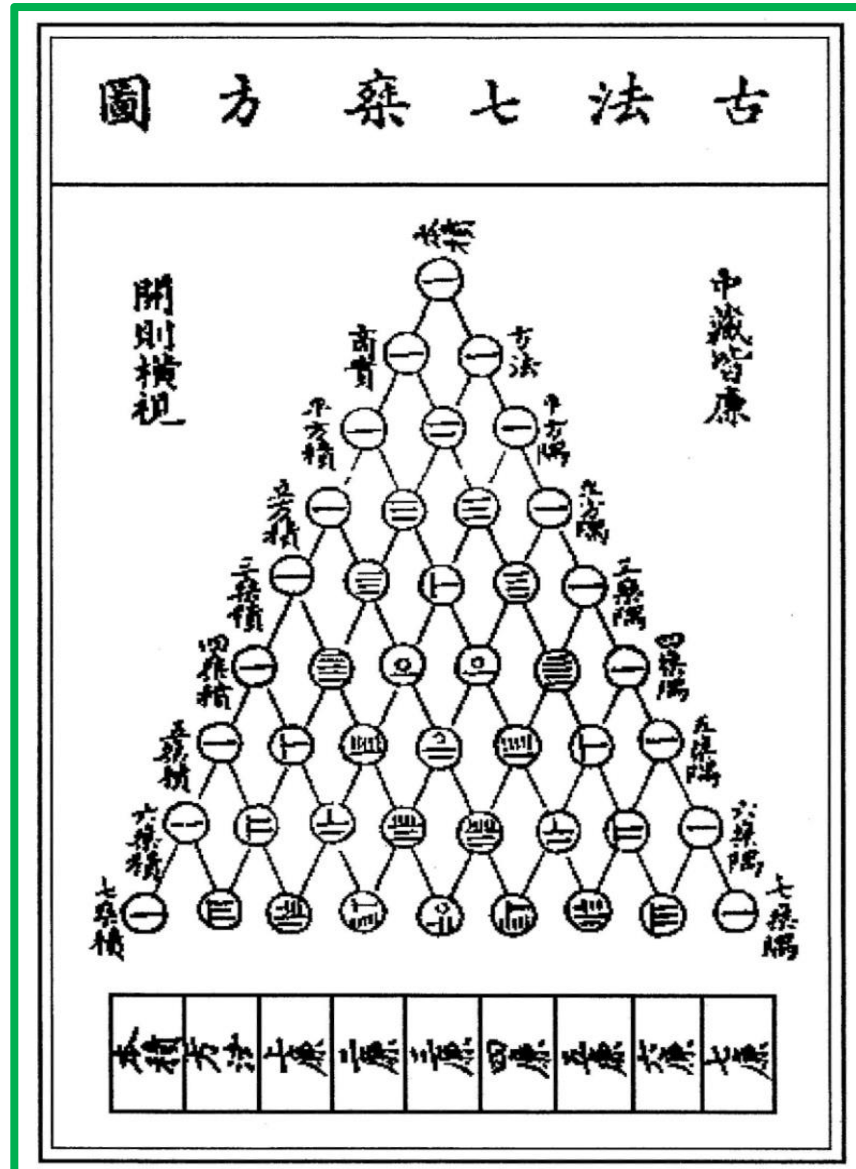
Arithmetical triangle of Ibn Munim

(c. AD 1200)

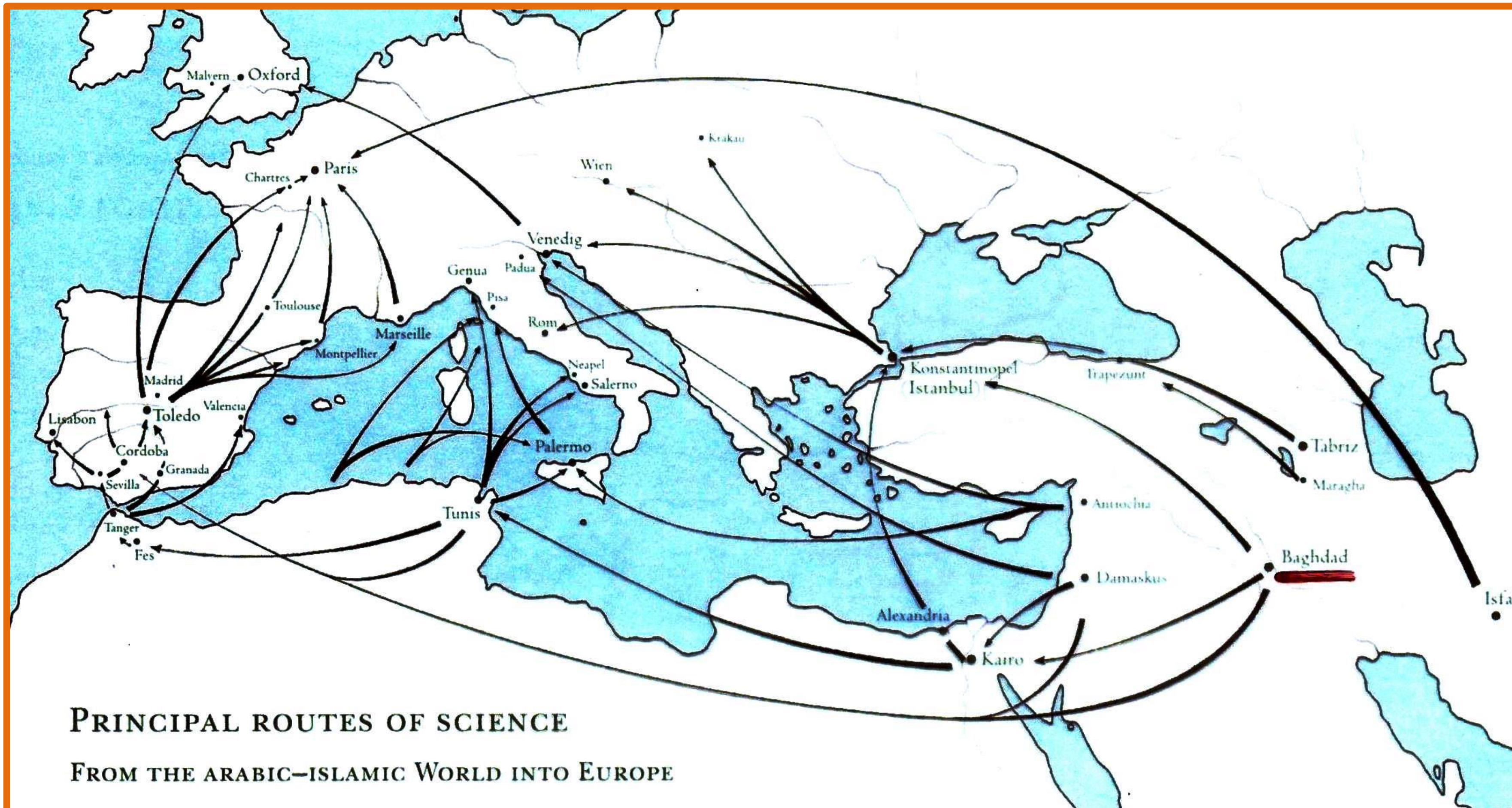
1	1									
10	9	1								
45	36	8	1							
120	84	28	7	1						
210	126	56	21	6	1					
252	126	70	35	15	5	1				
210	84	56	35	20	10	4	1			
120	36	28	21	15	10	6	3	1		
45	9	8	7	6	5	4	3	2	1	
10	1	1	1	1	1	1	1	1	1	1

Zhu Shijie (1303): *Sijuan yujian*

(Precious mirror of the four elements)

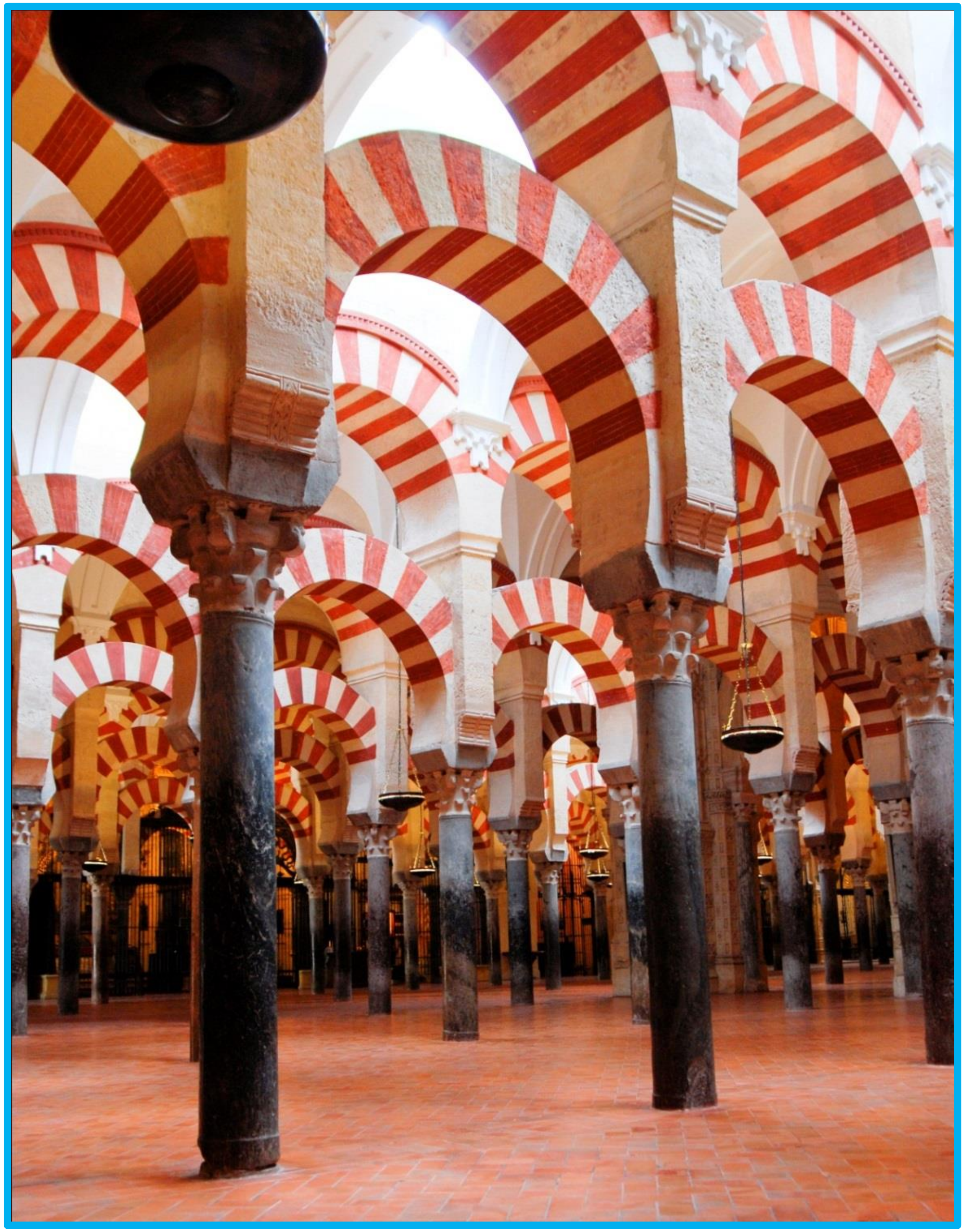


From Baghdad to Europe



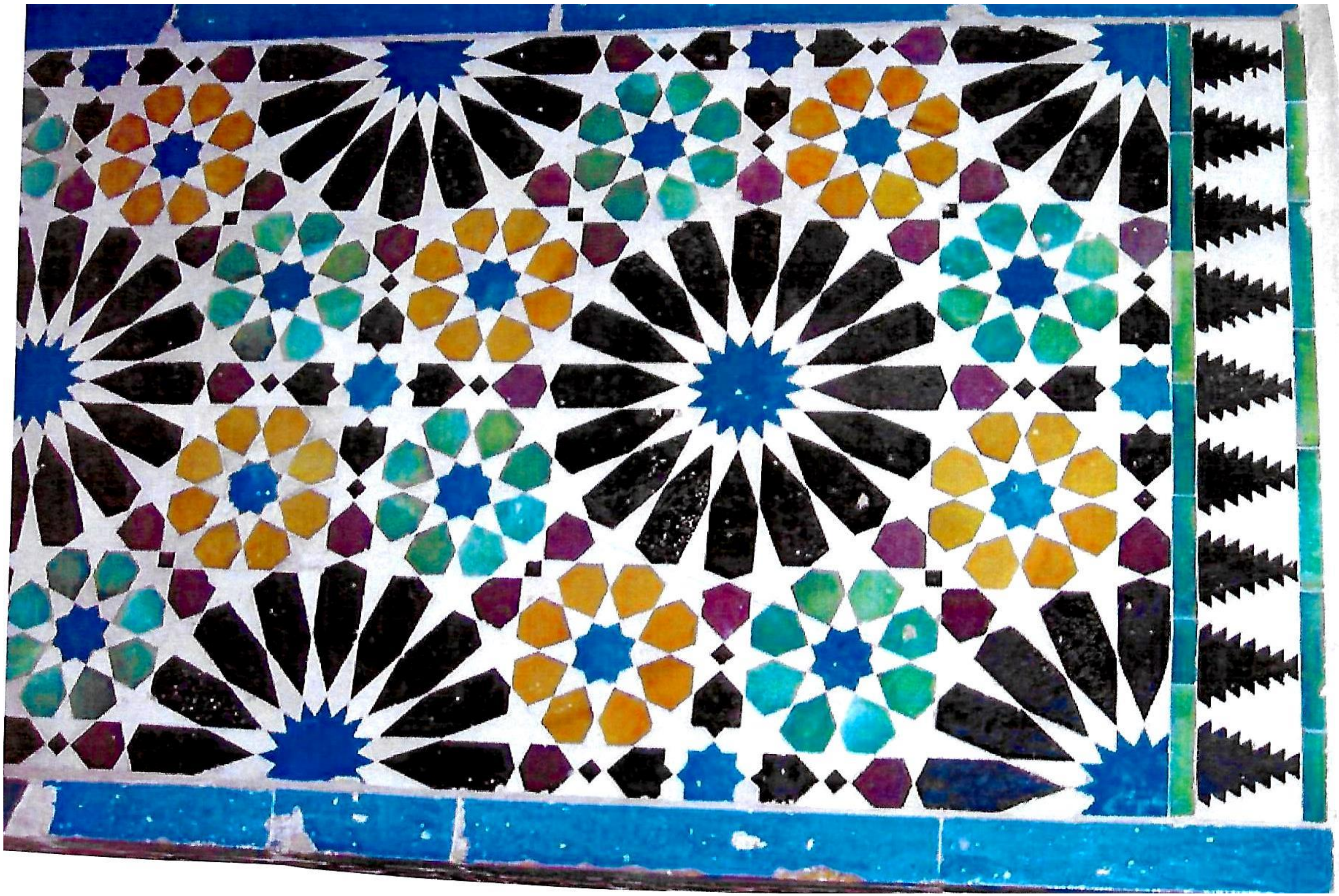
PRINCIPAL ROUTES OF SCIENCE

FROM THE ARABIC–ISLAMIC WORLD INTO EUROPE

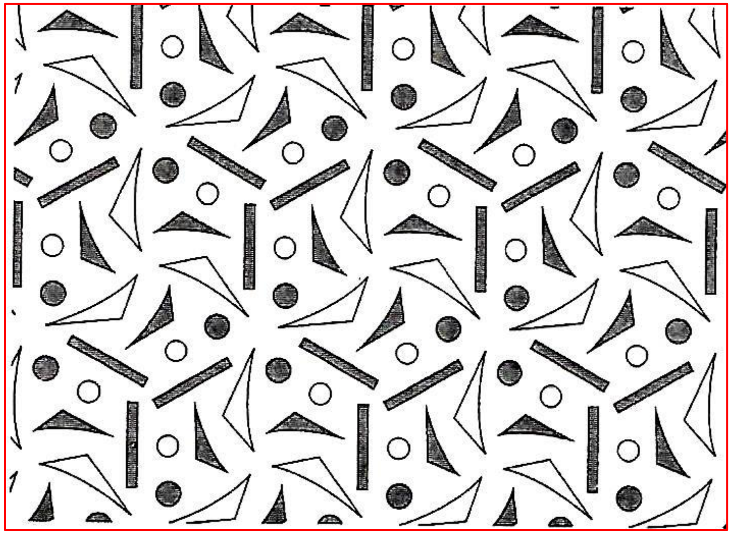


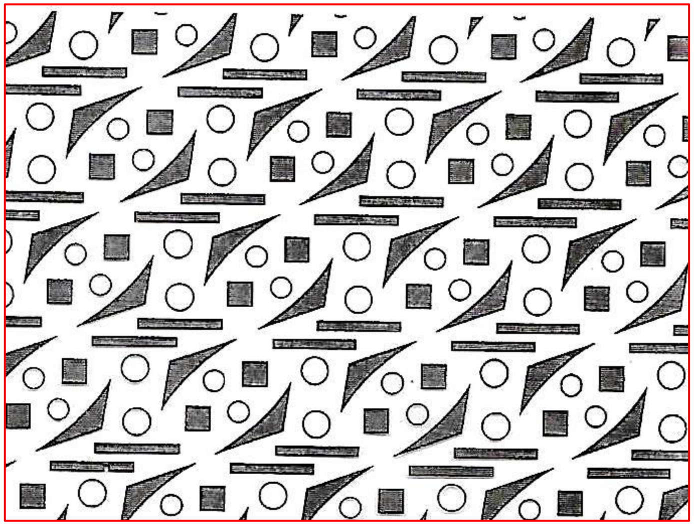
Southern Spain (Córdoba)

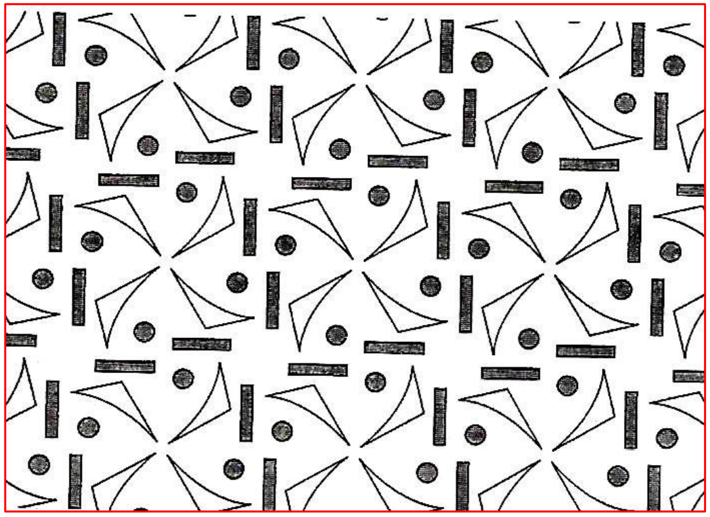
Arabic tiling (Alhambra)

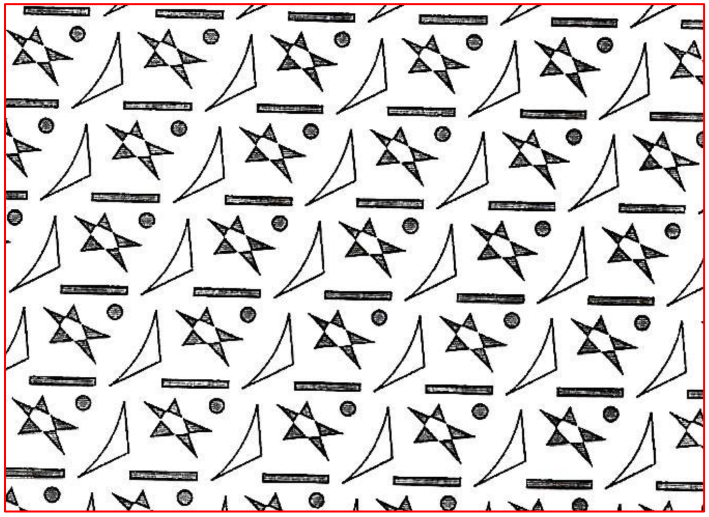


Wallpaper patterns (17 in all)









Jewish combinatorics:

Sefer Yetsirah

(Book of Creation)

(2nd-8th century)

Sefer Yetsirah

דברים, דבירם, דבימר, דבמיר, דבמרי, דברמי, דרבים, דרבמי, דרימב, דריבם, דרמבי, דרמיב,
דיברם, דירבם, דירמב, דיבמר, דימרב, דימבר, דמביר, דמברי, דמרבי, דמריב, דמיבר, דמירב,
בדימר, בדירם, בדרמי, בדרים, בדמרי, בדמיר, בירמד, בידרם, בירדם, בימדר, בימרד, בידמר,
במדרי, במדיר, במריד, במרדי, במדיר, במירד, בדמיר, בדמרי, בדרמי, בדרים, בדימר, בדירם,
רימדב, רימבד, רידבם, רידמב, ריבדם, ריבמד, רדמבי, רדבמי, רדיבם, רדימב, רדבים, רדמיב,
רבמדי, רבדמי, רבדים, רבידם, רבימד, רבמיד, רמידב, רמיבד, רמדיב, רמביד, רמבדי, רמדבי,
ימדרב, ימרדב, ימבדר, ימברד, ימרבד, ימדבר, ידברם, ידרבם, ידבמר, ידרמב, ידמבר, ידמרב,
יבדרם, יבדמר, יברדם, יברמד, יבמדר, יבמרד, ירדמב, ירדבם, ירמבד, ירבמד, ירבדם, ירדמב,
מדברי, מדרבי, מדביר, מדיבר, מדירב, מדריב, מבדיר, מבידר, מבריד, מבירד, מברדי, מבדרי,
מרדבי, מרבדי, מרדיב, מרביד, מריבד, מרידב, מידבר, מידרב, מיברד, מיבדר, מירדב, מירבד

Number of permutations of five letters
= 5! = 120

Sefer Yetsirah (2nd–8th century)

Suitable combinations of letters had power over Nature:

God drew them, combined them, weighed them, interchanged them, and through them produced the whole creation and everything that is destined to be created.

For two letters, *he combined the aleph with all the other letters in succession, and all the other letters again with aleph; bet with all, and all again with bet; and so the whole series of letters*.

Thus, there are (22 × 21)/2 = 231 formations in total.

Saadia Gaon (AD 892-942)

[After calculating $n!$ up to $7! = 5040$, and using $n! = (n - 1)! \times n$]

If you want to know the number of permutations of 8 letters, multiply the 5040 that you got from 7 by 8 and you will get 40,320 words; and if you search for the number of permutations of 9 letters, multiply 40,320 by 9 and you will get 362,880; and if you search for the number of permutations of 10 letters, multiply 362,880 by 10 and you will get 3,628,800 words; and if you search for the number of permutations of 11, multiply these 3,628,800 by 11 and you will get 39,916,800 words.

And if you want to know still larger numbers, you may operate according to the same method. We, however, stopped at the number of 11 letters, for the longest word to be found in the Bible [with no letter repeated] contains 11 letters.

Rabbi ibn Ezra (1090-1167)

Possible conjunctions of 7 ‘planets’: C(7, k) for k = 2, 3, . . . , 7.

It is known that there are seven planets. Now Jupiter has six conjunctions with the other planets. Let us multiply then 6 by its half and by half of unity. The result is 21, and this is the number of binary conjunctions (that is, C(7, 2) = 21.)

For k = 3, the answer is C(7, 3) = 35.

k=3:					
	567	467	367	267	167
		456	356	256	156
		457	357	257	157
			345	245	145
			346	246	146
			347	247	147
				234	134
				235	135
				236	136
				237	137
					123
					124
					125
					126
					127

total:

1 + 3 + 6 + 10 + 15 = 35

ben Gerson's *Maasei Hoshev* (1321)

Proposition 63:

If the number of permutations of a given number of different elements is equal to a given number, then the number of permutations of a set of different elements containing one more number equals the product of the former number of permutations and the given next number.

$$P(n+1) = (n+1) \times P(n)$$

$$(n+1)! = (n+1) \times n!$$